Canonical quantum gravity

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- **Lecture 1**: Introduction: The early beginnings (1984-1992)
- Lecture 2: Formal developments (1992-4)
- Lecture 3: Physics (1994-present)

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Today’s lecture

- Motivation: why do we insist on trying to quantize ordinary general relativity?
- Canonical quantization of GR, a quick tour.
- Ashtekar’s new variables.
- Loops and the loop representation.

One normally hears that there are problems when people try to quantize gravity. It has been tried since the 30’s, and a strong effort took place in the 50’s and 60’s, when people applied the technologies of quantum field theory that so successfully had handled the other interactions, to the gravitational field.

What really happened is that perturbative quantization techniques were applied, in which one assumes that the metric is the flat metric plus small perturbations, and quantizes the latter as an interacting theory of spin-2 fields living in flat space. The resulting theory diverges badly and is non-renormalizable, as can be shown by simple power counting arguments.

This led a group of people to draw an analogy to what had happened with the weak interactions. The earlier theory due to Fermi was also non-renormalizable. Yet, it was discovered that it was only an “effective” theory and that a more correct theory did account for the weak interactions and was renormalizable.

The analogy therefore would be that general relativity (which was developed in 1916 long before quantum mechanics) perhaps is not a fundamental theory of nature but an effective theory that works at low energies (very well). We therefore should not expect to quantize it, very much as one does not quantize the Navier-Stokes theory to describe a quantum fluid. What one needs -according to this point of view- is a more fundamental theory of gravity that yields general relativity at low energies, but that behaves much better upon quantization.

Over the years several candidates have been proposed.

This point of view is quite reasonable. However, it is not the only one that can be taken.

In particular, it should be noted that what fails is the idea of perturbatively quantizing general relativity, and more precisely a given perturbative scheme fails.

It is not true that if a theory is not quantizable perturbatively it cannot be quantized. DeWitt has studied several non-linear Sigma models that do not exist perturbatively but that have satisfactory quantum theories (constructed, for instance in the lattice).

Worse, perturbative schemes are specific. For instance, if one attempts to quantize general relativity in 2+1 dimensiones perturbatively by assuming that the metric is flat plus perturbations, one also concludes that the theory is non-renormalizable. Yet Witten taught us that the theory exists non-perturbatively and after the fact succesful perturbative schemes were found (for instance starting with a vanishing metric).

Moreover, there are good reasons why one may a priori expect that things are not going to be easy for quantum gravity:

a) The theory does not have a unique notion of time, fundamental ingredient in quantum mechanics.
b) The theory is diffeomorphism invariant, and we had (and still have) very little experience on how to do quantum field theory of diffeomorphism invariant theories, especially with local d.o.f.'s.
c) As we will see, even classically some of these issues are hard (e.g. observables).

These observations have convinced a group of people that one is not necessarily wasting one's time trying to understand better if general relativity can be quantized. Even if we ultimately fail and indeed there is a deeper theory that describes things, the knowledge acquired of how to deal with a,b,c) will surely be useful to understand the ultimate theory that describes gravitational interactions.

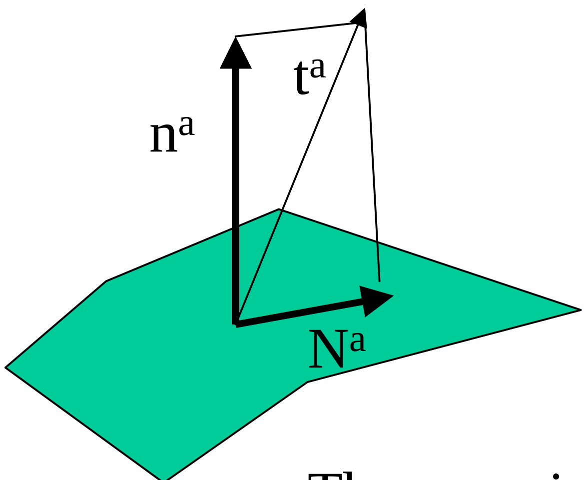
So in these lectures we will try to apply the rules of quantum mechanics to general relativity and see how far we can get. We will consider the quantization in the canonical approach. One could also attempt path integration (and some people are attempting this). However, canonical quantization has the advantage of being more explicit (you need to know what your space of states is, what your operators are and what your inner product is) and this is very useful when you are dealing with a theory one does not understand well.

Sketch of canonical quantization:
1) Pick a Poisson algebra of classical quantities.
2) Represent these quantities as quantum operators acting on a space of quantum states.
3) Implement any constraint you may have as a quantum operator equation and solve for the physical states.
4) Construct an inner product on physical states.
5) Develop a semiclassical approximation and compute expectation values of physical quantities.

So the first thing you need is a canonical formulation of your theory, in this case general relativity. This was first worked out by Dirac and Bergmann in the late 60's.

One starts by considering the Hilbert action $S = \int L dt = \int d^4x \sqrt{-g} R$

And considers a foliation of space-time into space and time,



$t^a = N n^a + N^a$

N lapse
N^a shift vector

$g_{ab} = g^4_{ab} + n_a n_b$

$q_a{}^b = \delta_a{}^b + n_a n^b$

g_{ab} spatial metric (+,+,+)

The canonically conjugate variable to g_{ab} is called π^{ab}

$\pi^{ab} = \sqrt{g}\,(K^{ab} - K g^{ab})$ $\quad K_{ab} = q_a^c q_b^d \nabla_a n_b = L_n q^{ab}$

$\dot{q}^{ab} = L_t q^{ab}$

$ds^2 = -N^2 dt^2 + q_{ab}(dx^a + N^a dt)(dx^b + N^b dt)$

Canonical momenta are densities. I will denote this from here on with a tilde (this turns out to have some importance later).

If one examines carefully the Hilbert action, one notices that there are no time derivatives of the lapse N or the shift N^a and therefore their canonically conjugate momenta vanish. These are conditions that have to be satisfied on any spatial surface. They are called constraints. There are four of them, three associated with the shift and one with the lapse. The three constraints associated with the shift are called “momentum” or “diffeomorphism” constraints. They are either denoted as a co-vector (density) C_a or some times, for convenience they are integrated against an fixed test vector and denoted $C(\vec{N}) = \int d^3x N^a(x) C_a(x)$

The one associated with the lapse is called the “Hamiltonian” constraint and is a (doubly densitized) scalar, which sometimes it is presented integrated against an arbitrary density of weight -1,

$$C(N) = \int d^3x \underset{\sim}{N}(x) \tilde{\tilde{C}}(x)$$

With the variables we are using, the explicit form of the constraints is,

$$C_a = D_a \tilde{\pi}^{ab}$$

$$\tilde{C} = g\,{}^3R - \tilde{\pi}^{ab}\tilde{\pi}_{ab} + \frac{1}{2}\tilde{\pi}^2$$

The vector or diffeomorhpism constraint has a simple geometric interpretation. If one considers its Poisson bracket with a function of the canonical variables, one finds,

$$\left\{ C(\vec{N}); f(g_{ab}; \tilde{\pi}^{ab}) \right\} = L_{\vec{N}} f(g_{ab}; \tilde{\pi}^{ab})$$

That is, the constraint is in the canonical language, the infinitesimal generator of diffeomorphisms on the spatial surface.

(to follow usual notation I will from now on use q^{ab} to represent the spatial metric, although some typos might exist)

In particular, the constraints themselves are covariant,

$$\left\{ C(\vec{N}), C(\vec{M}) \right\} = C([\vec{N}, \vec{M}])$$

$$\left\{ C(\vec{N}), C(M) \right\} = C(L_{\vec{N}} \vec{N})$$

$$\{ C(N), C(M) \} = C((N \partial_a M - M \partial_a N) g^{ab})$$

The diffeomorphism algebra is a “Lie” algebra, but the algebra involving the Hamiltonian constraint has non-constant structure functions.

If one performs a Legendre transform, one finds that the theory has a “Hamiltonian” that is a combination of the constraints.

$$H = N^a C_a + NC$$

If one considerst the Poisson bracket of the “Hamiltonian” with a function of the canonical variables, one gets the “time” evolution along the vector t^a we introduced, interpreted as a “shift” given by the diffeomorphism constraint and “evolution” perpendicular to the surface given by the Hamiltonian constraint.

Notice that we are dealing with an unusual theory in the sense that the Hamiltonian vanishes. This is in line with the fact that we introduced a fiducial notion of time by picking a foliation. The formalism is therefore telling us that it was just an arbitrary choice and it does not necessarily represent “time” evolution.

How would one go about quantizing this theory? One could start by picking as the classical Poisson algebra just the three-metric and its canonically conjugate momenta. One would then consider functions, for instance, of the three metric and represent,

$$\hat{q}_{ab}\;(q) = q_{ab}\;(q); \qquad \hat{\pi}^{ab}\;(q) = \frac{\partial}{\partial q_{ab}}\;(q)$$

One would then require that the wavefunctions be annihilated by the constraints, written as quantum operator equations. Geometrically this means that the wavefunctions be invariant under the symmetries of the theory.

Here one runs into trouble. The Hamiltonian constraint is a complicated, non-polynomial expression that needs to be regularized. Usual regularization procedures do not preserve the covariance of the theory or require external background structures.

Moreover, we have almost no experience on handling these kinds of wavefunctions. We know very little about the functional space we are dealing with and in particular do not have a measure of integration that would yield a physically valid inner product.

In addition to this, even if one ignores these difficulties, there is the issue of what to compute with a quantum theory like this. One should only consider of physical interest quantities that have vanishing Poisson brackets with all the constraints. Such quantities are usually called “observables” although in gravity Kuchar has suggested to call them “perennials”. NO such quantities are known for general relativity (in a compact manifold).

These difficulties stalled the progress on canonical quantization since the 60’s.

In 1984 Ashtekar proposed a new set of variables for studying canonical quantum gravity. The first step consists of using triads to encode the metric information,

$$\det q \, q^{ab} = \tilde{E}^a_i \tilde{E}^b_i$$

The canonically conjugate variable is related to the extrinsic curvature, $K_a{}^i = K_{ab} E^{bi}$

This presentation follows Barbero gr-qc/9410014

The constraints become,

$$\epsilon_{ijk} K^j_a \tilde{E}^{ak} = 0$$

New constraint (more variables)

$$D_a \left[\tilde{E}^a_k K^k_b - \delta^a_b \tilde{E}^c_k K^k_c \right] = 0$$

$$-\zeta \sqrt{q} R + \frac{2}{\sqrt{q}} \tilde{E}^{[c}_k \tilde{E}^{d]}_l K^k_c K^l_d = 0$$

Where ζ=-1 for Lorentzian and ζ=1 for Euclidean space-time

Formulating general relativity with triads instead of metrics was not new, it had been tried before and the problems are similar to using the metric formulation when one tries to build a quantum theory.

Ashtekar's new insight was to introduce a new variable canonically conjugate to the triad, via the canonical transformation,

$$\mathbf{A}^{\mathbf{i}}_{\mathbf{a}} = \text{È}^{\mathbf{i}}_{\mathbf{a}} + \text{ì}\,\mathbf{K}^{\mathbf{i}}_{\mathbf{a}}$$

Then the constraints become,

$$\tilde{G}_i \equiv \nabla_a \tilde{E}^a_i = 0$$

$$\tilde{V}_a \equiv F^i_{ab} \tilde{E}^b_i = 0$$

$$\tilde{\tilde{S}} \equiv -\zeta \epsilon^{ijk} \tilde{E}^a_i \tilde{E}^b_j F_{abk} + \frac{2(\beta^2\zeta - 1)}{\beta^2} \tilde{E}^a_{[i} \tilde{E}^b_{j]} (A^i_a - \Gamma^i_a)(A^j_b - \Gamma^j_b) = 0$$

So we see that if we choose the Immirzi parameter to be the imaginary unit (in the Lorentzian case) or one (in the Euclidean case), the constraints become polynomial functions of the fundamental variables.

The first constraint becomes a Gauss law, stating that the theory is invariant under frame rotations (Euclidean group in 3 dimensions, that is, SO(3)).

$$\tilde{G}_i \equiv D_a \tilde{E}^a_i = 0$$

The second constraint, which we know correspond to diffeomorphism invariance, are written simply as the vanishing of the Poynting vector.

$$\tilde{V}_a \equiv F^i_{ab} \tilde{E}^b_i = 0$$

The complicated Hamiltonian constraint can be made a simple expression quadratic in the momenta,

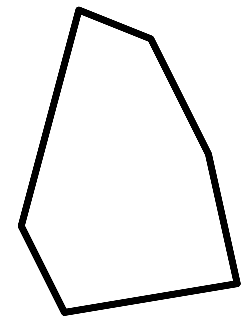
$$\epsilon^{ijk} \tilde{E}^a_i \tilde{E}^b_j F_{abk} = 0$$

Another interesting aspect of this reformulation is that we can think of general relativity as an unusual type of SO(3) Yang-Mills theory. In particular its (unconstrained) phase-spaces are the same, and at least one of the constraints is the same. General relativity can be viewed as Yang-Mills theory with some extra constraints (and a different Hamiltonian).

We therefore can think of attempting the quantization of general relativity as we would do for a Yang-Mills theory. In particular, it is natural to consider a polarization in which one considers wavefunctions of the connection $\Psi[A]$. Notice that this is significantly different from what one would have done in the metric representation.

In such a representation, the first constraint just requires that the wavefunctions be invariant under SO(3) gauge invariant, just like wavefunctions of Yang-Mills theory.

An example of wavefunction that is invariant under SO(3) gauge transformation is the trace of the holonomy of the connection along a loop, also called "Wilson loop",



$$W_\gamma[A] = \mathrm{Tr}\,P\exp\left(-\oint_\gamma dy^a A_a\right); \qquad \gamma \text{ closed loop}$$

Wilson loops are annihilated by the Gauss law, but not by the diffeomorphism constraint. The constraint has a simple geometric action, though, stating that the constraint shifts the loop,

$$C(\vec{N})W_\gamma[A] = W_{L_\gamma \gamma}[A]$$

Wilson loops are known to be an overcomplete (more later) basis for all gauge invariant functions *(Giles PRD24, 2160, (1981))* With the Wilson loops we solve the Gauss law constraint. One is left with the diffeomorphism and Hamiltonian constraint. It is clear that trying to solve the diffeomorphism constraint in terms of Wilson loops will not be too effective.

The way to handle this is by changing representation to the loop representation. Basing ourselves on the fact that Wilson loops are a basis we can formally expand any gauge invariant function of a connection as,

$$\Psi[\mathrm{A}] = \text{"}\sum_{g} \Psi[g]\mathrm{W}_{g}[A]\text{"}$$

At the moment this is only a formal expression, we will see later that we can get better control of it. It is analogous to what one does when one goes to the momentum representation in quantum mechanics

$$\Psi[\mathrm{x}] = \int \mathrm{dk}\Psi[\mathrm{k}]\exp(\mathrm{ikx})$$

The coefficients of the expansion are functions of loops. They are the “wavefunctions in the loop representation” given by the “inverse loop transform”

$$\Psi[g] = \int dA \Psi[A] W_g[A]$$

Why would we want to go the loop representation? Because in this representation it is immediate to write solutions to the diffeomorphism constraint. One simply has to consider functions of loops that are invariant when one applies a diffeomorphism to the loop. That is, they have to be what mathematicians call **knot invariants**.

This opens an unexpected connection between knot theory and quantum gravity, first noted by Rovelli and Smolin (1988). It also illustrates why using these new variables opens new perspectives on the problem not available with the traditional variables.

What about the Hamiltonian constraint? Let us go back to the connection representation for a moment, and consider the action of the Hamiltonian constraint on a Wilson loop,

$$\hat{H} W_\gamma[A] = \epsilon^{ijk} \hat{E}^a_i \hat{E}^b_j \hat{F}^k_{ab} W_\gamma[A]$$

The Hamiltonian involves products of operators that need to be regularized. Let us be simple minded and consider a point-splitting regularization with a fiducial background metric. Let us also consider a factor ordering in which the electric fields are to the right. We then have

$$\hat{H}_\epsilon W_\gamma[A] = \int d^3x\, d^3x\, d^3z\, f_\epsilon(x;y;z) \hat{F}^i_{ab} \frac{\delta}{\delta A^j_a(y)} \frac{\delta}{\delta A^k_b(z)} W_\gamma[A]$$

The action of the functional derivative on the Wilson loop is,

$$\frac{\delta}{\delta A_a^j(y)} W_\gamma[A] = \oint_\gamma dw^a \delta^3(y - w) Tr[P \exp(\int_0^w dz^b A_b)\tau^j Tr[P \exp(\int_w^0 dz^b A_b)]$$

The important aspect to notice is that the right hand side is given by an integral of the tangent vector of the loop and is fixed at the point y by the delta function. We can therefore schematically write,

$$\frac{\delta}{\delta A_a^j(y)} W_\gamma[A] = \dot{\gamma}^a(y) \times \text{ etc }^j$$

This has a remarkable implication: since in the Hamiltonian two functional derivatives are contracted with F_{ab}which is antisymmetric in the indices a,b, it implies that it vanishes. This is really not true, since the vectors are multiplied times objects that are distributional. It just shows that formally, if the loops are smooth (otherwise one could have two tangent vectors at a given point) the Hamiltonian constraint vanishes.

Jacobson and Smolin NPB1997

Now let's go back to the loop representation and consider wavefunctions $\Psi[\gamma]$ that vanish if the loop is not smooth and that are knot invariants. Such functions solve the Gauss law, the diffeomorphism constraint and (formally) the Hamiltonian constraint. When this was noticed, it was the first time that (formally) the Hamiltonian constraint (Wheeler-DeWitt equation) was solved in a generic situation. This generated a lot of interest in this approach.
Rovelli and Smolin (1988)

As we discuss, there are a lot of caveats with these observations, many of which have been overcome to a certain extent. We will spend most of lecture 2 dealing with them.

Let us consider another type of states that can be formally constructed and attracted a lot of ongoing interest in this formalism.

Let us go back to the connection representation and consider the following function of a connection,

$$\Psi_\kappa[A] = \exp\left(\kappa \int d^3x \, Tr\left(A \wedge \partial A + \tfrac{2}{3} A \wedge A \wedge A\right)\right)$$

That is, a state given by the exponential (up to a constant k) of the spatial integral of the Chern-Simons form built with the Ashtekar connection. Such a state is known to be gauge invariant and is also diffeomorphism invariant (it is a scalar built only with the fields, and no additional background structure).

The Chern-Simons form has been used to define some topological theories in 3 space-time dimensions. We will get back to this. At the moment let us just consider one property of these states,

$$\hat{E}^a_i(x)\Psi_\kappa[A] = \frac{\delta}{\delta A^i_a}\Psi_\kappa[A] = \frac{1}{\kappa}\epsilon^{abc}F^i_{ab}(x)\Psi_\kappa[A]$$

That is, for this state the “electric field” is proportional to the “magnetic field”.

Now, let us consider the Hamiltonian constraint of general relativity with a cosmological constant,

$$H = \epsilon^{ijk} \tilde{E}^a_i \tilde{E}^b_j F^k_{ab} + \frac{\Lambda}{6} \epsilon^{ijk} \tilde{E}^a_i \tilde{E}^b_j \tilde{E}^c_k$$

And if we consider a quantum state for which F~E, then the two terms in the Hamiltonian constraint are equal. This is what happens for the Chern-Simons state. If one chooses k=6/Lambda, then the constraint is identically satisfied.

In other words, the Chern-Simons state satisfies all the constraints of quantum gravity with a cosmological constant.

What does this state look like in the loop representation?

If we write the loop transform for the Chern-Simons state,

$$\Psi^{CS}[\gamma] = \int DA \exp\left(\frac{6}{\Lambda} Tr \int \left(A \wedge \partial A + \frac{2}{3} A \wedge A \wedge A\right)\right) W_\gamma[A]$$

Now, this integral has been considered by Witten (1988) in a completely different context. He was interested in computing the expectation value of a Wilson loop in a Chern-Simons topological field theory. He noted that the result of the integral is a knot invariant closely related to the celebrated Jones polynomial (it is a polynomial in k, or in our case in 1/Lambda) (it is called the Kauffman bracket).

So if we believe in the loop representation, we must conclude that the Kauffman bracket is a solution of all the constraints of general relativity with a cosmological constant.

B. Bruegmann,R. Gambini, J. Griego, C. Di Bartolo, JP 1992

So these were the "wild years". Many suggestive facts about the possibility of making breakthroughs in solving the quantum constraint equations of canonically quantized general relativity were discovered. But most of the results were formal and loaded with difficulties.

Let me mention one difficulty we glossed over in the presentation. We said that the Wilson loops were an "overcomplete" basis of gauge invariant states. Let me explain what this means. Suppose we are working in SO(3) (we are really not, that is the next caveat I'll discuss). In the fundamental representation, matrices of SO(3) are not generic 2x2 matrices, but they actually satisfy certain relationships, like for instance,

$$\mathrm{Tr}\mathbf{A}\,\mathrm{Tr}\mathbf{B} = \mathrm{Tr}(\mathbf{AB}) + \mathrm{Tr}(\mathbf{AB}^{-1})$$

Or, in terms of Wilson loops,

$$\mathbf{W}_{\gamma}[A]\mathbf{W}_{\eta}[A] = \mathbf{W}_{\gamma\circ\eta}[A] + \mathbf{W}_{\gamma\circ\eta^{-1}}[A]$$

Now, this does have implications. It means wavefunctions in the loop representation are not free quantities. We are talking about the coefficients in a basis of overcomplete vectors. That means that not any function of a loop can be considered a wavefunction of the gravitational field. In particular, it implies that "a function that is non-vanishing on smooth loops only" is not a good candidate.

As we see, just using functions of loops is a problem.

The other point we need to consider is that if we consider the expression of the (doubly densitized) metric in terms of the new variables

$$\tilde{\tilde{q}}^{ab} = \tilde{E}^a_i \tilde{E}^b_i$$

And think of the corresponding quantum operator, we see that the metric is, as a matrix, degenerate everywhere on loop states except at intersections. We therefore must consider such loops.

Finally, we saw that the canonical transformation defining the Ashtekar connection had the form

$$A_a^i = \text{È}_a^i + \text{ì}\, K_a^i$$

And that the “Immirzi” parameter had to be the imaginary unit for the Hamiltonian to have the simple form we have been considering. This implies that one is dealing with a theory of complex variables.

At a quantum mechanical level, we need to impose extra conditions to ensure that at the end of the day we are dealing with real general relativity. One possibility that was suggested was to request that observables (perennials) of the theory be real. At a quantum mechanical level this means they should be self-adjoint operators with respect to the inner product one selects. In particular, this can be used as a selection criterion for the inner product. Unfortunately, the lack of explicit observables makes this difficult. We will also see that it is harder to handle theories with non-compact groups as complex SO(3).

Conclusions:

- These first years were ripe with suggestive, but formal results.
- The results indicated attractive connections between canonical quantum gravity, knot theory, topological field theory and Yang-Mills theories.
- In the next lecture we will discuss how to set these results on a much firmer footing.