The Pennsylvania State University

The Graduate School

Department of Physics

PROBING QUANTUM GENERAL RELATIVITY THROUGH EXACTLY SOLUBLE MIDI-SUPERSPACES

A Thesis in

Physics

by

Monica Pierri-Galvao

Submitted in Partial Fulfillment
of the Requirements
for the Degree of

Doctor of Philosophy

August 1996

We approve the thesis of Monica Pierri-Galvao.

Date of Signature

Abhay V. Ashtekar
Eberly Professor of Physics
Thesis Adviser
Chair of Committee

Jane Charlton
Assistant Professor of Astronomy and Astrophysics

Curt Cutler
Assistant Professor of Physics

Murat Günaydin
Professor of Physics

Jorge Pullin
Assistant Professor of Physics

Howard Grotch
Professor of Physics
Head of the Department of Physics

Abstract

Quantization of general relativity faces certain problems because the theory is diffeomorphism invariant and, at the same time, has an infinity number of degrees of freedom. To gain insight in to these problems, in this thesis we will consider two models that share these difficulties but are exactly soluble: Einstein-Rosen waves and Gowdy T^3 space-times. These are vacuum solutions of 3+1-dimensional general relativity which admit certain symmetries. One can show that they are in one-to-one correspondence to axi-symmetric, 2+1-dimensional general relativity coupled with a massless Klein-Gordon field.

We will canonically quantize both models from the 2+1 perspective. For Einstein-Rosen waves, in contrast to previous works, we will construct the Hamiltonian formulation which carefully incorporates the asymptotically flat boundary conditions. This step involves several subtle issues which have ramifications also in the quantum theory. In the case of Gowdy models, a satisfactory quantum theory with a clear and coherent physical interpretation is lacking. We will fill this gap. Thus, in both cases, we will present complete solutions to the problem of quantization.

Finally, we will use the two quantum theories to analyze several conceptual and technical issues of quantum gravity.

Table of Contents

Acknowledgments

First and most important of all I want to thank my adviser, Prof. Ashtekar. For endless hours that he spent with me, and for always being open for discussions. Now, I have his prints on my thoughts and ideas in physics. I am most thankful for having this privilege.

I wish to thank Prof. Smolin for sharing his enthusiasm on long teas and discussions. He was always inspiring. Prof. Pullin, for teaching me almost all that I know in loop representation. I also would like to thank Prof. Cutler and Prof. Pullin for helpful comments on this thesis.

I wish to thank Alejandro Corichi for countless discussions and long conversations. I also would like to thank some other graduate students that were with me for some of these years with whom I shared ideas and concerns: Seth Major, Troy Schilling, Sumati Surya, Madhavan Varadarajan and Jose Zapata. For so many years there are so many people involved that it is difficult to make justice for all.

There is no way of expressing my gratitude for Felipe and Bruno. They were part of all, without them nothing of this would be possible. I also want to thank Hiron and Lucila, they have always been of great importance in all the steps of my life.

I thank Capes (Brazil) and The Pennsylvania State University for the financial support.

Chapter 1

Introduction

The problem of quantization of the gravitational field has certain features that are absent in other quantum field theories. This is not surprising . For, as Einstein taught us through general relativity, the gravitational field plays a dual role which is not shared by other interactions: in addition to encoding a specific physical interaction, it describes the "setting" where *all* interactions take place. This peculiarity leads to a host of new conceptual and technical difficulties. To face these, one needs to develop new strategies.

In broad terms, there exist two sets of approaches to deal with these problems. The first is perturbative and mimics the procedures which have been highly successful in particle physics. Here, one begins by splitting the physical space-time metric into two parts; a fiducial background metric (generally taken to be flat) which is to determine, in the first step, the underlying geometry and the remainder, which is to represent the gravitational field. One quantizes the latter on the background space-time given by the former and studies the interactions between the resulting spin-two quanta among themselves and with matter fields. At first sight, this strategy seems to by-pass all the conceptual difficulties that arise from the fact that the physical metric simultaneously plays two roles. However, as is well-known, this strategy does not work. Specifically, if one begins with general relativity as the starting point, the resulting quantum theory turns out *not* to be renormalizable.

A second approach, that still uses the basic technical ideas of standard quantum field theory but does not rely on a background space-time metric, is the canonical formulation. In this case, the topology of the manifold is fixed once and for all to be $\Sigma \times \mathrm{R}$, and the basic phase space variables are fields on Σ. General relativity in the canonical approach is then described by a suitable set of phase space variables with closed Poisson algebra, subject to constraints. As examples of such sets we have: The geometrodynamical variables, where the phase space variables are the spatial metric of Σ and its conjugate momentum (related to the extrinsic curvature) and the Ashtekar connection formulation, where the set of variables is given by a self-dual connection and a triad.

Quantization is accomplished by constructing a Hilbert space of quantum states with an appropriate inner-product such that the phase space variables are promoted to well-defined operators. However, the detailed procedure involves a number of non-trivial issues. Technically one is dealing with an infinite number of degrees of freedom which, moreover, are not all independent. Thus, there are non trivial constraints, which are difficult to deal with in the quantum theory: neither the procedure of solving the constraints classically, nor the method of imposing them as operator conditions on states is easy to implement. Another technical but also conceptual issue is 'the problem of time'. Since we are quantizing space-time geometry time is not fixed once and for all. The notion of time is intertwined with the field that is being quantized. So, how is time to be described in the quantum theory? If one is to fix gauge before quantizing, which choice of time should be made and are they all equivalent? Further specific questions arise depending on whether Σ is compact or asymptotically flat. In the compact case there is no distinction between dynamics and gauge. The total energy of the system vanishes because there are no boundary terms. Therefore the process of fixing the gauge involves deparametrization of the theory and in general there is no preferred way to implement it, i.e., it is not apriori clear which variable or combination of variables should be chosen as time. However, in the asymptotically flat context there is a clear distinction between gauge and dynamics. The generator of asymptotic unit time translations does not vanish on the constraint surface and provides a notion of time in the asymptotic region. However, there is still freedom on the choice of time in the interior region. How does one deal with it?

A basic assumption in the quantization approach based on Ashtekar coordinatization of the phase space is that the "Wilson-loop operators" – which correspond to traces of holonomies of a connection around space-like loops – should be well-defined. Is this a reasonable assumption? Apriori it is not clear, since in the definition of these operators, one appears to smear a quantum field along a *one* dimensional object (rather than three or four).

Let us assume for a moment that we are able to handle all the issues above and finally quantize general relativity. Then we expect to have answers to some basic questions, such as: Do singularities that may be present in the classical theory get softened by quantum effects? Are there operators in the final Hilbert space corresponding to space-time geometries? If so, is there adequate structure to analyze how the light cones fluctuate? More generally, can we tie the canonically quantized theory to the quantum description that emerges from covariant approaches? In the classical theory there is a positive energy theorem [1, 2, 3]. Does it continue to hold in the quantum theory? Is the true ground state "peaked around" Minkowski space-time? Another question which plays an important role in semi-classical considerations is: Are there "coherent states" which are peaked at classical solutions?

To gain insight into some of these issues, a number of models have been discussed in the literature. In linearized gravity, for example, one can explore the issue of infinite number of degrees of freedom. However, the analysis relies entirely on a background structure. The Bianchi models have a finite number of degrees of freedom due to the spatial symmetries. They are useful to address technical and conceptual questions concerning quantum cosmology. Perhaps a more striking model is presented by 2+1 dimensional vacuum general relativity which, like the 4-d theory, is fully diffeomorphism invariant. Moreover, one can show that it corresponds, in fact, to a symmetry reduction of 3+1 dimensional general relativity. More precisely it is equivalent to 4-d gravity with one Killing vector field hypersurface orthogonal and of constant norm. Quantization of this model has shed light on the notion of observables, role of discrete symmetries, etc. These models have also given us considerable insights into the technical problems that arise due to the underlying diffeomorphism invariance. For example, since we have no Poincaré group to help us, the problem of finding the correct inner-product on the space of quantum states requires a new strategy. The 2+1 model has provided a method which, moreover, is free of ambiguities, e.g., in the deparametrization procedure.

However, none of these models has the two main characteristics of general relativity together, i.e., diffeomorphism invariance and an infinite number of degrees of freedom. An obvious strategy would be to consider again symmetry reductions which, however, are mild enough to leave behind local degrees of freedom. Therefore, it is natural to consider reductions with one Killing field without further restrictions on the norm or the twist. If the Killing field is time-like it will provide a preferred time slicing of the space-time. Hence, it does not give the opportunity to investigate the issue of time. Thus, we will concentrate on reductions with space-like Killing vector fields. In this case the 3+1 dimensional vacuum Einstein's equations are equivalent to 2+1 dimensional Einstein's equations coupled to a non-linear sigma model [4]. In fact there are an infinite number of degrees of freedom. Precisely two per space-time point. It has been investigated at the classical level in the compact [5] and asymptotically flat context [3] but it is a hard problem to deal with because the matter fields and gravity are non-trivially coupled. The coupling between gravity and matter fields persists even if the Killing vector is required to be hypersurface orthogonal. The reduced system is 2+1-dimensional gravity coupled to a massless scalar field. However, if we require that there be another Killing field which commutes with the first one (not necessarily hypersurface orthogonal), the problem decouples. The non-trivial equations of the reduced system describe two coupled scalar fields propagating in flat background. Because of the non-linearities of the matter equation it is still complicated to deal with. The next obvious restriction is to assume that the Killing vectors are hypersurface orthogonal, then the situation simplifies dramatically. For, now, in effect, all the non-linearities go away. There will be only one non-trivial Einstein's equation that is equivalent to a non-interacting scalar field propagating on a flat 2+1 dimensional space-time. Since this situation is simple enough to be exactly soluble, it provides a concrete arena to examine the issues of quantum gravity and to see how they can be resolved in practice.

In the spatially non-compact case, such space-times were considered already in the thirties by Einstein and Rosen [6], with $\mathrm{R} \times U(1)$ isometry group. Thus, they represent (one polarization) cylindrical gravitational waves. Their canonical quantization was considered in a remarkable paper by Kuchař [7] in 1971. The problem was considered again from a 2+1 perspective in [8]. However, the boundary conditions were not carefully treated. An appropriate treatment will play a significant role in the quantum theory, both conceptually and technically. The obvious corresponding compact case is to take $U(1) \times U(1)$ isometry group acting on a compact manifold $\Sigma = \mathrm{T}^3$. Such space-times were initially considered by Gowdy [9] and later canonically quantized by Berger [10]. However, the quantization procedure adopted there does not appear to be appropriate for the physical problem. Specifically, there is creation of 'particles' that has no physical meaning because it is associated to a Hamiltonian that arises from the deparametrization procedure.

In this thesis we shall return to the canonical quantization of such midi-superspaces. We will consider both, the non-compact and the compact case, from the 2+1 perspective. For the Einstein-Rosen waves, at the classical level, we will make an appropriate treatment of the canonical formalism in the asymptotically flat context. In the quantum theory, there are ultra-violet divergences that were also noticed before in [8] but were not appropriately addressed. We will provide a better regularization procedure for these divergences. In the compact case, we will obtain a correct physical representation for the quantum theory. It will consist of a 'single' Hilbert space and therefore, there will be no production of (unphysical) particles. More importantly, in both models, we will use the resulting theory to analyze the conceptual and technical problems of quantum gravity raised earlier in this chapter.

The plan of this thesis is the following: In chapter 2 we will consider the case Σ non-compact [11]. It is important that we look at this system from the $2+1$ dimensional perspective because we can assume asymptotic flatness and only then the Hamiltonian formalism is adequate. The boundary term in the action corresponds to the total energy of the system and does not vanish on the constraint surface. Therefore there is a distinction between gauge and dynamics and we can apply the standard gauge fixing procedure. Then, we will see that there is a 'natural' Hilbert space for the gauge fixed model. In the quantum theory, we will investigate the issues raised earlier in this section.

In chapter 3 we will consider the case Σ compact [12]. We will see that before gauge fixing the two models have formally the same Lagrangian. But, the topological differences in the isometry groups and in the spatial slices introduce some technical and conceptual changes. For instance, now, there is no generator of dynamics on the constraint surface because of the absence of boundary terms in the action. Hence, we will have to use the deparametrization approach in order to extract the true degrees of freedom of the theory. Also, the space-time has an initial singularity and a new global constraint is left after deparametrization that will be carried on to the quantum theory. Again we will see that there is a 'natural' Hilbert space for the reduced model. But now, the physical Hilbert space will be the subspace of the Hilbert space corresponding to the kernel of the global constraint. In the quantum theory we will again analyze the questions raised previously. At appropriate points, whenever applicable we will compare the situation with the Einstein-Rosen waves case.

Finally, In chapter 4 we will summarize the main results and point out open questions. In the appendix A, for the convenience of the reader, we will review the reductions of 4 dimensional Einstein's equations with Killing vector fields.

Chapter 2

Einstein-Rosen Waves

In this chapter, we will quantize Einstein-Rosen waves using the canonical formalism for 2+1 gravity coupled to matter field. In section 2.1 we consider the classical Hamiltonian formulation and isolate the true degrees of freedom by gauge fixing procedure. Because we are in the asymptotically flat situation, by treating the boundary conditions carefully, we can distinguish gauge from dynamics. In particular, the true degrees of freedom are naturally subject to non-trivial dynamics (without the need of any "deparametrization".) In section 2.2, we calculate the classical Wilson loop functions and express them in terms of the true degrees of freedom. Quantization is taken up in section 2.3. Finally, questions raised in chapter 1 are analyzed within this solution.

2.1 Hamiltonian Formulation

2.1.1 The midi-superspace

Let us begin with a precise specification of our midi-superspace. For definiteness, we will work in the 2+1-dimensional formulation. Thus, we will consider asymptotically flat, axi-symmetric solutions of 2+1-dimensional general relativity coupled to zero rest mass scalar-fields (where the rotational Killing field is hypersurface orthogonal). The underlying manifold M will be topologically R^3 and the space-time metric will have signature (–,+,+). For simplicity, we will assume that all fields under consideration are C^∞.

Denote by σ^a the rotational Killing field. Hypersurface orthogonality of σ^a implies that the space-time metric g_{ab} has the form:

$$g_{ab} = h_{ab} + R^2\, \nabla_a\sigma\, \nabla_b\sigma \tag{2.1}$$

where R is the norm of the Killing field and σ is the "angular coordinate"; $\nabla_a\sigma = R^{-2} g_{ab}\sigma^b$. The field h_{ab} so defined is a metric of signature (–, +) on the 2-manifolds orthogonal to σ^a. Let us introduce a space-like foliation of this 2-manifold by lines $t =$ const and a dynamical vector field $t^a = Nn^a + N^r\hat{r}^a$, where n^a is the unit, time-like normal to the foliation and $\hat{r}^a$ the unit (outgoing) vector field within each slice. The pair N, N^r constitutes the lapse and the shift. If we now introduce a radial coordinate r on any one leaf such that $r = 0$ at the axis (i.e., where $R = 0$) and r tends to infinity at spatial infinity, the 2-metric h_{ab} can be written as:

$$h_{ab} = (-N^2 + (N^r)^2)\nabla_a t\, \nabla_b t + 2N^r\nabla_{(a} t\, \nabla_{b)} r + e^\gamma \nabla_a r\, \nabla_b r\,, \tag{2.2}$$

where N, N^r and γ are functions of r and t. It is because of axi-symmetry, that the 3-metric g_{ab} has only four independent components and they are functions only of two variables.

Thus, our midi-superspace consists of five functions, $(N, N^r, \gamma, R, \psi)$ of r and t, on the space-time manifold M where ψ is the zero rest mass scalar field (which is also

Lie-dragged by the rotational Killing field). The five fields are subject to the following field equations:

$$G_{ab} = T_{ab} \quad \text{and} \quad g^{ab}\nabla_a\nabla_b\psi = 0\,, \tag{2.3}$$

where G_{ab} is the Einstein tensor of g_{ab} which is determined by the fields (N, N^r, γ, R) via (2.2) and T_{ab} is the stress-energy tensor of the scalar field ψ:

$$T_{ab} = \nabla_a\psi\,\nabla_b\psi - \tfrac{1}{2}(g^{cd}\nabla_c\psi\nabla_d\psi)g_{ab}\,. \tag{2.4}$$

(Here, we have used a normalization that arises naturally in the reduction from the 3+1 theory to the 2+1. From the 2+1 perspective, it is natural to regard $\phi := \psi/\sqrt{8\pi G}$ as the scalar field.)

Asymptotic flatness and regularity at the axis imply certain boundary conditions on our dynamical fields. We first note that g_{ab} reduces to a Minkowskian metric when $N = 1, N^r = 0, \gamma = 0, R = r$ and $\psi = 0$. The general asymptotic flatness conditions can be written as:

$$\begin{aligned} N &= 1 + N_1(r,t), \qquad N^r = N^r_o(t) + N^r_1(r,t) \\ \gamma(r,t) &= \gamma_\infty(t) + \gamma_1(r,t), \qquad R(r,t) = r(1 + R_1(r,t)) \end{aligned} \tag{2.5}$$

where, on any $t = \text{const}$ surface, $N_1, N^r_1, \gamma_1, R_1$ and the scalar fields ψ are of asymptotic order $O(1/r)$. (We will say that a function $f(r)$ is of asymptotic order $1/r$ if $rf(r), r^2 f'(r)$ and $r^3 f''(r)$ admits limits as r tends to infinity, where a prime denotes a derivative with respect to r.) While the conditions imposed on N, N^r, R and ψ are the obvious ones, the condition on the field γ seems surprising at first. For, even at infinity, γ is not required to approach its Minkowskian value, 0. The reason is that the asymptotic value of γ contains the information about mass: If $\gamma_\infty \neq 0$, the spatial metric has a deficit angle at infinity which measures the ADM mass [2, 3]. Thus, there is a striking contrast with asymptotic flatness in 3+1 dimensions; the space-time metrics in our midi-superspace do *not* approach a fixed Minkowskian metric at infinity. Note finally that these boundary conditions are somewhat simpler than those used in [3] where general 2+1-dimensional space-times were considered. Here, we can exploit the fact that we are now working in a highly restrictive context of cylindrical waves.

Finally, regularity at the axis is ensured by requiring that N^r, γ and R vanish there for all t. (Recall also that by assumption, N, N^r, γ, R^2 and ψ are C^∞ everywhere and, in particular, at $r = 0$.)

2.1.2 Phase Space

Let us begin with the 3-dimensional action with appropriate boundary terms:

$$\begin{aligned} S(g, \psi) \quad &:= \quad \frac{1}{16\pi G} \int_{M'} d^3x \, \sqrt{g}[\mathcal{R} - g^{ab} \nabla_a \psi \nabla_b \psi] \\ &+ \quad \frac{1}{8\pi G} \oint_{\partial M'} d^2x [K\sqrt{h} - K_o \sqrt{h_o}] \, , \end{aligned} \tag{2.6}$$

where M' is an open set in M; $\partial M'$, its boundary in M; $\mathcal{R}$, the scalar curvature of g; K and h, the trace of the extrinsic curvature of, and the determinant of the intrinsic metric on $\partial M'$ induced by g_{ab}; and, K_o and h_o are the corresponding fields induced by the Minkowski metric $\overset{\circ}{g}_{ab}$ (obtained by setting $N = 1, N^r = 0, \gamma = 0, R = r$ and $\psi = 0$).

To pass to the Hamiltonian formulation, one performs a 2+1-decomposition. Let us substitute in (2.6) the form of the metric given by Eqs. (2.1) and (2.2). Then, the action reduces to the standard form:

$$S = \frac{1}{8G} \int dt \left(\int dr (p_\gamma \dot{\gamma} + p_R \dot{R} + p_\psi \dot{\psi}) \; - \; H[N, N^r] \right) \tag{2.7}$$

The Hamiltonian H is given by:

$$H[N, N^r] = \frac{1}{8G} \int dr (NC + N^r C_r) \; + \frac{1}{4G}(1 - e^{-\gamma_\infty/2}) \tag{2.8}$$

where C and C_r are functions of the canonical variables,

$$\begin{aligned} C &= e^{-\gamma/2}(2R'' - \gamma' R' - p_\gamma p_R) + \frac{1}{2} R\, e^{-\gamma/2} (\frac{{p_\psi}^2}{R^2} + \psi'^2), \\ C_r &= e^{-\gamma}(-2{p'}_\gamma + \gamma' p_\gamma + R' p_R) + e^{-\gamma} p_\psi \psi' , \end{aligned} \tag{2.9}$$

and γ_∞ is the value of γ at $r = \infty$. (Here primes denote derivatives with respect to r.)

As expected, the lapse and shift functions N, N^r appear as Lagrange multipliers; they are not dynamical variables. Thus, the phase-space Γ consists of three canonically-conjugate pairs, $(\gamma, p_\gamma; R, p_R; \psi, p_\psi)$, on a 2-manifold Σ which is topologically R^2. The boundary conditions on the configuration variables (γ, R, ψ) have already been discussed. The conditions on the momenta can be deduced from their definitions in terms of these fields and their time derivatives. At infinity, p_γ and p_R fall-off as $O(1/r^2)$ while p_ψ falls off as $O(1/r)$. (Note that these conditions imply that action $\int drp_\gamma \delta\gamma, \int drp_R \delta R$ and $\int drp_\psi \delta\psi$ of the momenta on the tangent vectors $\delta\gamma, \delta R, \delta\psi$ to our configuration space are all finite, so that we have a well-defined (weakly non-degenerate) symplectic structure.) There are two first class constraints, $C = 0$ and $C_r = 0$, obtained by varying the action with respect to the Lagrange multipliers N and N^r. The Hamiltonian is given by H. (It is because of the underlying axi-symmetry that we have only one diffeomorphism constraint, C_r.)

Let us begin by analyzing the canonical transformations generated by the constraints. For this, we have to first smear the constraints and obtain well-defined functions on the phase space, say, $C[N_g] := \int drN_g C$ and $C[N_g^r] = \int drN_g^r C_r$. Using the boundary conditions on the phase space variables, it is straightforward to verify that these functions are well-defined *and* differentiable on the phase space if N_g is of asymptotic order $O(1/r)$ and N_g^r admits a limit at infinity. (From now on, the subscript g on smearing fields will indicate that they satisfy these boundary conditions.) Since the constraints are of first class, and since we are in the asymptotically flat context, the canonical transformations generated by these constraints can be regarded as "gauge" in an appropriate sense. As one might expect, $C[N_g]$ generates "bubble time evolutions" via lapses which go to zero at infinity while $C[N_g^r]$ generates spatial diffeomorphisms which are bounded at infinity. The situation with the Hamiltonian constraint is the same as the one we normally encounter in the 3+1-dimensional theory. For the diffeomorphism constraint, on the other hand, the situation is quite different since the diffeomorphisms generated by $N_g^r \hat{r}^a$ are not necessarily asymptotically identity. This is, however, the standard situation in 2+1 dimensions (see e.g., [2, 3]): In 2+1 dimensions, there are no asymptotic Killing fields corresponding to spatial translations and the ADM 2-momentum vanishes.

To obtain genuine time translations, we have to allow lapses which tend to 1 at infinity and on the axis. When this is done, the constraint function $C[N]$ continues to exist everywhere on the phase space. However, due to the presence of the first two terms involving derivatives of γ and R in the expression of C, the function $C[N]$ fails to be differentiable. To make it differentiable, one has to add a surface term. As one might expect, this is precisely the surface term in the expression (2.8) of the Hamiltonian. Thus, the function which generates the canonical transformation corresponding to (asymptotically unit) time translation is precisely the Hamiltonian $H[N]$ (obtained by setting $N^r = 0$ in Eq. (2.8)). On physical states –i.e., when the constraints are satisfied– the numerical value of the Hamiltonian is given by the surface term in (2.8):

$$E = \frac{1}{4G}(1 - e^{-\gamma_\infty/2}), \tag{2.10}$$

As usual, in the space-time picture, the evolution generated by the Hamiltonian on the phase space corresponds to motions along the vector field t^a.

Let us summarize the discussion of this subsection. Because we are in the asymptotically flat context, there is a clean separation between gauge and dynamics. As usual, when it comes to physical interpretation, the "gauge transformations" of general relativity have a somewhat different status from that in Yang-Mills theory. It is not that the diffeomorphisms generated by $C[N_g]$ and $C[N_g^r]$ are "unphysical". Rather, they are "redundant" when it comes to extracting the physical content of the theory. As we will see below, we can gauge fix these constraints and extract the true degrees of freedom. The Hamiltonian generates "time evolution" among these gauge fixed points. Knowing this evolution, we can reconstruct the entire solution; motions generated by constraints are not needed and are in this sense "redundant".

2.1.3 Gauge Fixing

Since the canonical transformations generated by $C[N_g]$ and $C[N_g^r]$ are to be regarded as gauge, as in Yang-Mills theory, to gauge fix the system we need to extract one point from each orbit of the corresponding Hamiltonian vector fields. This is achieved by imposing gauge fixing conditions which, together with the constraints, constitute a second class system. As in [7], we will choose these conditions to make the space-time geometry transparent. Let us demand:

$$R(r) = r \quad \text{and} \quad p_\gamma(r) = 0\,. \tag{2.11}$$

The first condition is motivated by the fact that, in any solution to the field equations (satisfying our boundary conditions), the gradient $\nabla_a R$ of the norm of the Killing field $\partial/\partial\sigma$ is space-like everywhere on M [13]. Since furthermore $R \sim r$ at the axis and at infinity, it is natural to use R itself as the radial coordinate. After this condition is imposed, R will no longer be a dynamical variable. The second gauge fixing condition will remove γ from our list of dynamical variables. Thus, if these conditions are admissible, the true degrees of freedom will all reside in the field ψ, in accordance with our general expectation that in 2+1 dimensions, all the local degrees of freedom are carried by matter fields.

To see if our gauge fixing conditions are admissible, let us compute their Poisson brackets with the constraints. We have:

$$\begin{aligned} \{R(r) - r, C_r[N_g^r]\} &= N_g^r e^{-\gamma} R' \\ \{p_\gamma, C[N_g]\} &= \left[\frac{N_g}{2}\left(-p_\gamma p_R + \frac{p_\psi^2}{2R} + \frac{R}{2}\psi'^2\right) - N_g{}' R'\right] e^{-\gamma/2}, \quad (2.12) \end{aligned}$$

where, as before, N_g^r and N_g are pure gauge lapses and shifts. If $N_g^r \neq 0$ and $N_g \neq 0$, the right sides of (2.12) do not vanish at any point on the intersection of the surfaces defined by constraints and gauge fixing conditions (2.11). Hence, as needed, the gauge fixed surface intersects the gauge orbits transversely.

The question now is whether we can choose lapse and shift such that the dynamical evolution generated by the Hamiltonian $H[N, N^r]$ preserves the gauge conditions. More precisely, since the Hamiltonian $H[N, N^r]$ weakly commutes with the constraints $C[N_g], C[N_g^r]$, we know that the dynamical evolution it generates maps entire gauge orbits to entire gauge orbits. The question is if we can select N, N^r such that the image under evolution of any gauge fixed point on the constraint surface is another gauge fixed point. General considerations from symplectic geometry imply that if such a pair exists, it is unique. We will now establish the existence. Let us begin with the Poisson brackets between the gauge conditions and the Hamiltonian:

$$\{R(r) - r, H[N, N^r]\} \approx N^r e^{-\gamma}$$

$$\{p_\gamma(r), H[N, N^r]\} \;\approx\; \left[\frac{N}{4r}\left(p_\psi^2 + r^2\psi'^2\right) - N'\right] e^{-\gamma/2}\,, \tag{2.13}$$

where $\approx$ stands for equality modulo constraints and gauge conditions. We seek N and N^r which satisfy our boundary conditions (namely, $N = 1 + O(1/r)$ and $N^r = N^r_o + O(1/r)$ at infinity) and for which the right hand sides of (2.13) vanish (modulo constraints and gauge conditions). The only solutions are:

$$N = \exp\left[-\frac{1}{4}\int_r^\infty dr_1 r_1 \left(\frac{(p_\psi)^2}{r_1^2} + (\psi')^2\right)\right] \quad \text{and} \quad N^r = 0. \tag{2.14}$$

Finally, let us extract the true degrees of freedom of the theory. In order to accomplish this, we need to eliminate redundant variables by solving the set of second class constraints (2.9) and using the gauge conditions (2.11). By setting $R = r$ and $p_\gamma = 0$ in (2.9), we can trivially solve for γ and p_R in terms of ψ and p_ψ (using the Hamiltonian and the diffeomorphism constraints respectively). The result is:

$$\gamma(R) \;=\; \frac{1}{2}\int_0^R dR_1 R_1 \left(\frac{{p_\psi}^2}{R_1^2} + \psi'^2\right), \tag{2.15}$$

$$p_R \;=\; -p_\psi \psi' \tag{2.16}$$

Substituting (2.15) in (2.14), we can also express the lapse N in terms of γ. Thus, as expected, the true degrees of freedom reside just in the matter variables. Indeed, the space-time metric is now completely determined by ψ and p_ψ:

$$g_{ab} = e^{\gamma(R,t)}\left(-e^{-\gamma_\infty}\nabla_a t\,\nabla_b t + \nabla_a R \nabla_b R\right) + R^2 \nabla_a \sigma\,\nabla_b \sigma\,, \tag{2.17}$$

where, from now on, γ will only serve as an abbreviation for the right side of (2.15).

2.1.4 Reduced Phase Space

It is obvious from the above discussion that the reduced phase space $\bar{\Gamma}$ can be coordinatized by the pair $(\psi(R), p_\psi(R))$. The (non-degenerate) symplectic structure on the reduced phase space $\bar{\Gamma}$ is the pull-back of the symplectic structure on Γ. Thus,

$$\{\psi(R_1), p_\psi(R_2)\} = \delta(R_1, R_2) \tag{2.18}$$

on $\bar{\Gamma}$. Next, let us write the reduced action by substituting (2.11), (2.15) and (2.16) in (2.7),

$$S[\psi, p_\psi] = \frac{1}{8G} \int dt \left[\int dR (p_\psi \dot{\psi}) \; -2(1 - e^{-\gamma_\infty/2}) \right], \tag{2.19}$$

where, as before, $\gamma_\infty = \gamma(r = \infty)$. By varying the action (2.19) with respect to ψ and p_ψ we then obtain equations of motion:

$$\dot{\psi} = e^{-\gamma_\infty/2} \frac{p_\psi}{R} \quad \text{and} \quad \dot{p}_\psi = e^{-\gamma_\infty/2} (R\psi')' . \tag{2.20}$$

Due to the presence of $\exp(-\gamma_\infty/2)$ factors, these equations are highly non-linear. However, using (2.20) it is straightforward to check that $\gamma_\infty(t)$ is a *constant of motion.* Hence, given any *one* solution, we can define a new time coordinate T on M via a constant rescaling: $T := (\exp -\gamma_\infty/2)\, t$. Then, the field ψ satisfies the following *linear* second-order equation of motion:

$$-\frac{\partial^2\psi}{\partial T^2} + \frac{\partial^2\psi}{\partial R^2} + \frac{1}{R}\frac{\partial\psi}{\partial R} = 0. \tag{2.21}$$

This is *exactly* the Klein-Gordon equation for a scalar field propagating on a Minkowskian background $\overset{\circ}{g}_{ab}$, given by:

$$\overset{\circ}{g}_{ab} = -\nabla_a T \nabla_b T + \nabla_a R \nabla_b R + R^2 \nabla_a \sigma \nabla_b \sigma\,. \tag{2.22}$$

Thus, a remarkable simplification has occurred. We can just solve for a free scalar field ψ in Minkowski space $(M, \overset{\circ}{g}_{ab})$, *define* a function γ through (2.15), and construct a curved metric g_{ab} through (2.17). Then the pair (g_{ab}, ψ) satisfies the non-linear Einstein-Klein-Gordon equations.

This decoupling is not surprising from the space-time perspective. Indeed, it has been exploited repeatedly in the literature. However, it is illuminating to see how the decoupling comes about from a phase space perspective especially since the dynamics of the true degrees of freedom is driven only by the boundary term Hamiltonian which, furthermore, seems quite complicated at first sight. Note also that, while from a space-time perspective the passage between t and T involves only a constant rescaling, since the constant varies from solution to solution, from a phase space perspective it is a rather complicated, "q-number" transformation. Thus, in quantum theory, if one variable in the pair (t, T) is taken as a "time-parameter", the other will be a genuine operator. It is therefore instructive to contrast the two notions of time. By construction, t can be identified with the affine parameter along the Hamiltonian vector field defined by (2.8) on the phase space. Given any dynamical trajectory, we obtain a space-time metric g_{ab} and

t can then be thought of a time coordinate on M with the property that $\partial/\partial t$ generates an *unit* time translation at infinity. The parameter T, on the other hand does not have a direct and simple physical interpretation in our phase space framework. Its most direct interpretation comes from the fiducial Minkowskian metric $\overset{\circ}{g}_{ab}$ on M. Even at infinity, the norm of the vector field $\partial/\partial T$ varies from one physical metric g_{ab} to another. For the decoupling procedure, on the other hand, it is natural to fix, once and for all, the Minkowskian metric $\overset{\circ}{g}_{ab}$ on M and regard g_{ab} simply as a "derived" quantity. Then T does have a natural interpretation of time. Finally, note that this somewhat peculiar situation arose because, in 2+1 dimensions, the physical metrics g_{ab} do not approach a fixed Minkowskian metric even at infinity (or alternatively, because in 3+1 dimensions, cylindrical waves fail to be asymptotically flat in the conventional sense.)

We will conclude this section with a remark. To begin with, one can ignore the broad physical problem of interest and focus just on a free scalar field satisfying the wave equation on the Minkowskian background $(M, \overset{\circ}{g}_{ab})$. The phase space for this system is the same as our reduced phase space and the Hamiltonian is given by γ_∞. However, γ_∞ does *not* have a direct physical interpretation in terms of the original, coupled system; the physical energy of our system is given by (2.10).

2.2 Holonomy

As pointed out in chapter 1, the approach to (3+1-dimensional) quantum general relativity based on Ashtekar phase space variables [14] is based on the assumption that traces of holonomies of a certain connection are well-defined operators in the quantum theory. We would like to investigate the status of this assumption in the context of our midi-superspace. Therefore, in this section, we will make a short detour to compute the holonomy in question in the classical theory. Readers who are not familiar with this approach to quantum gravity may skip this section without loss of continuity.

In the first-order (Palatini) formalism for $2+1$ general relativity the fundamental variables are triads e_a^I and connection 1-forms which take values in the Lie-algebra of $SU(1,1)$. Let us denote the $SO(2,1)$ connection by ${}^3A_a^I$ and its pull-back to the 2-dimensional slice Σ by A_a^I, where $I, J, \cdots = 0,1,2$ are internal indices with respect to a basis τ_I in the Lie algebra of $SU(1,1)$. The internal indices are raised and lowered with a Minkowski metric η_{IJ} with signature $(-,+,+)$.

2.2.1 $SO(2,1)$ Connection

To obtain the internal connection for the space-time metric (2.17), we need to fix the internal (i.e., $SU(1,1)$) gauge. This is accomplished by fixing the triads e_a^I. Our choice will be:

$$e_a^I \tau_I = \sqrt{2} e^{(\gamma(R,t)-\gamma_\infty)/2} (\nabla_a t)\, \tau_0 + \sqrt{2} e^{\gamma(R,t)/2} (\nabla_a R)\, \tau_1 + \sqrt{2} R (\nabla_a \sigma)\, \tau_2 \qquad (2.23)$$

It is straightforward to check that the space-time metric (2.17) is recovered via $g_{ab} = \eta_{IJ} e_a^I e_b^J$ with the convention $\eta_{IJ} = 2\,Tr(\tau_I \tau_J)$.

The triad determines the (Christoffel symbols and the) internal connection ${}^3A_a^I$ uniquely. Its pull-back to the spatial slice Σ turns out to be:

$$A_a = A^I_a \tau_I = \frac{\dot{\gamma}}{2} e^{\gamma_\infty/2} (\nabla_a R)\, \tau_2 + e^{-\gamma/2} (\nabla_a \sigma)\, \tau_0. \tag{2.24}$$

Note, however, that since R, σ fail to be smooth at $R = 0$ our connection also fails to be smooth there. However, our boundary conditions do ensure that all physical fields are smooth at the origin. Thus, this singularity is merely a reflection of a bad choice of gauge (which has in effect introduced a "source" at the origin). We can remedy this situation by a gauge transformation. The general form of gauge transformations is:

$$A'_a = g A_a g^{-1} - (\partial_a g) g^{-1} \quad \text{with} \quad g = e^{\tau_I \Lambda^I (R,\sigma)}. \tag{2.25}$$

By choosing the transformation parameters to be $\Lambda^0 = e^{-\gamma(0)/2} \sigma$ and $\Lambda^1 = \Lambda^2 = 0$, we obtain a smooth connection as desired:

$$A'_a = A'^I_a \tau_I = \frac{\dot{\gamma}}{2} e^{\gamma_\infty/2} \nabla_a R \, [\cos\sigma\, \tau_2 - \sin\sigma\, \tau_1] + [e^{-\gamma/2} - 1] \nabla_a \sigma\, \tau_0. \tag{2.26}$$

2.2.2 Holonomy computation

The holonomy of A'^I_a along a loop η is given by a path ordered exponential of the integral of A'^I_a along η:

$$U_\eta[A] := \mathcal{P} \exp \left(\oint_\eta A_a dS^a \right). \tag{2.27}$$

For quantum considerations, it turns out that the most interesting loops are the integral curves of the rotational Killing vector σ^a. Note that, along these curves, only the second term in the expression (2.26) of the connection contributes. Since the internal vector in this term is constant, this part of the connection is effectively Abelian. Recall that in the case of an Abelian connection the path ordered exponential reduces to an ordinary exponential. Hence, if η is chosen to be the integral curve of the Killing field with radius R_o, the holonomy can be easily evaluated. We have:

$$U_\eta[A'] = \cos\left[\pi\left(1 - e^{-\gamma(R_o)/2}\right)\right] - 2\tau_0 \sin\left[\pi\left(1 - e^{-\gamma(R_o)/2}\right)\right] \tag{2.28}$$

where we have used the fundamental representation of $SU(1,1)$. For our purposes, it will suffice to consider these particular loops.

Of special interest to the quantization program under consideration are the functions $T^0_\eta[A]$ of connections defined by the trace of the holonomy. Taking the trace of (2.28) yields

$$T^0_\eta[A'] = 2\cos\left[\pi\left(1 - e^{-\gamma(R_o)/2}\right)\right]. \tag{2.29}$$

Note, incidentally, that if η is chosen to be the loop at infinity, $T^0_\eta[A]$ reduces to a simple function of the total energy of the coupled system. For the reduced system, $\gamma(R_o)$ represents precisely the energy of ψ in a box of radius R_o (where ψ is regarded as a scalar field propagating on the Minkowskian background.) The question of whether the T^0_η can be promoted to a well-defined operator will therefore reduce to the question of whether the operator corresponding the energy of a scalar field in a box can be satisfactorily regulated.

2.3 Quantum Theory

2.3.1 Quantization

The reduced phase space of subsection 2.1.4 serves as the natural point of departure for quantization. Since the constraints have been solved, the algebra $\mathcal{A}$ of observables is easy to construct. The obvious complete set of classical observables is given by the smeared fields and momenta, $\psi(f) := \int drf(r)\psi(r)$ and $p_\psi(g) := \int drg(r)p_\psi(r)$, where f, g belong to the Schwartz space $\mathcal{S}$ of smooth test functions of rapid decay at infinity. Thus, the quantum algebra $\mathcal{A}$ is generated by operators $\hat{\psi}(f)$ and $\hat{p}_\psi(g)$, subject to the canonical commutation relations:

$$[\hat{\psi}(f), \hat{\psi}(g)] = 0 \quad [\hat{p}_\psi(f), \hat{p}_\psi(g)] = 0 \quad [\hat{\psi}(f), \hat{p}_\psi(g)] = i\int_\Sigma fgI. \tag{2.30}$$

Our task is to find a representation of $\mathcal{A}$. As is well-known there exist an infinite number of unitarily inequivalent representations of the algebra $\mathcal{A}$. Our additional physical requirements that will suggest a suitable choice are that the Hamiltonian $\frac{1}{4G}[1-\exp(-\frac{\gamma_\infty}{2})]$ be promoted to a well-defined Hamiltonian operator $\hat{H}$ and that the physical states be invariant under the rotational symmetry generated by $\partial/\partial\sigma$.

For technical simplicity, we will regard $\hat{\psi}$ and $\hat{p}_\psi$ as operator-valued distributions in two (space) dimensions and incorporate rotational symmetry by restricting the states to be axi-symmetric at the very end. Our experience from low dimensional, interacting scalar quantum field theories now suggests that we use as our Hilbert space $\mathcal{H} = L^2(\mathcal{S}', d\mu)$ where $\mathcal{S}'$ is the space of all tempered distributions on R^2, and μ a suitable measure thereon. (For details, see, e.g., [15]). Since γ_∞ is the Hamiltonian of the free scalar field in Minkowski space, to make the quantum Hamiltonian operator well-defined, it is natural to use for μ the standard Gaussian measure for a free, massless scalar field with covariance $\frac{1}{2}\triangle^{-\frac{1}{2}}$, where $\triangle$ is the Laplacian on R^2 with respect to the flat metric

$$\overset{\circ}{q}_{ab} = \nabla_a R \nabla_b R + R^2 \nabla_a \theta \nabla_b \theta. \tag{2.31}$$

Thus, μ is defined by

$$\int_{\mathcal{S}'} d\mu \; e^{i\int d^2x f(\vec{x})\tilde{\psi}(\vec{x})} = e^{-\frac{1}{2}\int d^2x f(\vec{x})\triangle^{-\frac{1}{2}} f(\vec{x})}, \tag{2.32}$$

where $\tilde{\psi} \in \mathcal{S}'$. (Heuristically, "$d\mu = e^{-\frac{1}{2}\int d^2x(\psi\triangle^{1/2}\psi)}\mathcal{D}\psi$".) The action of the basic operators is then given by:

$$\hat{\psi}(f)\cdot\Psi(\psi) = \left(\int d^2x f\psi\right)\Psi(\psi) \quad \text{and} \quad \hat{p}_\psi\cdot\Psi(\psi) = -i\hbar\int d^2x g \frac{\delta}{\delta\psi}\Psi(\psi). \tag{2.33}$$

where Ψ belong to the dense subspace of cylindrical functions in $\mathcal{H}$. The operators $\hat{\psi}(f)$ and $\hat{p}_\psi(f)$ admit self-adjoint extensions to $\mathcal{H}$. We will see below that the Hamiltonian is also represented by a self-adjoint operator and that, like its classical counterpart, it is positive.

This choice of representation is also suggested by the mathematical equivalence between our physical system and a free massless scalar field on Minkowski space defined by $\overset{\circ}{g}_{ab}$. Thus, although our viewpoint is somewhat different, our final choice of representation is the same as that of Refs [7, 8].

In a more familiar terminology, our representation can be obtained by introducing an operator-valued distribution $\hat{\psi}(\vec{x},T)$ in the fictitious Minkowskian background $(M,\overset{\circ}{g}_{ab})$:

$$\hat{\psi}(\vec{x},T)=\frac{1}{2\sqrt{2}\pi}\int\frac{d^2k}{(\omega_k)^{1/2}}\left[\hat{A}(\vec{k})\ e^{i(\vec{k}\cdot\vec{x}-\omega_kT)}+\hat{A}^\dagger(\vec{k})\ e^{-i(\vec{k}\cdot\vec{x}-\omega_kT)}\right], \tag{2.34}$$

where $\omega_k=\sqrt{\vec{k}\cdot\vec{k}}$, and $\hat{A}(\vec{k})$ and $\hat{A}^\dagger(\vec{k})$ are the standard creation and annihilation operators. The Hilbert space $\mathcal{H}$ can be generated by repeated actions of creation operators on the vacuum. There is a well-defined self-adjoint operator $\hat{L}_\sigma$ on $\mathcal{H}$ which represents the total angular momentum along the Killing field $\partial/\partial\sigma$. The physical Hilbert space $\mathcal{H}_P$ is the eigenspace of $\hat{L}_\sigma$ with zero eigenvalue. Since zero is a discrete eigenvalue, $\mathcal{H}_P$ is a sub-space of $\mathcal{H}$.

The physical Hilbert space can also be obtained more directly by using, instead of (2.34), an operator valued distribution in which the zero angular momentum constraint has already been incorporated, namely,

$$\hat{\psi}(R,T)=\int_0^\infty dk\left[f_k^+(R,T)\hat{A}(k)+\ f_k^-(R,T)\hat{A}^\dagger(k)\right]. \tag{2.35}$$

Here $f_k^+(R,T)=[f_k^-(R,T)]^*=\frac{1}{\sqrt{2}}J_0(kR)e^{-i\omega_kT}$, where, from now on $J_n(kR)$ will denote the n-th order Bessel function of the first kind. Note that $f_k^+(R)$ are solutions of the equation of motion (2.21) and provide an orthonormal basis for the one-particle Hilbert space with respect to the Klein-Gordon inner-product. (Our normalization is such that the creation and annihilation operators satisfy the commutation relations

$[\hat{A}(k), \hat{A}^\dagger(k')] = \delta(k,k')$.) The physical Hilbert space $\mathcal{H}_P$ can be generated by repeatedly acting on the vacuum by the creation operators $\hat{A}^\dagger(k)$. In what follows, we will use both the two dimensional as well as the one dimensional descriptions given by (2.34) and (2.35).

We will conclude this subsection with two remarks.

1) Since the physical Hilbert space has a Fock structure, it is tempting to refer to the quanta created by $\hat{A}^\dagger(k)$ as (scalar) "particles" and we will often do so. Note, however, that from the point of view of the coupled Einstein-Klein-Gordon system we began with, this description is gauge dependent. The system has one local degree of freedom and we chose to put it in the scalar field. Another gauge choice could put it in the gravitational field and the interpretation of quantum states would then be different.

2) We now have the full Hilbert space of states. So, it is natural to examine if one can generate a picture of space-*time* –as opposed to just spatial– quantum geometry in spite of our use of the canonical approach. As one might expect from our gauge-fixing procedure, the answer is in the affirmative. In the fixed chart (T, R, σ) on M, the metric operator can be (heuristically) written as:

$$\text{“ } \hat{g}_{ab} =: e^{\hat{\gamma}(R,T)} : (-\nabla_a T \nabla_b T + \nabla_a R \nabla_b R) + R^2 \nabla_a \sigma \nabla_b \sigma \text{ ”}, \tag{2.36}$$

where, as usual, the double-dots indicate normal ordering. (The reason behind the qualification "heuristic" and the quotes will become clear in section 2.3.3.)

We can now ask if there are well-defined semi-classical states peaked at classical solutions. The answer is again in the affirmative. Consider, in the Fock space, a coherent state $|\Psi_c>$ which is peaked at a classical solution $\psi_c(R,T)$ of (2.21). In the configuration representation, these are Gaussians for which the uncertainty in the field operator and its momentum are "shared equally", the product of the two uncertainties being minimum *for all times* T. On these states, the expectation value of the metric operator (2.36) is well-defined and is given just by

$$< \Psi_c|\hat{g}_{ab}|\Psi_c >= e^{\gamma[\psi_c,p_{\psi_c}]}(-\nabla_a T\nabla_b T + \nabla_a R\nabla_b R) + R^2\nabla_a\sigma\nabla_b\sigma, \qquad (2.37)$$

where $\gamma[\psi_c, p_{\psi_c}]$ is the right side of (2.16), evaluated on the initial data of the classical solution ψ_c. Thus, every coherent states in our physical Hilbert space $\mathcal{H}_P$ remains peaked at a classical scalar field ψ_c *and* a metric g_{ab}, satisfying the coupled Einstein-Klein-Gordon equation. While the result is technically rather simple, conceptually it is somewhat surprising. For, the coupled system satisfies highly non-linear equations and the wave packets do not disperse in spite of these non-linearities.

2.3.2 Hamiltonian and Time

Recall that the classical Hamiltonian is given by $H = \frac{1}{4G}\left[1 - \exp(-\frac{1}{2}\gamma_\infty)\right]$. Since γ_∞ is the Hamiltonian of a free scalar field in Minkowski space, the normal-ordered operator $:\hat{\gamma}_\infty:$ admits the standard self-adjoint extension which, for simplicity, we will denote also by $:\hat{\gamma}_\infty:$. Then, the standard spectral theorems ensure that

$$\hat{H} := \frac{1}{4G}(1 - e^{-\frac{1}{2}:\hat{\gamma}_\infty:}) \equiv \frac{1}{4G}(1 - e^{-\int kdk\hat{A}^\dagger(k)\hat{A}(k)}) \tag{2.38}$$

is a well-defined, self-adjoint operator. Since $:\hat{\gamma}_\infty:$ is a non-negative, unbounded operator and since $f(\lambda) = (1 - e^{-\frac{\lambda}{2}})$ takes values in $[0, 1]$ for $\lambda \in [0, \infty]$, it follows that the spectrum of H is given by $[0, 1/4G]$. If we consider states in $\mathcal{H}_P$ with higher and higher frequency, the expectation value of $\hat{\gamma}_\infty$ –i.e., the energy in the field from the mathematical, Minkowskian perspective– increases unboundedly. However, the expectation value of the *physical* Hamiltonian $\hat{H}$ remains bounded and tends to the limit $1/4G$. Thus, the situation is completely analogous to that in the classical theory [2].

Let us now examine the ground state. Since $|0>$ is the unique ground state of $:\hat{\gamma}_\infty:$ on $\mathcal{H}_P$, it follows immediately that it is also the unique ground state of $\hat{H}$. Since $|0>$ is, in particular, a coherent state, it is peaked at a classical solution to the coupled system. As one might expect, the solution is: $\psi = 0$ and $g_{ab} = \overset{\circ}{g}_{ab}$. Thus, the quantum ground state is peaked on Minkowski space-time. The ground state geometry is thus quite tame, there is no evidence of wild fluctuations at the Planck scale.

What is the situation with general coherent states? Given a coherent state $|\Psi_c> := \exp[\int dk\psi_c(k)\hat{A}^\dagger(k)] \cdot |0>$, peaked at a classical solution ψ_c, we have:

$$[\exp -\tfrac{1}{2} :\hat{\gamma}_\infty:] \cdot \Psi_c = [\exp \int dk e^k \psi_c(k)\hat{A}^\dagger(k)] \cdot |0> =: \Psi'_c \tag{2.39}$$

where, $\psi'_c(k) = e^k\psi_c(k)$. Thus, the image is again a coherent state but its peak is shifted. Therefore, the expectation value of the Hamiltonian in a coherent state Ψ_c is given by:

$$\frac{<\Psi_c\,,\,\hat{H}\cdot\Psi_c>}{<\Psi_c\,,\,\Psi_c>} \;=\; \frac{1}{4G}\,[1-\exp\,\tfrac{1}{\hbar}\int dk(e^{-\hbar k}-1)|\psi_c(k)|^2\,]\,, \tag{2.40}$$

where, to bring out the quantum effects, we have restored the factors of $\hbar$. (Recall also that, from the perspective of the 2+1-dimensional theory, the scalar field has to be rescaled by factors involving $\sqrt{G}$. The net effect is to replace $\hbar$ in (2.40) by $\hbar G$ which has the physical dimension of length.) By contrast, the classical energy (2.10) of the solution to the Einstein-Klein-Gordon equation determined by ψ_c is $E(\psi_c) = \frac{1}{4G}[1 - \exp - \int dkk|\psi_c(k)|^2]$. If we expand out $\exp \hbar k$ in (2.40), the leading term yields the classical answer. In general, the classical energy is a good approximation to the expectation value of the quantum Hamiltonian if $\psi_c(k)$ is concentrated on low frequencies. Quantum corrections (of order $(G\hbar)$ and higher) become more and more significant if the support of $\psi_c(k)$ is shifted to higher and higher frequencies.

Next, let us consider the issue of time. Recall that, in the classical theory, the Hamiltonian evolution is tied to time t, the affine parameter along the Hamiltonian vector field in the phase space. Each dynamical trajectory gives rise to a space-time and t can then be interpreted as a time coordinate in *that* space-time, $\partial/\partial t$ being an unit asymptotic time translation. From the decoupling viewpoint, on the other hand, it is the variable T that arises naturally; it represents time in the fixed Minkowskian background. What is the situation in the quantum theory? Now, our measure μ on $\mathcal{S}'$ which dictates the Hilbert structure is rooted in the flat 2-geometry induced by $\mathring{g}_{ab}$ or, alternatively, in the positive and negative frequency decomposition with respect to the Minkowskian time T. Indeed, since the field equation (2.20) in terms of t is non-linear, positive frequency decomposition with respect to t is not meaningful apriori. Thus, while t and T are on equal footing in the classical theory, our choice of representation breaks this symmetry in the quantum theory.

We can mimic the situation in the classical theory and introduce a dynamical parameter λ –analogous to the classical t– associated with the Hamiltonian:

$$i\hbar \frac{\partial \Psi}{\partial \lambda} = \hat{H} \cdot \Psi \, . \tag{2.41}$$

However, unlike in the classical theory, now a solution to the dynamical equation does *not* define a hyperbolic space-time and hence we can not interpret λ as a time parameter in the familiar sense, i.e., in space-time terms. However, a key simplification occurs if we restrict ourselves to coherent states Ψ_c . Since each of these states is peaked at a classical space-time, we can ask if, given any one of these states, we can interpret λ as a time parameter in the corresponding classical space-time. The answer is in the affirmative. In fact λ can be identified with the time coordinate t of that space-time! Thus, as one might have hoped, the familiar notion of time re-emerges in the semi-classical regime. In the full quantum theory, however, the dynamical parameter defined by the Hamiltonian does not have a simple space-time interpretation.

We will conclude this discussion with a remark. There is an obvious alternative form for the Hamiltonian: We can further normal-order $\hat{H}$ and define a new Hamiltonian $\hat{H}' = :\hat{H}:$. One can verify that $\hat{H}'$ is densely defined and admits a self-adjoint extension. It also annihilates $|0>$. Furthermore, the expectation values of $\hat{H}'$ on a coherent state $|\Psi_c>$ equals the classical energy of ψ_c. It thus appears to be an attractive alternative. However, its spectrum is the *entire real line*! This comes about because the overall normal ordering ensures that, while acting on n-particle states, only the first $n+1$ terms in the expansion of the exponential in $\hat{H}'$ have non-vanishing contributions. Thus, for example, on 1-particle states, $\hat{H}'$ has the same action as $\frac{1}{8} :\hat{\gamma}_\infty:$ which is unbounded above. Similarly, on two particle states, it is unbounded below. Given that the classically allowed energy values lie in the interval $[0, 1/4G]$, we can not take $\hat{H}'$ as the physically admissible quantum analog of the classical Hamiltonian.

2.3.3 Metric operator

Since we are dealing with a system with an infinite number of degrees of freedom, operators corresponding to physical observables have to be regulated. For the Hamiltonian, this was achieved via normal ordering. In this section, we will focus on the metric operator.

A formal expression for the metric operator was already given in (2.36), where regularization again consisted of normal ordering. Consider the sub-space of $\mathcal{H}_\mathcal{P}$ which is spanned by finite linear combinations of coherent states. It is easy to show that the sub-space is dense and that the matrix elements of the metric operator $\hat{g}_{ab}$ are well-defined on it. Thus, the formal expression (2.36) does lead to a well-defined *quadratic form*; in a field theory terminology, $\hat{g}_{ab}$ exists in the LSZ sense. However, this does *not* imply that $\hat{g}_{ab}$ is well-defined *as an operator* on this sub-space. Note that this is *not* a peculiarity of quantum field theory; one encounters such situations already in non-relativistic quantum mechanics. Consider, for example, a 1-dimensional harmonic oscillator. The operator $\exp(\alpha a^\dagger a^\dagger)$ has finite matrix elements on the basis $|n>$ for all complex numbers α.

However, if $|\alpha| > 1$, the norm $|| e^{\alpha a^\dagger a^\dagger} |n> ||$ diverges for any $|n>$, whence the operator fails to be defined on the sub-space spanned by these basis vectors.

It turns out that the situation with the metric operator is quite analogous (which is the reason behind the quotes in (2.36)). To see this, let us begin with the first non-trivial term in the expansion of $\hat{g}_{RR}$ or $\hat{g}_{TT}$. Setting for simplicity $T = 0$ in (2.35), we have:

$$\begin{aligned} :\hat{\gamma}(R): \; &= \; \frac{1}{2} \int dk_1 \int dk_2 \Big[2F_+(R, k_1, k_2) \left(\hat{A}^\dagger(k_1) \hat{A}(k_2) \right) \\ &+ \; F_-(R, k_1, k_2) \left(\hat{A}(k_1) \hat{A}(k_2) \, + \, \hat{A}^\dagger(k_1) \hat{A}^\dagger(k_2) \right) \Big] , \end{aligned} \tag{2.42}$$

where

$$F_\pm(R, k_1, k_2) = \pm k_1 k_2 \int_0^R r dr \; \left(J_0(k_1 r) J_0(k_2 r) \pm \; J_1(k_1 r) J_1(k_2 r) \right) . \tag{2.43}$$

For any fixed R, one can regard the coefficient $F_-(R, k_1, k_2)$ of $\hat{A}^\dagger(k_1)\hat{A}^\dagger(k_2)$ as a "potential 2-particle state" in the Fock space. However, a direct calculation shows that its norm is ultra-violet divergent. This immediately implies that the norm $|| :\hat{\gamma}(R): |0> ||$ also diverges, whence the operator fails to be well-defined on the vacuum state. Further calculations show that the same result holds for any coherent state.

What is the origin of this divergence? Recall that $:\hat{\gamma}(R,T):$, obtained by promoting (2.15) to an operator, has the same functional form as the restriction of the Hamiltonian of a scalar field to a box of size R. That is,

$$:\hat{\gamma}(R):=\frac{1}{2}\int_0^\infty dr\theta(R-r)\ :(\frac{\hat{p}_\psi^2}{r}+r(\hat{\psi}')^2):, \qquad (2.44)$$

where $\theta(R-r)$ denotes the Heaviside step-function, which equals 1 if $r<R$ and 0 otherwise. Normal ordering softens the singularity that arises from the fact that fields are being multiplied at the same point. However, this turns out to be insufficient because of two simultaneous pathologies: the operator contains products of derivatives of the field $\hat{\psi}(R,T)$ and these are integrated on a region with *sharp* boundary.

Now, a natural strategy to obtain a well-defined metric operator in such circumstance is to soften the sharp boundary of the box. This can be achieved by replacing the Heaviside function $\theta(R-r)$ in (2.44) with a smooth function $f_R(r)$ which equals 1 for $r\leq R-\epsilon$, then it smoothly decreases to zero and equals zero for $r\geq R+\epsilon$, where ϵ is a small parameter. An example of such a regulator is:

$$f_R(r)=\begin{cases}1, & \text{if } r\leq R-\epsilon,\\ \exp\left(-\frac{4\epsilon^2}{[r-(R+\epsilon)]^2}+1\right), & \text{if } R-\epsilon\leq r\leq R+\epsilon,\\ 0, & \text{if } r\geq R+\epsilon.\end{cases}$$

Now, in Minkowskian field theories, while one can begin with such a regulator, after suitable renormalization, one has to take the regulator away to ensure Poincaré invariance. In the present case, however, we need only respect the rotational symmetry and hence there is no apriori need to take the limit $\epsilon\to 0$. Indeed, the Planck length is now a natural candidate for ϵ.

Let us therefore fix a regulator f_R and consider the smeared version of (2.42):

$$:\hat{\gamma}(f_R,T):=\frac{1}{2}\int dk_1\int dk_2 \qquad \Big[2F_+(f_R,k_1,k_2)\left(\hat{A}^\dagger(k_1)\hat{A}(k_2)e^{i(k_1-k_2)T}\right)$$
$$+F_-(f_R,k_1,k_2)\left(\hat{A}(k_1)\hat{A}(k_2)e^{-i(k_1+k_2)T}\right.$$
$$\left.+\ \hat{A}^\dagger(k_1)\hat{A}^\dagger(k_2)e^{i(k_1+k_2)T}\right)\Big] \tag{2.45}$$

where,

$$F_\pm(f_R,k_1,k_2)=\pm k_1k_2\int_0^\infty f_R(r)r\left(J_0(k_1r)J_0(k_2r)\pm\ J_1(k_1r)J_1(k_2r)\right). \tag{2.46}$$

The rest of this section is devoted to showing that this operator is well-defined so long as the smearing function f_R belongs to the Schwartz space $\mathcal{S}$.

The proof is technically simpler if we adopt the 2-dimensional version of the Fock space introduced before (see (2.34)). For, we can then mimic the proofs of analogous statements from [16]. Now, we can take as our smearing fields, elements $f_R(\vec{x})$ of the Schwartz space on R^2. (Thus, the results will in fact be slightly more general than what is needed; $f_R(r)$ above is a special case of $f_R(\vec{x})$.)

Let us then write the smeared version of the operator (2.44) expressed in terms of the creation and annihilation operators given by (2.34). We have:

$$\begin{aligned} :\hat{\gamma}(f_R,T): \ &= \ \frac{1}{8\pi}\int d^2k_1\int d^2k_2\ \Big[2G_+(f_R,\vec{k}_1,\vec{k}_2)\ \hat{A}^\dagger(\vec{k}_1)\hat{A}(\vec{k}_2)e^{i(\omega_{k_1}-\omega_{k_2})T} \\ &- \ G_-(f_R,\vec{k}_1,\vec{k}_2)\ \left(\hat{A}(\vec{k}_1)\hat{A}(\vec{k}_2)\ e^{-i(\omega_{k_1}+\omega_{k_2})T}\right. \\ &+ \ \left.\hat{A}^\dagger(\vec{k}_1)\hat{A}^\dagger\ (\vec{k}_2)\,e^{i(\omega_{k_1}+\omega_{k_2})T}\right)\Big] \end{aligned} \tag{2.47}$$

where

$$G_{\pm}(f_R, \vec{k}_1, \vec{k}_2) = \pm \left(\frac{\omega_{k_1}\omega_{k_2} + \vec{k}_1 \cdot \vec{k}_2}{(\omega_{k_1}\omega_{k_2})^{1/2}} \right) f(\vec{k}_1 \mp \vec{k}_2)\,, \tag{2.48}$$

and $f(\vec{k}_1 \pm \vec{k}_2)$ is the fourier transform of the smearing function,

$$f(\vec{k}_1 \pm \vec{k}_2) = \frac{1}{2\pi} \int d^2x f_R(\vec{x}) e^{i(\vec{k}_1 \pm \vec{k}_2)\cdot\vec{x}}. \tag{2.49}$$

Let us begin by showing that the action of the operator (2.47) is well-defined on the vacuum state. Since $\hat{A}(\vec{k})$ annihilates the vacuum state, we have:

$$|| :\hat{\gamma}(f_R): |0> ||^2 = \int d^2k_1 \int d^2k_2 \; |G_-(f_R, \vec{k}_1, \vec{k}_2)|^2 \,. \tag{2.50}$$

It follows immediately from (2.48) that this integral has no infra-red divergences. Therefore, from now on, let us concentrate only on the ultra-violet behavior of the integrand. The factor in the round brackets is ultra-violet divergent. The multiplicative factor f provides a damping, but only for large $|\vec{k_1} + \vec{k_2}|$. However, using simple algebra one can bound $G_-(f_R, \vec{k}_1, \vec{k}_2)$ of Eq (2.48) by

$$|G_-(f_R, \vec{k}_1, \vec{k}_2)| \leq \frac{|\vec{k}_1 + \vec{k}_2|^2 |f(\vec{k}_1 + \vec{k}_2)|}{\sqrt{\omega_{k_1}\omega_{k_2}}}. \tag{2.51}$$

Now, because the smearing function $f_R(\vec{x})$ belongs to the Schwartz space, its Fourier transform $f(\vec{k}_1 + \vec{k}_2)$ falls faster than any polynomial in $|\vec{k_1} + \vec{k_2}|$. This in turn implies that $G_-(f_R, \vec{k}_1, \vec{k}_2)$ is square integrable. Note that the smearing function plays a crucial role in this argument. Had we replaced $f_R(\vec{x})$ by the Heaviside function θ the corresponding Fourier transformed function $f(\vec{k}_1 + \vec{k}_2)$ would behave as $1/|\vec{k}_1 + \vec{k}_2|$ which would not be sufficient to ensure square-integrability of $G_-(f_R, \vec{k}_1, \vec{k}_2)$ (see (2.51)). Finally, as a side remark, note that the procedure followed above to prove that $G_-(f_R, \vec{k}_1, \vec{k}_2)$ is square integrable does not go through for G_+ because of the minus sign in the argument of the function $f(\vec{k}_1 - \vec{k}_2)$ (see (2.48)).

Next, one can show that the action of this operator is in fact well-defined on a generic n-particle state on the Fock space,

$$|\Psi_n\!>= \int d^2k_1 \cdots d^2k_n \; g^{(n)}(\vec{k}_1, \cdots, \vec{k}_n) \hat{A}^\dagger(\vec{k}_1) \cdots \hat{A}^\dagger(\vec{k}_n)|0>, \tag{2.52}$$

where $g^{(n)}(\vec{k}_1, \cdots, \vec{k}_n) =< \vec{k}_1, \cdots, \vec{k}_n|\Psi_n>$, and $\int d^2k|g^{(n)}(\cdots, \vec{k}, \cdots)|^2 < \infty$. Now the terms involving annihilation operators will also contribute. The final result that we obtain is that $|| : \hat{\gamma}(f_R) : |\Psi_n> ||$ is finite provided that $|\Psi_n>$ is a state such that $\int d^2k|\vec{k}|^2|g^{(n)}(\cdots, \vec{k}, \cdots)|^2 < \infty$. (This restriction is coming from the "particle number preserving term" in (2.47).) Since finite linear combinations of these states form a dense subset of the Hilbert space, we have now established that the operator $:\hat{\gamma}(f_R):$ is densely defined on $\mathcal{H}_{\mathcal{P}}$.

By inspection, it also symmetric on this space. We will now show that it admits a self-adjoint extension to $\mathcal{H}_P$. For this, by a theorem due to Von-Neumann [17], it is sufficient to exhibit on $\mathcal{H}_P$ an anti-linear operator $\hat{C}$ with $\hat{C}^2 = 1$ which leaves the domain of $: \hat{\gamma}(f_R) :$ invariant and commutes with it. We can take $\hat{C}$ to be the complex-conjugation operator on $\mathcal{H}_P = L^2(\mathcal{S}', d\mu)$. It is straightforward to show that $\hat{C}$ commutes with $\hat{\psi}(\vec{x}, T)$ whence $\hat{C}\hat{A}(\vec{k}) = \hat{A}(-\vec{k})\hat{C}$, and, $\hat{C}\hat{A}^\dagger(\vec{k}) = \hat{A}^\dagger(-\vec{k})\hat{C}$. Finally, since $G_\pm(f_R, \vec{k}_1, \vec{k}_2)$ is real and equals $G_\pm(f_R, -\vec{k}_1, -\vec{k}_2)$, it follows that $\hat{C}$ satisfies the conditions of Von-Neumann's theorem. Again, for notational simplicity, we will denote the self-adjoint extension also by $:\hat{\gamma}(f_R):$.

We can now return to the metric. Since $:\hat{\gamma}(f_R):$ is a self-adjoint operator on $\mathcal{H}_P$, it follows that $\exp :\hat{\gamma}(f_R):$ is also self-adjoint. Thus, we can now give meaning to the formal expression (2.36) and define a regulated operator for the full space-*time* metric:

$$\hat{g}_{ab}(f) = e^{:\hat{\gamma}(f_R,T):}(-\nabla_a T\nabla_b T + \nabla_a R\nabla_b R) + R^2\nabla_a\sigma\nabla_b\sigma\,, \tag{2.53}$$

within canonical quantization. In the classical theory, the existence theorems ensure that a space-time metric can be recovered from the canonical framework. There is, however, no such general result in the quantum theory. Our success can be traced back to the use of a well-suited gauge fixing procedure. (Whether a different choice of gauge will give equivalent results is far from being clear.)

At first, it is somewhat confusing that while we do not need a smearing function to obtain a well-defined quadratic form, we need one to obtain a well-defined operator. Note however, that the situation is rather similar even in the classical theory! The metric component $\exp\gamma(R)$ is a well-defined functional on (a dense sub-space of) the reduced phase space. However, precisely because of the sharpness of the boundary, this functional *fails* to give rise to a well-defined Hamiltonian vector field. To obtain a Hamiltonian vector field, one again needs to soften the boundaries using a smearing function. The fact that the unsmeared functional is well-defined is analogous to the fact that, in the quantum theory, the quadratic form is well-defined without smearing. The smeared quantum operator is the analog of the smeared classical observable with a well-defined Hamiltonian vector field. From this perspective, in fact it would have been surprising if a self-adjoint metric operator had existed without smearing; it would then have defined a 1-parameter group of motions on the Hilbert space which would have no classical counterpart.

2.3.4 Quantum geometry

We will now briefly investigate three consequences of the results obtained in the last three sub-sections.

The first concerns the issue of vacuum fluctuations of geometry. To compute these, we need a well-defined operator; quadratic forms do not suffice. Let us therefore consider the regulated metric operator (2.53). Note first that, even though the expression now contains a regulator, the vacuum expectation value of the operator is still $\overset{\circ}{g}_{ab}$, the Minkowski metric. However, because of vacuum fluctuations, there is a non-zero probability of finding other geometries as well.

A qualitative measure of these probabilities is given by the uncertainty. Since the metric has a single non-trivial component, $\exp\gamma$, we have

$$\left[\delta\left(e^{:\hat{\gamma}(f_R,T):}\right)\right]^2 := <0|(e^{:\hat{\gamma}(f_R,T):})^2|0> - <0|e^{:\hat{\gamma}(f_R,T):}|0>^2 \ . \tag{2.54}$$

The right side is a measure of the fluctuation of the metric around the mean $\overset{\circ}{g}_{ab}$.

An immediate consequence of the above result is the existence of the fluctuations of the light cone. To see, this, consider a vector k^a in the tangent space of a point (T, R, σ) which is null with respect to g^o_{ab}. Now, due to the vacuum fluctuations of the metric operator, the value of the norm of k^a is uncertain and, since the fluctuation can have either sign, there is in general a non-zero probability for k^a to be space-like or time-like. The exception occurs if the vector k^a is radial, i.e., orthogonal to $\partial/\partial\sigma$. Then, because of the specific form (2.53) of the metric operator, k^a continues to be null. (Similar considerations obviously apply to time-like and space-like vectors.) This simple example illustrates that, contrary to an oft-expressed view, the canonical framework *is* capable of addressing space-*time* issues such as the fluctuations of the causal structure.

The second feature we wish to discuss concerns the commutator of the metric operators at the same value of T. Again, in this calculation, quadratic forms do not suffice and we must use the regulated operator (2.53). A straightforward calculation yields:

$$[:\hat{\gamma}(f_R):, :\hat{\gamma}(g_{R'}):] = \frac{i}{2}\int d^2x \quad \left(f(\vec{x})\nabla^a g(\vec{x}) - g(\vec{x})\nabla^a f(\vec{x})\right) \times \\ :(\hat{p}_\psi(\vec{x})\nabla_a\hat{\psi}(\vec{x}) + \nabla_a\hat{\psi}(\vec{x})\hat{p}_\psi(\vec{x})): \, . \qquad (2.55)$$

Thus, the commutator does *not* vanish; the non-vanishing contribution comes from the smeared boundary at the smaller of R and R'. At first the result seems surprising since $\hat{\gamma}(f_R)$ and $\hat{\gamma}(g_{R'})$ dictate the "value" of the metric operator at points R and R' which can be widely separated (and have the same value of T). However, the result does not contradict any physical principle. For, although the basic field operators $\hat{\psi}$ and $\hat{p}_\psi$ associated with such points do commute, the metric operator is a *non-local* functional of these.

Indeed, the result has a classical analog. As we pointed out at the end of the last sub-section, the unsmeared metric g_{ab} does not define a Hamiltonian vector field on the reduced phase space. Hence, to evaluate Poisson brackets, we are forced to use the smeared metric. Then, it is easy to verify that the Poisson brackets between the functionals $\gamma(f_R)$ and $\gamma(g_{R'})$ fail to vanish even when R and R' are widely separated. In fact these Poisson bracket just mirror the commutators given above.

The last point we wish to discuss concerns the holonomies computed in section 2.2. We found that the expression of the holonomy involves the exponential of the integral of the connection along a loop on Σ. Now, as we indicated in chapter 1, there is a canonical quantization program which is based on the assumption that the quantum analogs of these holonomies are well-defined operators. The present model provides a good testing ground for the validity of this assumption.

To see that the issue is non-trivial, let us first recall the situation in the well-understood Maxwell theory, say in 2+1 dimensions. There, the connection is generally promoted to an operator-valued distribution and the holonomies (of real connections) fail to be well-defined in the standard Fock representation. For, in a 2+1-dimensional theory, the operator-valued connection has to be smeared with 2-dimensional test fields while loops have only 1-dimensional support. In the present case, we are also using a Fock representation. A natural question therefore arises: Is the situation then analogous to the Maxwell theory? If so, the basic assumption mentioned above would fail to hold in our solution.

Now, because of axi-symmetry, smearing along a path in the radial direction in effect corresponds to a 2-dimensional smearing. Hence, the acid test is provided by loops $R =$ const where one can not take advantage of axi-symmetry. Can the classical expression (2.29) of the trace of the holonomy along such a loop, η, be promoted to a well-defined, regulated operator? Following the procedure we used in subsection 2.3.3, we find that the answer is in the affirmative. The quantum operator is given by:

$$\hat{T}^0_\eta = 2\cos\left[\pi\left(1 - e^{-\frac{1}{2}:\hat{\gamma}(f_R):}\right)\right], \tag{2.56}$$

The standard spectral theorems ensure that the operator on the right is well-defined, self-adjoint and has spectrum bounded between -1 and $+1$. Thus, the situation is very different from that in the Maxwell case. Indeed, in the present case, it is the scalar field that is subject to Fock quantization. The connection –like the metric– is a *non-local* functional of the elementary scalar field; its expression involves 2-dimensional integrals of the basic fields. It is because of this that the trace of the holonomy can be promoted to a well-defined operator on $\mathcal{H}_P$. As in the case of the metric, if we were interested only in quadratic forms, there would be no need to use any smearing fields; they are needed only if one wishes to obtain genuine operators.

Chapter 3

Gowdy T^3 Models

In this chapter we will quantize Gowdy T^3 models using the canonical formalism for 2+1 gravity coupled to matter field. In section 3.1, we consider the classical Hamiltonian formulation and isolate the true degrees of freedom by deparametrization. The coordinate choice is such that will lead to the same sort of simplifications as in the Einstein-Rosen case. However, there will be several differences due to compactness of the spatial slices. For instance, the Hamiltonian of the reduced system will be explicitly time-dependent. There will be an initial singularity on this space-time. Also, this model will have an additional global constraint left on the reduced phase-space. These facts will considerably change the quantum theory in comparison to Einstein-Rosen waves. The quantization will be performed on section 3.2.

3.1 Hamiltonian Formulation

In this section we will define our midi-superspace and obtain the action for this model. Then, we will introduce appropriate coordinate conditions in order to deparametrize the theory. Finally the reduced system will be obtained.

3.1.1 The midi-superspace

As in chapter 2 we begin by specifying our midi-superspace from the 2+1 perspective. Thus, we will consider solutions of 2+1-dimensional general relativity with $U(1)$

isometry group coupled to zero rest mass scalar-fields (where the $U(1)$ Killing field is hypersurface orthogonal). The underlying manifold M will be topologically $\mathrm{T}^2 \times \mathrm{R}$ and the space-time metric will have signature (–,+,+).

Denote by σ^a the Killing field. Hypersurface orthogonality of σ^a implies that the space-time metric g_{ab} has the form:

$$g_{ab} = h_{ab} + \tau^2 \nabla_a \sigma \nabla_b \sigma \tag{3.1}$$

where τ is the norm of the Killing vector field and σ is an angular coordinate with range $0 \leq \sigma < 2\pi$ such that $\sigma^a \nabla_a \sigma = 1$. The field h_{ab} so defined is a metric of signature (–,+) on the 2-manifolds orthogonal to σ^a. Let us introduce a generic slicing by compact space-like hypersurfaces labelled by $T = \text{const}$ and a dynamical vector field $T^a = Nn^a + N^\theta \hat{\theta}^a$ where n^a is a unit normal to the slices and $\hat{\theta}^a$ is the unit vector field within each slice orthogonal to σ^a. The pair N, N^θ constitute the lapse and shift. If we now introduce an angular coordinate such that $\hat{\theta}^a \nabla_a \theta = 1$ and $0 \leq \theta < 2\pi$. Then, the 2-metric can be written as:

$$h_{ab} = (-N^2 + (N^\theta)^2) \nabla_a T \nabla_b T + 2N^\theta \nabla_{(a} T \nabla_{b)} \theta + e^\gamma \nabla_a \theta \nabla_b \theta, \tag{3.2}$$

where N, N^θ and γ are functions of θ and T. It is because of the underlying $U(1)$ symmetry that the 3-metric g_{ab} has only four independent components and they are functions only of two variables. Moreover due to compactness of the spatial slice they are periodic functions of θ.

Thus, our midi-superspace consists of five functions, $(N, N^\theta, \gamma, \tau, \psi)$ of T and θ, (which are periodic in θ), where ψ is the zero rest mass scalar field. These five fields are subject to the following field equations:

$$G_{ab} = T_{ab}, \quad \text{and} \quad g^{ab}\nabla_a\nabla_b\psi = 0\,, \tag{3.3}$$

where G_{ab} is the Einstein tensor of g_{ab} which is determined by the fields (N, N^θ, γ, R) via (3.1) and (3.2) and T_{ab} is the stress-energy tensor of the scalar field:

$$T_{ab} = \nabla_a\psi\nabla_b\psi - \tfrac{1}{2}(g^{cd}\nabla_c\psi\nabla_d\psi)g_{ab}\,. \tag{3.4}$$

Note that T_{ab} satisfies the *strong energy condition*, i.e., $T_{ab}\lambda^a\lambda^b \geq -\frac{1}{2}T$, where λ^a is any unit time-like vector field and $T = T^a_a$. Now due to compactness of the spatial slice and the *strong energy condition*, the Hawking-Penrose theorems tell us that the space-time described above will generically have a singularity. This is a major difference from Einstein-Rosen cylindrical waves. There the spatial slices were asymptotically flat and at the classical level we were mainly concerned with an appropriate description of the asymptotic structure. Another difference related to compactness of the spatial slices is that now $\nabla_a\tau$ has to be time-like on M [18] whereas on the non-compact case it is space-like (it is for this reason that at chapter 2 τ is denoted by R).

Before we conclude this section we would like to comment on the coordinatization of the superspace that is being adopted here. Let us look at this system from a $3+1$ perspective for a moment. In this case the midi-superspace (γ, R, ψ) refers to vacuum general relativity. There exists in the literature coordinatizations for this model which differ somewhat from ours [9, 10]. But the route to quantization is not as direct as the one that we will obtain here. More details of this comparison will be given later on.

3.1.2 Canonical Form of the Action

Let us begin with the 3-dimensional action:

$$S(g,\psi) := \frac{1}{16\pi G}\int_M d^3x\sqrt{g}[\mathcal{R} - g^{ab}\nabla_a\psi\nabla_b\psi], \tag{3.5}$$

where $\mathcal{R}$ is the scalar curvature of g. There are no boundary terms because the spatial slice is compact. To pass to the Hamiltonian formulation, one performs a 2+1-decomposition. Let us substitute the form of the metric given by Eqs. (3.1) and (3.2) in (3.5). Then, the action reduces to the standard form

$$S = \frac{1}{8G}\int dT\left(\int d\theta(p_\gamma\dot{\gamma} + p_\tau\dot{\tau} + p_\psi\dot{\psi}) \ - H[N,N^\theta]\right), \tag{3.6}$$

The Hamiltonian H is given by:

$$H[N,N^\theta] = \frac{1}{8G}\int d\theta(NC + N^\theta C_\theta) \tag{3.7}$$

where C and C_θ are functions of the canonical variables:

$$\begin{aligned} C &= e^{-\gamma/2}(2\tau'' - \gamma'\tau' - p_\gamma p_\tau) + \frac{1}{2}\tau\, e^{-\gamma/2}(\frac{{p_\psi}^2}{\tau^2} + \psi'^2), \\ C_\theta &= e^{-\gamma}(-2p'_\gamma + \gamma' p_\gamma + \tau' p_\tau) + e^{-\gamma}p_\psi\psi'. \end{aligned} \tag{3.8}$$

(Here primes denote derivatives with respect to θ.)

As expected the lapse and shift N, N^θ appear as Lagrange multipliers; they are not dynamical variables. Thus, the phase-space Γ consists of three canonically-conjugate pairs of periodic functions of θ, $(\gamma, p_\gamma; \tau, p_\tau; \psi, p_\psi)$ on a 2-manifold Σ which is topologically T^2. By varying the action with respect to lapse and shift we obtain as usual two first class constraints $C = 0$ and $C_\theta = 0$. (Because of the underlying symmetry of the space-time, the σ component of the diffeomorphism constraint C_σ vanishes identically.) Therefore, the Hamiltonian of the system vanishes on the constraint surface.

Note that there is no distinction between gauge and dynamics in the compact case. The situation is different from the one we encountered in the Einstein-Rosen waves. There, the space-time is asymptotically flat and a non-vanishing Hamiltonian generates dynamics. The standard gauge fixing procedure is therefore well-suited. In Gowdy space-times, Σ is spatially compact, the Hamiltonian vanishes on the constraint surface and hence one needs to deparametrize the theory. In other words it is necessary to select a variable or combination of variables on the phase-space to play the role of time. In this sense the compact case is conceptually more complex.

3.1.3 Deparametrization

Deparametrization is accomplished by imposing 'gauge conditions' on the phase-space variables (the choice of phase-space variables that will play the role of coordinates) such that they extract one point from each orbit of the Hamiltonian vector fields corresponding to the constraints, except for one. This remaining orbit will be the Hamiltonian vector field that will describe evolution. In another words, we have to select one

Hamiltonian constraint to represent evolution and gauge fix the others. We will choose coordinates conditions to make the space-time geometry transparent. Let us demand:

$$\tau'(\theta) = 0 \quad \text{and} \quad p_\gamma = -1. \tag{3.9}$$

The first condition will allow us to regard $\tau(\theta)$ as the time parameter and is motivated by the fact, noted in subsection 3.1.1, that the field equations imply that $\nabla_a \tau$ is time-like everywhere in M. The second condition states that the scalar density p_γ should equal -1 in our chart (θ, σ) on Σ. It can be restated as $\int^\theta p_\gamma(\theta_1) d\theta_1 = -\theta$. It will serve to remove γ from our list of dynamical variables. Thus, if these choices are admissible, the true degrees of freedom will all reside in the field ψ, in accordance with our general expectation that in 2+1 dimensions, all the local degrees of freedom are carried by matter fields. We should add that the conditions (3.9) were also motivated from the ones adopted previously for Einstein-Rosen waves. A comparison shows that $\tau(\theta)$ is formally replacing the function $R(r)$ in the action. However their roles are quite different due to the restriction on their gradients. Just as we chose $R(r)$ as a radial coordinate, now we choose $\tau(\theta)$ as a time coordinate. So we might expect to achieve the same technical simplifications on this model.

To see if the coordinate condition is acceptable we have to show that the Poisson brackets $\{\tau'(\theta), H[N, N^\theta]\}$ and $\{p_\gamma + 1, H[N, N^\theta]\}$ vanish for a unique choice of lapse and shift, or equivalently $\dot\tau = 1$ and $\dot p_\gamma = 0$. Explicitly,

$$\dot{\tau}(\theta) = \{\tau, H[N, N^\theta]\} = -Ne^{-\gamma/2}p_\gamma + N^\theta e^{-\gamma}\tau' = 1$$

$$\dot{p}_\gamma(\theta) = \{p_\gamma, H[N, N^\theta]\} = \frac{1}{2}NC + N^\theta C_\theta - \left(Ne^{-\gamma/2}\tau'\right)' + \left(N^\theta e^{-\gamma}p_\gamma\right)' = 0\,. \quad (3.10)$$

As needed the right sides vanish for a unique choice of lapse and shift and do not vanish in any other circumstance. The only solutions are:

$$N = e^{\gamma/2} \quad \text{and} \quad N^\theta = 0. \quad (3.11)$$

Hence the choice (3.9) selects uniquely a Hamiltonian vector field that represents evolution and fixes the remainder gauge freedom. The Hamiltonian vector field corresponding to the lapse and shift given by (3.11) generates evolution along the one-parameter family of points labelled by $\tau(\theta) = T$.

Finally, let us extract the true degrees of freedom of the theory. In order to accomplish this, we need to eliminate redundant variables by solving the set of second class constraints (3.8) and gauge conditions (3.9). By setting $\tau = T$ and $p_\gamma = -1$ in (3.8), we can trivially solve for p_T and for γ in terms of ψ and p_ψ (using the Hamiltonian and the diffeomorphism constraints respectively). The result is:

$$p_T = -\frac{1}{2}T\left(\frac{{p_\psi}^2}{T^2} + \psi'^2\right), \quad (3.12)$$

$$\gamma(\theta) = \int^\theta d\theta_1 p_\psi \psi' \quad (3.13)$$

Substituting (3.13) in (3.11), we can also express the lapse N in terms of ψ and p_ψ. Thus, as expected, the true degrees of freedom reside just in the matter variables. Indeed, the space-time metric is now completely determined by ψ and p_ψ:

$$g_{ab} = e^{\gamma}\,(-\nabla_a T\,\nabla_b T\; + \nabla_a\theta\nabla_b\theta) + T^2\nabla_a\sigma\,\nabla_b\sigma\,, \tag{3.14}$$

where, from now on, γ will only serve as an abbreviation for the right side of (3.13). One can show that the curvature invariant $R_{abcd}R^{abcd}$ for this space-time metric blows up at $T = 0$ (for $\gamma \neq const$). Thus, as expected, there is an initial singularity.

Recall that the phase-space variables are periodic functions of θ. Therefore, from (3.13) we obtain a global constraint, that we will denote by P_θ:

$$P_\theta := \gamma(2\pi) - \gamma(0) \equiv \int_0^{2\pi} d\theta_1 p_\psi \psi' = 0. \tag{3.15}$$

Note that this extra global constraint arises because the spatial slices are compact. The presence of this extra constraint will make the quantum theory considerably different from the asymptotically flat case described in section 2.3.

3.1.4 Reduced Phase Space

As in the case of Einstein-Rosen waves the reduced phase space $\bar{\Gamma}$ can be coordinatized by the pair $(\psi(\theta), p_\psi(\theta))$. However, now the canonical variables are subject to the global constraint $P_\theta = 0$. Therefore, the physical reduced phase space is non-linear and has the structure of a manifold instead of the usual vector space. Because of this non-linearity it is appropriate to postpone the reduction by the constraint to the quantum theory. Then, it will be imposed as an operator condition on the quantum states.

The (non-degenerate) symplectic structure on the reduced phase space $\bar{\Gamma}$ is the pull-back of the symplectic structure on Γ. Thus,

$$\{\psi(\theta_1), p_\psi(\theta_2)\} = \delta(\theta_1, \theta_2) \tag{3.16}$$

on $\bar{\Gamma}$. Next, let us write the reduced action by substituting (3.9), (3.12) and (3.13) in (3.6),

$$S[\psi, p_\psi] = \frac{1}{8G} \int dT \left[\int_0^{2\pi} d\theta (p_\psi \dot{\psi}) - \frac{1}{2} T \left(\frac{{p_\psi}^2}{T^2} + {\psi'}^2 \right) \right], \tag{3.17}$$

where we have disregarded surface terms. The reduced Hamiltonian, that we shall denote by H, is explicitly time dependent. Note that the global constraint is conserved under time evolution because $\{P_\theta, H\} = 0$. By varying the action (3.17) with respect to ψ and p_ψ we obtain the second-order equation of motion for ψ:

$$-\frac{\partial^2 \psi}{\partial T^2} + \frac{\partial^2 \psi}{\partial \theta^2} - \frac{1}{T} \frac{\partial \psi}{\partial T} = 0. \tag{3.18}$$

This is exactly the Klein-Gordon equation for a scalar field propagating on a flat background $\overset{\circ}{g}_{ab}$, given by:

$$\overset{\circ}{g}_{ab} = -\nabla_a T \nabla_b T + \nabla_a \theta \nabla_b \theta + T^2 \nabla_a \sigma \nabla_b \sigma. \tag{3.19}$$

So we have achieved the same conceptual simplification as in the Einstein-Rosen model, i.e., the decoupling of the system. However it is technically different because now the topology of the fictitious background is not R^3, the flat space is not globally Minkowskian. As we will see, it is a wedge of Minkowski space-time identified, such that the boundary of the compact spatial slices is time dependent and starts at a singularity. The topology of the fictitious background and the geometry of the singularity on $\overset{\circ}{g}_{ab}$ is more transparent if we change to a coordinate system where the metric is explicitly flat, i.e.,

$$\overset{\circ}{g}_{ab} = (-\nabla_a x_0 \nabla_b x_0 + \nabla_a \theta \nabla_b \theta) + \nabla_a x_2 \nabla_b x_2. \tag{3.20}$$

The two sets of coordinates are related by $x_0 = T\cosh(\sigma - \pi)$ and $x_2 = T\sinh(\sigma - \pi)$. Let us concentrate on the plane defined by x_0 and x_2. Due to the angular range of σ, the x_2 coordinate will be identified, i.e., $x_2(0) = x_2(2\pi)$. At $T = 0$ the length of the orbit of $\partial/\partial\sigma$ goes to zero and thus the orbit reduces to a point and the two-surface (torus) is a circle (on θ). Therefore, the space-time has a singularity at $x_0 = T = 0$. In terms of surfaces $T = const$, as T increases the spatial slices (tori) are expanding.

Although the fictitious metric has a singularity the initial value problem for the scalar field ψ is well posed $\forall T > 0$. The explicit time-dependence of the Hamiltonian reflects the expanding background and the existence of the singularity. So, differently from the asymptotically flat case, there is a ('mild') back-reaction of the gravitational field. We can nonetheless solve first for a scalar field ψ propagating on a flat background $\overset{\circ}{g}_{ab}$, define a function γ via (3.13), and construct a curved metric g_{ab} through (3.14).

A remark is in order. In the coordinatization adopted in [10] the reduced system also decouples, but naively it is equivalent to a scalar field, denoted there by $B_{-}(\theta, t)$ ($\psi = \ln \tau - 2\sqrt{3}B_{-}$), propagating on a curved background. In the light of this result, the fact that we got a scalar field propagating on a flat background might seem surprising. However, changing the time coordinate t via $\ln 2T = 2t$ one can reinterpret the equation of motion of $B_{-}(\theta, T)$ to that of a scalar field propagating on the flat background $\overset{\circ}{g}_{ab}$.

Finally, for reasons that will be clear in the next section, let us introduce a covariant description of the reduced phase-space $\bar{\Gamma}$. In this approach the phase-space consists of the real solutions of the Klein-Gordon equation (3.18), i.e.,

$$\psi(\theta, T) = \sum_{m=-\infty}^{\infty} f_m(\theta, T) A_m + f_m^*(\theta, T) A_m^* \tag{3.21}$$

where A_m's are arbitrary constants and

$$\begin{aligned} f_0(\theta, T) &= \frac{1}{2}(\ln T - i) \\ f_m(\theta, T) &= \frac{1}{2} H_0^{(1)}(|m|T) e^{im\theta} \\ &= \frac{1}{2}\left(J_0(|m|T) + iN_0(|m|T)\right) e^{im\theta} \quad \text{for} \quad m \neq 0 \end{aligned} \tag{3.22}$$

where $H_0^{(1)} = J_0 + iN_0$ is the 0th-order Hankel function of the 1st kind and J_0 and N_0 are the 0th-order Bessel function of the first and second kind respectively. (* denotes complex conjugation.).

The symplectic structure is given by:

$$\Omega(\psi_1, \psi_2) = \int_0^{2\pi} T d\theta \, (\psi_2 \partial_T \psi_1 - \psi_1 \partial_T \psi_2) \, . \tag{3.23}$$

This covariant approach is completely equivalent to the canonical description outlined on the first paragraph of this subsection. We will refer to the covariant phase space as the 'space of real solutions' and denote by V. Note that because the symplectic structure is conserved ($\dot{\Omega} = 0$), generaically the scalar field ψ to diverge at $T = 0$.

3.2 Quantum Theory

In this section we will start by constructing a fiducial Hilbert space. Then, by imposing the global constraint as an operator condition on the states, we will obtain the physical Hilbert space. Finally, we will investigate the quantum geometry.

3.2.1 Fiducial Hilbert Space

In this subsection we will ignore the global constraint (3.15) and quantize the system. The Hilbert space thus obtained will be called fiducial Hilbert space. The physical Hilbert space is the subspace of the fiducial Hilbert space corresponding to the kernel of the global constraint and will be constructed on the next subsection.

The overall procedure was discussed in subsection 2.3.1. Recall that in order to quantize the (unconstrained) theory the phase space variables $(\psi(\theta), p_\psi(\theta))$ are smeared with test fields and represented by operators satisfying the commutation relation corresponding to their Poisson brackets. Quantization is then accomplished by constructing a Hilbert space of quantum states that corresponds to a *-representation of this observable algebra. The appropriate representation will be selected by imposing physical requirements. We will demand the time dependent Hamiltonian (3.17) and the global constraint (3.15) to be promoted to well-defined operators.

Recall that the fictitious background (M, $\overset{\circ}{g}_{ab}$) is a suitably identified wedge of flat space-time and that the spatial slices have a time dependent boundary. Moreover there is no global time-like Killing vector field. Therefore, the procedure to obtain a representation, followed in subsection 2.3.1, is not directly applicable. We will instead adopt a prescription that is outlined in [19]. We should add that for linear theories both procedures are general and it is just a matter of one being more convenient then the other for the given problem.

The first assumption to obtain a representation for the observable algebra is that, as in subsection 2.3.1, the Hilbert space will have the structure of a symmetric Fock-space $\mathcal{F}$, based on some one-particle Hilbert space $\mathcal{H}$. The one-particle Hilbert space can be constructed from the space of real solutions V in the following way. First one introduces on V a complex structure $\mathcal{J}$ which is compatible with the symplectic structure (3.23), i.e., $(\psi_1, \psi_2) := \Omega(\mathcal{J}\psi_1, \psi_2)$ is a positive-definite inner product on V. Then, one can define on the complex vector space $(V, \mathcal{J})$ the inner-product:

$$< \psi_1|\psi_2 >:= \frac{1}{2}\Omega(\mathcal{J}\psi_1, \psi_2) + \frac{1}{2}i\Omega(\psi_1, \psi_2). \tag{3.24}$$

Finally, the one-particle Hilbert space is the Cauchy completion of the complex inner-product space $(V, \mathcal{J}, < \mid >)$. Furthermore, to complete the construction of the Fock space one introduces positive and negative frequency decomposition on $\mathcal{H}$ via:

$$\psi^+ \quad := \quad \frac{1}{2}(\psi - i\mathcal{J}\psi) \tag{3.25}$$

$$\psi^- \quad := \quad \frac{1}{2}(\psi + i\mathcal{J}\psi) \tag{3.26}$$

such that $\psi = \psi^+ + \psi^-$. Thus, a field operator in the Fock space is represented by $\hat{\psi}(\psi) = \hat{A} + \hat{C}$, where $\hat{C}$ and $\hat{A}$ are creation and annihilation operators associated with the positive and negative frequency decomposition.

In summary, to obtain a representation all the work can be focussed on finding an appropriate complex structure on V. In our model there is a natural complex structure that arises from the general solution (3.21). Using the fact that the solution in (3.21) naturally occurs in pairs, we define a complex structure $\mathcal{J}$ for our model as:

$$\mathcal{J}\,\alpha \ln T := -\alpha \quad \text{and} \quad \mathcal{J}\,\alpha := \alpha \ln T \qquad \text{for} \quad m = 0 \tag{3.27}$$

$$\mathcal{J}\,J_0(|m|T) := -N_0(|m|T) \quad \text{and} \quad \mathcal{J}\,N_0(|m|T) := J_0(|m|T) \qquad \text{for} \quad m \neq 0 \tag{3.28}$$

(α is a constant.) Note that $J^2 = -1$ as required. Thus, the "positive frequency" part can be obtained by using (3.21), (3.25), (3.27) and (3.28):

$$\psi^+ = \sum_{m=-\infty}^{\infty} \frac{1}{2} f^*_m(\theta, T) A^*_m, \tag{3.29}$$

and, from (3.26), the negative frequency part is just the complex conjugate of (3.29). Moreover one can show that the complex-structure, given by (3.27, 3.28), is in fact compatible with the symplectic structure and therefore completely determines the inner-product on the one particle Hilbert space $\mathcal{H}$. Following the prescription we can write the field operator $\hat{\psi}$ in terms of creation and annihilation operators corresponding to the positive and negative frequency decomposition defined on $\mathcal{H}$ by the complex structure (3.27, 3.28):

$$\hat{\psi}(\theta,T) = \sum_{m=-\infty}^{\infty} \left(f_m(\theta,T)\hat{A}_m + f_m^*(\theta,T)\hat{A}_m^\dagger \right) \tag{3.30}$$

A comparison with the classical solution (3.21) shows that one could obtain this field operator naively by promoting the constants of motion from the explicit solution to operators. In fact this is also a motivation for choosing (3.27, 3.28) as the complex structure. Furthermore, as needed, one can show that the normal ordered Hamiltonian and global constraint are well-defined operators.

There is an important result from the above construction. Note that the complex-structure is time *independent*, therefore, there is no mixing between positive and negative parts or analogously, creation and annihilation operators as time passes. Thus, in contrast to ref.[10], we conclude that there is no-creation of 'particles'.

3.2.2 Physical Hilbert Space

The space $\mathcal{F}_{phys}$ of physical states is the subspace of the fiducial Hilbert space $\mathcal{F}$ defined by:

$$:\hat{P}_\theta: |\Psi>_{phys} = 0. \tag{3.31}$$

In order to have a better understanding of this condition let us express the global constraint in terms of creation and annihilation operators:

$$: \hat{P}_\theta : |\Psi >_{phys} = 2 \sum_{m=-\infty}^{\infty} m \hat{A}^\dagger_m \hat{A}_m \, |\Psi >_{phys} = \; 0. \tag{3.32}$$

Thus, the physical states are states such that the total angular momentum in θ-direction vanishes. Therefore, obviously the (usual) vacuum state defined by $\hat{A}_n|0>= 0$, $\forall\, k$ and states with particles in the zero mode (i.e. $m = 0$) belong to the physical Hilbert space. Explicitly, a generic physical state with N particles is given by:

$$|^N \Psi >_{phys} = \prod_{i=1}^{N} |m_i > \quad \text{such that} \quad \sum_{i=1}^{N} m_i = 0. \tag{3.33}$$

where $|m_i>$ represents a one-particle state with angular momentum m_i. Note that, the space of physical states does not inherit the Fock space structure of $\mathcal{F}$. For instance, except for the zero mode, none of the one-particle states of $\mathcal{F}$ belongs to $\mathcal{F}_{phys}$. Nonetheless because the orbit of P_θ is compact (or $: \hat{P}_\theta :$ has discrete spectrum) $\mathcal{F}_{phys}$ is a closed subspace of $\mathcal{F}$ and hence has a natural Hilbert space structure. Moreover, the operators from $\mathcal{F}$ can be projected to $\mathcal{F}_{phys}$. We will denote the projection operator by $\mathcal{P}$.

Let us now investigate the dynamics of the system. The Hamiltonian, from expression (3.17), can be promoted to the operator:

$$\hat{H}(T) = \frac{1}{16G} \int_0^{2\pi} d\theta T : \left(\frac{\hat{p}_\psi^2}{T^2} + (\hat{\psi}')^2 \right) : . \tag{3.34}$$

Expressing in terms of creation and annihilation operators, we obtain:

$$\hat{H} \;\; = \;\; \frac{\pi}{32GT} : (\hat{A}_0 + \hat{A}_0^\dagger)^2 :$$

$$+ \quad \frac{T}{64G}\sum_{m=-\infty}^{\infty} m^2 \left[2\hat{A}^{\dagger}_m\hat{A}_m(H_1^{(1)}H_1^{(2)} + H_0^{(1)}H_0^{(2)})\right.$$

$$\left. + \quad \hat{A}_m\hat{A}_{-m}\left((H_0^{(1)})^2 + (H_1^{(1)})^2\right) + \hat{A}^{\dagger}_m\hat{A}^{\dagger}_{-m}\left((H_0^{(2)})^2 + (H_1^{(2)})^2\right)\right], \quad (3.35)$$

where $H_0^{(2)}(|m|T) = [H_0^{(1)}(|m|T)]^*$ and $H_1^{(1,2)}(|m|T) = -\frac{1}{|m|}\dot{H}_0^{(1,2)}(|m|T)$. By inspection one can easily conclude that the action of the Hamiltonian operator on a physical state gives back a physical state, thus it leaves $\mathcal{F}_{phys}$ invariant. Note that the vacuum state is *not* an eigenvector of the Hamiltonian with zero eingenvalue. However, this Hamiltonian arises from our choice of deparametrization, therefore apriori there is no direct physical significance to this result. In contrast to the Einstein-Rosen model, where the vacuum was physically defined, here it is not clear how to define the 'real', physical vacuum. This is a consequence of the arbitrariness of the deparametrization procedure.

In order to investigate the time parameter that arises from the quantum theory let us write the Schrödinger equation associated with (3.34):

$$i\hbar\frac{\partial}{\partial T}|\Psi>_{phys}(T) = \hat{H}(T)|\Psi>_{phys}(T). \quad (3.36)$$

One can interpret this equation by saying that this evolution takes place on the fictitious background $\mathring{g}_{ab}$ where $\frac{\partial}{\partial T}$ has a space-time interpretation. The physical space-time is a derived quantity, i.e., there is no physical metric to start with, thus apriori $\frac{\partial}{\partial T}$ has no physical interpretation. Nonetheless, because of the classical decoupling we can still write the metric operator by using the chart (T, θ, σ). This is remarkable because we can conclude that in fact, although there is no background physical metric, the simplification of this system is such that provides a time parameter for the quantum theory.

3.2.3 Quantum Geometry

Let us start investigating semi-classical geometries. In order to do this, we shall return to the fiducial Hilbert space $\mathcal{F}$. First we promote the classical expression of the space-time metric (3.14) to an operator on $\mathcal{F}$. Formally,

$$\text{“ } \hat{g}_{ab} =: e^{\hat{\gamma}(\theta,T)} : (-\nabla_a T\, \nabla_b T\, + \nabla_a \theta \nabla_b \theta) + T^2 \nabla_a \sigma\, \nabla_b \sigma \text{ ”} . \qquad (3.37)$$

As in the Einstein-Rosen case, the states yielding semi-classical geometry will be the usual coherent states for the field operators. Similarly, the matrix elements of this operator on coherent states on $\mathcal{F}$ are well-defined. In particular, the expectation value on a coherent state $|\Psi_c>$, gives the classical expression (3.14) evaluated on the classical field configuration ψ_c, explicitly

$$< \Psi_c | \hat{g}_{ab} | \Psi_c > =: e^{\int^\theta T \dot{\psi}_c \psi'_c d\theta_1} : (-\nabla_a T\, \nabla_b T\, + \nabla_a \theta \nabla_b \theta) + T^2 \nabla_a \sigma\, \nabla_b \sigma \, . \qquad (3.38)$$

An immediate consequence of this result is that the coherent state provides us an example to show that the singularity persists in the quantum theory. Specifically, we can compute a scalar formed out from the Riemann tensor, promote to a (normal ordered) quantum operator and calculate its expectation value on a coherent state. We will, then, obtain the corresponding divergent classical value.

The only non-trivial metric operator component is $\exp(\int^\theta T \dot{\hat{\psi}} \hat{\psi}' d\theta_1)$. Note that the exponent has the same functional form as the angular momentum for a scalar field in a box (if one recovers the integral on σ that is omitted due to symmetry). Similarly to the energy in a box that was extensively discussed in subsection 2.3.3, one can show that it is not a well-defined operator on $\mathcal{F}$. Thus, it has to be regulated. Using basically the same procedure we obtain the regulated metric operator on $\mathcal{F}$:

$$\hat{g}_{ab}(f_\theta) = e^{\int_0^{2\pi} f_\theta(\theta_1) T :\dot{\hat{\psi}} \hat{\psi}': d\theta_1} \left(-\nabla_a T \, \nabla_b T + \nabla_a \theta \nabla_b \theta\right) + T^2 \nabla_a \sigma \, \nabla_b \sigma \,, \tag{3.39}$$

where the regulator $f_\theta(\theta_1)$ equals 1 for $\theta_1 \leq \theta - \epsilon$, then it smoothly decreases to zero and equals zero for $\theta_1 \geq \theta - \epsilon$. Also, $f_{2\pi}(\theta) = f_0(\theta)$. As we see, again the Planck length comes into play naturally in the quantum theory

Although we have obtained a well-defined operator on $\mathcal{F}$, it is not an operator on the physical space because it does not commute with the global constraint : $\hat{P}_\theta$:. However, as we pointed out before, it can be projected to the physical space. Therefore the physical regulated metric operator in $\mathcal{F}_{phys}$ is given by:

$$[\hat{g}_{ab}(f_\theta)]_{phys} = \mathcal{P} \, \hat{g}_{ab}(f_\theta) \, \mathcal{P}. \tag{3.40}$$

As in the Einstein-Rosen waves, there are interesting features associated to the metric operator. Because the calculations are similar, we will avoid the repetition by pointing out only the final results. First there will be quantum fluctuations of the light-cone. Specifically, for a given quantum state of the system (gravity coupled to scalar field), the norm of a null vector will fluctuate between positive, null and negative values. Second the commutator between two non-trivial metric operators is non-vanishing, this is a consequence, as before, of the non locality of the metric components with respect to the (basic) scalar field. Finally, the holonomy operator is well-defined in this model as well. Regarding this operator there are some differences that are worth pointing out in more detail. Recall that in the Einstein-Rosen waves the first internal gauge choice that we made led to a badly-behaved connection at the origin (2.24). Therefore, a gauge transformation was necessary. After that, the component of the connection along $\nabla_a R$ was not Abelian anymore, therefore the holonomy along a loop $\sigma = const$ was not trivial to compute. It was given by the path-ordered exponential. Thus, we explicitly computed only the holonomy along a loop defined by $R = const$ because in this case the component of the connection that was contributing was Abelian. However, now using the same (first) internal gauge choice, we obtain a well-defined connection because of the topology of the spatial slices. Moreover, the holonomy along both generators of the torus can be easily computed, because the respective contributing components of the connection are Abelian. The corresponding operators are given by:

$$\hat{T}^0_\eta = 2\cosh\left[\pi e^{-:\hat{\gamma}(f_\theta):/2}\right] :, \tag{3.41}$$

for a loop η with tangent vector given by $\dot{\eta}^a = \sigma^a$ (along the integral curve of the Killing field), and

$$\hat{T}^0_\eta = 2\cosh\left[4G\,\hat{H}\right], \tag{3.42}$$

where now the loop η has tangent vector given by $\dot{\eta}^a = \theta^a$ and $\hat{H}$ is the Hamiltonian operator (3.34). Note that in the Einstein-Rosen case we calculated the holonomy for a loop along the integral curve of the Killing field and it needed to be regulated. Here, the operator (3.41) has to be regulated as well to yield a well-defined operator on $\mathcal{F}$. Moreover it does not commute with the global constraint, therefore it has to be projected to $\mathcal{F}_{phys}$. On the other hand, the holonomy operator (3.42) does not require any regularization procedure (other than normal ordering). Moreover, it commutes with the global constraint. Therefore it is automatically a well-defined operator on $\mathcal{F}_{phys}$, as expected (see discussion in the end of subsection 2.3.4).

3.2.4 Physical Coherent States

In this section we will obtain the physical coherent states. In the fiducial Hilbert space, the coherent states were exactly the usual coherent states for the field operator. However, in general, the field operator $\hat{\psi}(\theta, T)$, or equivalently the annihilation and creation operators, $\hat{A}_m$ and $\hat{A}^\dagger_m$ (for $m \neq 0$), do not leave the physical Hilbert space invariant. Therefore, to obtain a coherent state on $\mathcal{F}_{phys}$ we will adopt the following strategy: First we will obtain a set of 'basic' operators such that any operator on $\mathcal{F}_{phys}$ can be expressed as a combination of them. They play a role on $\mathcal{F}_{phys}$ analogous to $\hat{A}_m$ on $\mathcal{F}$. Then, having obtained these operators, we will seek states where they will be peaked on their classical value. These will be the coherent state on $\mathcal{F}_{phys}$ because by construction all other normal ordered operators will be peaked on their classical value as well.

Let us obtain the set of 'basic' operators on $\mathcal{F}_{phys}$. Note, first, that classically the phase space modulo the global constraint P_θ can be coordinatized by the infinite set:

$$\beta^{[N]}_{\{m_i\}} = \prod_{i=1}^{N} A_{m_i} \quad \text{and} \quad \left[\beta^{[N]}_{\{m_i\}}\right]^* = \prod_{i=1}^{N} A^*_{m_i} \quad \text{with} \quad \sum_{i=1}^{N} m_i = 0 \qquad (3.43)$$

(The infinite set A_m, A_m^* is subject to *one* global constraint, thus we are still left with an infinite set.) Now, the 'basic' set of operators can be obtained by promoting (3.43) to operators on $\mathcal{F}_{phys}$:

$$\begin{aligned} \hat{\beta}^{[N]}_{\{m_i\}} &= \prod_{i=1}^{N} \hat{A}_{m_i} \quad \text{with} \quad \sum_{i=1}^{N} m_i = 0 \\ \left[\hat{\beta}^{[N]}_{\{m_i\}}\right]^\dagger &= \prod_{i=1}^{N} \hat{A}^\dagger_{m_i} \quad \text{with} \quad \sum_{i=1}^{N} m_i = 0 \end{aligned} \tag{3.44}$$

We can now ask if there exists a state on $\mathcal{F}_{phys}$ with the following property,

$$\hat{\beta}^{[N]}_{\{m_i\}} |\beta_c> = \beta^{[N]}_{\{m_i\}} |\beta_c> . \tag{3.45}$$

This will be the coherent state that we are looking for. The answer is that there exists such state and it is given explicitly by:

$$\begin{aligned} |\beta_c> &= |0> + \beta^{[1]}_0 |1(0)> + \sum_{m=-\infty}^{\infty} \beta^{[2]}_{\{m\}} |1(m), 1(-m)> + \cdots \\ &+ \cdots + \sum_{\{J\}} \frac{\beta^{[N]}_{\{J\}}}{\sqrt{N_1!} \cdots \sqrt{N_I!}} |N_1(1), \cdots, N_I(I), \cdots > + \cdots \end{aligned} \tag{3.46}$$

where the sum stands for all possible sets $\{J\}$ such that

$$\sum_{J=-\infty}^{\infty} J N_J = 0 \quad \text{and} \quad \sum_{J=-\infty}^{\infty} N_J = N . \tag{3.47}$$

We have changed the notation slightly in order to adopt the basis of the number operator. To make the notation clear, let us give an example of two possible sets of this sum with three particles, i.e., $N = 3$:

$$\frac{1}{\sqrt{2}}\beta^{[3]}_{(1,1,-2)}|2(1), 1(-2) > \quad \text{and} \quad \beta^{[3]}_{(12,-4,-8)}|1(12), 1(-4), 1(-8) > . \tag{3.48}$$

The expectation value of the physical metric operator given by (3.39) on $|\beta_c >$ yields by construction:

$$< \beta_c|[\hat{g}_{ab}]_{phys}|\beta_c >= [g_{ab}(\beta_c, \beta_c^*)]_{phys}, \tag{3.49}$$

i.e., classical geometries on the phase space with the global constraint implemented. Note that, as before, there is no need to have the regulator in this case.

Another interesting result is that one can show that the physical coherent state (3.46) is the projection to $\mathcal{F}_{phys}$ of the coherent state ($< \Psi_c|\hat{\psi}|\Psi_c >= \psi_c$) on $\mathcal{F}$, i.e.,

$$|\beta_c >= \mathcal{P}|\Psi_c > . \tag{3.50}$$

In particular, note that if ψ_c is a classical solution corresponding to the zero mode, that we will denote by ${}^0\psi_c$, then $|{}^0\beta_c >= |{}^0\Psi_c >$, ie., the physical coherent state for this classical solution is equivalent to the (usual) coherent state.

As a consequence of (3.50) the expectation value of any operator projected to $\mathcal{F}_{phys}$ on a physical coherent state has the following correspondence with respect to the operator and coherent states on $\mathcal{F}$:

$$< \beta_c|\hat{O}_{phys}|\beta_c >=< \Psi_c|\hat{O}_{phys}|\Psi_c >=< \beta_c|\hat{O}|\beta_c > . \tag{3.51}$$

As a last remark, let us calculate the expectation value of the physical metric operator on a physical coherent state corresponding to a (physical) classical 'zero mode' solution, i.e. $< {}^0\beta_c|[\hat{g}_{ab}]_{phys}|{}^0\beta_c >$. Using (3.51) and the comment after expression (3.50) we obtain that this expectation value equals: $< {}^0\Psi_c|\hat{g}_{ab}|{}^0\Psi_c >$. But, from the classical expression of the metric (3.14), it can be easily seen that the zero mode yields the flat metric (3.19). Therefore,

$$< {}^0\beta_c|[\hat{g}_{ab}]_{phys}|{}^0\beta_c >=\overset{\circ}{g}_{ab} . \tag{3.52}$$

Note that the expectation value on vacuum state also yields the flat metric. This corresponds to the fact that in the classical theory, the stress-energy tensor of the zero mode scalar fields vanishes identically.

Chapter 4

Summary and Discussion

The proposal of this thesis was to discuss issues of quantum gravity using two exactly soluble models. The remarkable feature of these two models is that although they have many technical similarities they are fundamentally different: one is a cosmological model, while the other, asymptotically flat. This allowed us to examine a wide variety of problems that are present in the full theory. We will now summarize our results starting with Einstein-Rosen waves and then we will discuss Gowdy models making comparisons whenever appropriate. We will also point out open questions at appropriate places.

In the Einstein-Rosen case we saw that, as usual, the treatment of the Hamiltonian formalism in the asymptotically flat context leads to a non-vanishing total energy on the constraint surface. Therefore there is a non-vanishing generator of dynamics of the system. As a consequence, the gauge fixing procedure is technically the same as for Yang-Mills theory, i.e., the Poisson bracket between gauge conditions and the generator of gauge transformations (i.e. constraints) has to be different from zero and that between the gauge conditions and the generator of dynamics (i.e. the Hamiltonian) has to vanish. There is no need to "deparametrize" the system. This approach to the classical theory differs from previous works on this model [7, 8] and consequently the quantum model has also new features.

After gauge fixing we were able to solve the non-linear system of constraint for the true degrees of freedom. The reduced model is very simple. The infinite number of degrees of freedom are represented by the scalar field with dynamics determined by a Hamiltonian that is an exponential function of the Hamiltonian for the scalar field propagating on a Minkowskian background. Although the dynamical equation is not linear, we showed that after a time rescaling (that is solution dependent!), it is exactly the same as the one for a scalar field propagating on flat background. This decoupling is remarkable but only serves as a useful artifact in the process of quantization. There is no physical content in it. In particular, the reduced system describes the full non-linear theory in which matter field is coupled to gravity.

To obtain the quantum theory we searched for an appropriate measure on the space of square-integrable distributions of the scalar field. The choice relies on the physical requirement that the Hamiltonian had to be promoted to a well-defined operator. A suitable measure that fulfills this requirement –also endorsed by the classical decoupling– is the usual Gaussian measure. Therefore, the Hilbert space is exactly that of a free scalar field theory.

Then we investigated the issue of time in the quantum theory. There is a clear distinction between the classical and the quantum problem of time. Namely, although the model is non-linear, classically it can be put in the linear form and then we can obtain the space-time solution for the scalar field $\psi(R,T)$. Thus, the space-time metric is obtained from this solution by simple integration. Then, by a solution dependent rescaling of the time parameter, we write the line-element using as time coordinate the parameter t corresponding to the Hamiltonian vector field that generates asymptotic unit time translation. Because the interplay between the linear and non-linear theory involves the scalar field, at the quantum level it is an operator relation and has direct physical interpretation only on coherent states. Thus, in the full quantum theory, the two choices of time are inequivalent. By choosing Gaussian measure we implicitly assumed the existence of a time-like Killing vector field and in fact it was taken to be $\partial/\partial T$. Thus, T is a time parameter that naturally arises in the quantum theory.

As we pointed out in the introduction, because the space-time itself is subject to quantization, it is not surprising that there is no time parameter associated with the physical Hamiltonian given once and for all. The remarkable result regarding this model is that the system, nonetheless, ended up with a time parameter. This enabled us, in particular, to write the full space-time metric operator. It would be interesting to investigate how much of this result is tied to the gauge choice. If it turns out to be true only for this gauge choice, then one could think that it is more an artifact of the gauge choice than a feature of the model itself. Therefore we could have some hope of finding an appropriate gauge choice for the full theory such that it provides a fictitious time parameter. This would be a preferred gauge choice.

Another interesting result is that the Hilbert space admits states corresponding to entire classical solutions. Specifically, the expectation value of the metric and the scalar field operator on these states satisfy the non-linear field equations. They are exactly the usual coherent states of the scalar field. In particular, the vacuum state is a coherent state. The expectation value of the metric on the vacuum state gives the flat metric. Also the vacuum state is an eigenvalue of the Hamiltonian of the system with zero eigenvalue.

These two facts together guarantees for us that the vacuum of the fictitious free field theory can be taken as the vacuum of the coupled system.

We saw that the Planck length enters in this model in an indirect way. Although the matrix elements of the metric operator on coherent states are finite, it is not a well-defined operator on the Hilbert space. It is basically because it involves an integration of the fields in a region with sharp boundary. The regularization procedure is to smear this boundary. Obviously it requires a length parameter to determine the thickness of the smearing. This is provided by the Planck length. If this were (physically) field theory in Minkowski background then, this regulator would violate Poincaré invariance, but our system (gravity coupled to matter field) does not have the Poincaré group as a symmetry.

Having obtained an operator representing the space-time metric it was then natural to investigate fluctuations of the light-cone. Classically, for each given space-time metric, there is a light-cone in the tangent space of any point in the manifold that is defined by vectors with null norm. In the quantum theory, given a state, we obtained that, for an external observer the norm of a vector can fluctuate between negative, zero or positive values. This is a striking result from the point of view of the canonical approach. It shows that although we started with a split in spatial slices we can in the end recover covariant effects. Another feature is the non-commutativity of two metric operators. This is due to the non-locality of the metric operator in terms of the basic scalar fields and does not imply violation of causality.

Let us turn to the Gowdy model. First we want to point out that our choice of midi-superspace variables was motivated by Einstein-Rosen waves and differs from the literature on cosmological models. Our intention was to obtain the same type of simplifications as in the Eisntein-Rosen case and in fact this is achieved.

We saw that although this model had formally the same Lagrangian as the previous one (with τ replaced by R), because the spatial slices are compact the action is different, i.e., it does not have a boundary term. In the absence of a non-vanishing Hamiltonian we need to deparametrize the theory. Then $\tau(\theta)$ was chosen as the time coordinate. In the Einstein-Rosen case $R(r)$ was chosen as the radial coordinate. This difference has a counterpart in space-time language: In the Gowdy models, due to spatial compactness, the gradient of $\tau(\theta)$ is time-like everywhere on M whereas in the Einstein-Rosen waves $\nabla_a R(r)$ is space-like. As is usual in the deparametrization approach, the momentum canonically conjugate to $\tau(\theta)$ turned out to be the Hamiltonian of the system. In the Einstein-Rosen waves, by contrast, $p_R(r)$ does not play a special role. Another interesting fact is that after solving for the true degrees of freedom $p_\tau(\theta)$ has the same functional form (apart from the obvious changes on the coordinates) in terms of the scalar fields as the previous $\gamma(r)$ and not as $p_R(r)$.

The reduced system was again remarkably simple. The infinite number of true degrees of freedom are represented again by the scalar field. However, there are some fundamental differences from the asymptotically flat case. Firstly, the reduced Hamiltonian is explicitly time-dependent. Secondly, there exists a new global constraint that requires the total angular momentum in θ-direction to vanish. We decided to carry it over as an operator condition on the quantum theory. Next, the space-time now has an initial singularity. Finally, although the decoupling occurs again now the scalar field propagates on a wedge of flat space-time with certain identifications, rather than on a full Minkowski space-time. An analysis of this identified background space-time showed that the spatial slices are tori with time-expanding boundary that start at a singular circle.

Because of the topological differences of the fictitious flat background and the absence of a time-like Killing vector field, the procedure to obtain the quantum representation is different. Now, the representation is directly related to the choice of a complex structure. This approach is more suitable for linear theories and does not rely on the staticity of the background whereas the approach used before (to find a suitable measure) is more convenient if there is a static Killing vector field (but can be extended also to non-linear theories). The complex structure for our model was chosen accordingly to physical requirements. This way we constructed a fiducial Hilbert space. An interesting problem is to work out the corresponding measure in the space of quantum states $L^2(S', d\mu)$.

In contrast to ref. [10] , we obtained one fixed Hilbert space, instead of a one-parameter family of Hilbert spaces. This is a consequence of the fact that our complex structure is time independent. Another implication is that there is no creation of 'particles'. Indeed, one could obtain particle creation even in flat space-time just by a choice of time dependent complex structure. Here we do not have external fields, i.e., we are not quantizing fields in a curved space-time. Therefore, apriori there is no reason to expect creation of particles or even to interpret the quanta of the scalar field as physical particles.

The physical Hilbert space is the subspace of the fiducial state space defined by the kernel of the global constraint. There are no one-particle physical states, except for the zero mode. We saw that because the orbits of the classical constraint vector field are closed, the quantum physical operators can be obtained by simple projection into the physical subspace. We also obtained semi-classical states. The expectation value of the metric operator in such a state corresponds to classical geometry.

The issue of time in this model is technically simpler but conceptually more complex. As in the Einstein-Rosen case the gauge choice is such that the problem decouples and a time parameter arises naturally in the quantum theory. It is the time parameter of the fictitious background. The conceptual problem is that while we had a distinction between gauge and dynamics and a notion of time at spatial infinity before, now the time parameter and the Hamiltonian are artifacts of our deparametrization, i.e., they are not singled out by any physical reason. But, as is well known, this problem is intrinsic to the theory itself and is also present in the classical theory. The key point is that we were able to find a *gauge* which selects a fictitious background and provides a global time parameter for the quantum theory as well.

In this model, as we pointed out, there is an initial singularity in all classical solutions, including the flat fictitious background. Nonetheless we were able to quantize the system consistently. Note, however, that the quantization did not cure this break down of the classical theory. But, symmetry reductions freeze out degrees of freedom, and hence it is not clear that the problem will persist in the full quantum theory.

A gauge choice that has been considered in the literature [20] is to use the matter field as the *clock*. Since we have gravity coupled to a scalar field it seems natural to investigate this gauge. However, we obtained that this choice is valid only in a severely restricted region of the phase-space. (This was also pointed out in [21], where this region is called the 'clock regime'.) If one has to leave out the remaining part of the phase space, it is not clear that the quantum theory in this gauge would be meaningful. We also tried to choose the matter field as clock in the asymptotically flat context. But there the situation is even worse. There is no lapse that is asymptotically one such that the gauge choice is preserved under time evolution.

A supportive result for the non-perturbative program based on Ashtekar's self-dual connections is that the operator corresponding to the trace of the holonomy of a loop around the axis of symmetry is well-defined on the Hilbert space of both models. The holonomy along a loop in the perpendicular direction, due to axi-symmetry, corresponds to a 2-dimensional smearing. Therefore, it is expected to be well-defined. In the Gowdy model, we were also able to compute this holonomy as a simple exponential, and we verified that, in fact, the operator corresponding to its trace is well-defined without need of regularization (other than the usual normal ordering). It is not clear, however, how the first result will change if we adopt a different gauge choice and it may well be that a smearing of the loop would be necessary then. Here we saw that there is no need of such smearing because the metric is a surface integral of the basic fields.

Thus, the two models can be successfully used to probe a number of features of the quantum theory. The key aspect is that they are simple, exactly soluble models. They make potential problems transparent and suggest methods to deal with them.

References

[1] E. Witten, Commun. Math. Phys. **80**, 381 (1981).

[2] A. Ashtekar and M. Varadarajan, Phys. Rev. D **40**, 4944 (1994).

[3] M. Varadarajan, Phys. Rev. D **52**, 2020 (1995).

[4] D. Kramer, H. Stephani, E. Herlt and M. MacCallum, *Exact Solutions of the Einstein Field Equations*, (Cambridge U.P. Cambridge, 1980), p 326.

[5] V. Moncrief, Ann. Phys. **167**, 118 (1986).

[6] A. Einstein and N. Rosen, J. Franklin Inst., **223**, 43 (1937).

[7] K. Kuchař, Phys. Rev. D **4**, 955 (1971).

[8] M. Allen, Class. Quantum Grav. **4**, 149 (1987).

[9] R. H. Gowdy, Phys. Rev. Lett. **27**, 826 (1971); Ann. Phys. **83**, 203 (1974).

[10] B. K. Berger, Ann. Phys. **83**, 458 (1974).

[11] A. Ashtekar and M. Pierri, 'Probing Quantum General Relativity through Exactly Soluble Midi-Superspaces I', paper in preparation.

[12] A. Ashtekar and M. Pierri, 'Probing Quantum General Relativity through Exactly Soluble Midi-Superspaces II', paper in preparation.

[13] B.K. Berger, P.T. Chruściel, V. Moncrief, Ann. Phys. **237**, 322 (1995).

[14] A. Ashtekar, *Lectures on Non-perturbative canonical gravity*, notes prepared in collaboration with R.S. Tate (World Scientific, Singapore, 1991).

[15] J. Glimm and A. Jaffe, *Quantum Physics* (Springer-Verlag, New York, 1987).

[16] J. T. Cannon and A. M. Gaffe, Common. Math. Pays. **17**, 261 (1970).

[17] M. Reed and B. Simon, *Methods of Modern Mathematical Physics*, Vol II, theorem X.3 (Academic Press, New York, 1975).

[18] P. T. Chruściel, Ann. Phys. **202**, 100 (1990).

[19] A. Ashtekar and A. Magnon, Proc. R. Soc. Lond. A **346**, 375 (1975).

[20] L. Smolin: in the Brill Feschrift Proceedings, ed B-l Hu and T. Jacobson (CUP, Cambridge, 1993).

[21] C. Rovelli, 'Outline of a Generally Covariant Quantum Field Theory and a Quantum Theory of Gravity, preprint University of Pittsburgh (1995).

[22] R. Geroch, J. Math. Phys., **12**, 918 (1971).

Appendix A

Symmetry Reductions

In this Appendix we will introduce the reductions of 4-dimensional vacuum Einstein's equations for a space-time $({}^4M, {}^4g_{ab})$ with symmetries. Because of the Killing fields in the 4-d problem it seems reasonable to expect that Einstein's equations can be reduced to a lower dimensional perspective, namely ${}^4M$ 'divided out' by the symmetry group G. We will see that with one Killing field it reduces to a set of equations for a metric on a 3-dimensional manifold coupled to matter fields. Scalar functions defined from the Killing vectors will play the role of the matter fields. Our treatment for this topic will follow ref.[22]. Then, we will assume that the Killing field is hypersurface orthogonal. In this case the 4-dimensional problem reduces to 3-dimensional Einstein's equation coupled to a massless scalar field, that is related to the norm of the Killing field. Finally, we will introduce another Killing vector that commutes with the first one. There will be a remarkable simplification. The problem will decouple, i.e., first we solve for the scalar field propagating in a flat background and then obtain the gravity part.

Reduction of 4-d Vacuum Einstein's Equation with One Killing Field

Let us consider a 4-d space-time $({}^4M, {}^4g_{ab})$ with a space-like Killing vector field denoted by ξ^a. The norm and the twist of ξ^a are given, respectively, by

$$\lambda = \xi^a \xi_a, \tag{A.1}$$

$$\omega_a = \epsilon_{abcd} \xi^b \, {}^4\nabla^c \xi^d, \tag{A.2}$$

where ${}^4\nabla$ is the derivative operator of ${}^4g_{ab}$. As is well known in the absence of matter fields ${}^4g_{ab}$ satisfies vacuum Einstein's equations:

$${}^4\mathcal{R}_{ab} = 0, \tag{A.3}$$

where ${}^4\mathcal{R}_{ab}$ is the Ricci tensor of ${}^4g_{ab}$ and $\mathcal{L}_\xi \, {}^4g_{ab} = 0$. Because of the presence of the Killing field one can define a projection from ${}^4M$ to a 3-d manifold M that corresponds to the manifold of orbits of the Killing vector field. Furthermore there is a one-to-one correspondence between tensors in ${}^4M$ and tensors in M. In particular, there exits an induced metric $\tilde{g}_{ab}$ on M given by:

$$\tilde{g}_{ab} = {}^4g_{ab} - \frac{1}{\lambda} \xi_a \xi_b. \tag{A.4}$$

We denote the derivative operator compatible with $\tilde{g}_{ab}$ by $\tilde{\nabla}$ and the Ricci tensor by $\tilde{\mathcal{R}}_{ab}$.

Then it follows that Eqs.(A.3) for ${}^4g_{ab}$ can be expressed completely in terms of the fields $(\tilde{g}_{ab}, \lambda, \omega_a)$ on M, as

$$\begin{aligned}
\tilde{\mathcal{R}}_{ab} &= \frac{1}{2\lambda^2}(\tilde{\nabla}_a\omega)(\tilde{\nabla}_b\omega) - \tilde{g}_{ab}(\tilde{\nabla}^c\omega)(\tilde{\nabla}_c\omega) + \frac{1}{2\lambda}\tilde{\nabla}_a\tilde{\nabla}_b\lambda - \frac{1}{4\lambda^2}(\tilde{\nabla}_a\lambda)(\tilde{\nabla}_b\lambda) \\
\tilde{\nabla}^2\lambda &= \frac{1}{2\lambda}(\tilde{\nabla}^c\lambda)(\tilde{\nabla}_c\lambda) - \frac{1}{\lambda}(\tilde{\nabla}^c\omega)(\tilde{\nabla}_c\omega) \\
\tilde{\nabla}^2\omega &= \frac{3}{2\lambda}(\tilde{\nabla}^c\lambda)(\tilde{\nabla}_c\omega) \qquad \text{(A.5)}
\end{aligned}$$

where we have used that (A.3) implies that locally $\omega_a = \tilde{\nabla}_a\omega$. Thus, 4-dimensional vacuum general relativity is equivalent to a metric $\tilde{g}_{ab}$ on a 3-d manifold M and two scalar fields λ and ω subject to the set of equations (A.5). The remarkable result is that one can show that this set of equations is equivalent to 3-d gravity coupled to matter field. The basic procedure is to rescale the 3-metric by the norm of the Killing field. Then, by appropriate redefinition of the two scalar fields, the resulting system can be identified with 3-d gravity coupled to a $SO(2,1)$ non-linear sigma model. In summary, in this case not only the reduction is possible but also one recovers a lower dimensional set of Einstein's equations.

Restriction to Hypersurface Orthogonality

Let us assume further that the Killing vector field is hypersurface orthogonal, i.e. $w_a = 0$, then (A.5) reduces to:

$$\tilde{\mathcal{R}}_{ab} = \frac{1}{2\lambda}\tilde{\nabla}_a\tilde{\nabla}_b\lambda - \frac{1}{4\lambda^2}(\tilde{\nabla}_a\lambda)(\tilde{\nabla}_b\lambda) \qquad \text{(A.6)}$$

$$\tilde{\nabla}^2\lambda = \frac{1}{2\lambda}(\tilde{\nabla}^c\lambda)(\tilde{\nabla}_c\lambda) \qquad \text{(A.7)}$$

We will show that this set is equivalent to a scalar field coupled to 3-d gravity. First we make a conformal transformation on the metric $\tilde{g}_{ab}$,

$$g_{ab} = \lambda \tilde{g}_{ab}. \tag{A.8}$$

The Ricci tensor $\mathcal{R}_{ab}$ with respect to g_{ab} can be expressed in terms of $\tilde{\mathcal{R}}_{ab}$ and the conformal factor λ. Then, if we substitute on this expression $\tilde{\mathcal{R}}_{ab}$ as a function of λ via (A.6) and also use (A.7) we obtain the following equation for $\mathcal{R}_{ab}$ in terms of λ and ∇ (the derivative operator compatible with g_{ab}),

$$\mathcal{R}_{ab} = \frac{1}{2\lambda^2} \nabla_a \lambda \nabla_b \lambda. \tag{A.9}$$

If we redefine λ by: $\lambda = e^{\psi}$. The set of eqs. (A.9, A.7) yields,

$$\mathcal{R}_{ab} - \frac{1}{2} \nabla_a \psi \nabla_b \psi = 0 \tag{A.10}$$

$$g^{ab} \nabla_a \nabla_b \psi = 0, \tag{A.11}$$

Recall that 3-d Einstein's equations with matter field can be written as:

$$\mathcal{R}_{ab} = (T_{ab} - T g_{ab}), \tag{A.12}$$

plus the equations satisfied by the matter field. T_{ab} is the stress-energy tensor of the matter field and $T = g^{ab} T_{ab}$ its trace. Therefore one can easily verify that (A.10) is equivalent to 3-dimensional general relativity coupled to a scalar field with stress-energy tensor given by:

$$T_{ab} = \nabla_a \phi \nabla_b \phi - \frac{1}{2} \left(g^{cd} \nabla_c \phi \nabla_d \phi \right) g_{ab}, \tag{A.13}$$

where $\phi = \psi / \sqrt{2}$.

To summarize, we have shown that 4-d Einstein's equations (A.3) in the presence of a space-like hypersurface orthogonal Killing vector field is equivalent to 3-d gravity coupled to a massless scalar field, that is related to the norm of the Killing vector.

Although the 3-dimensional reduction of these systems can bring new insights to the 4-dimensional problem, the set of equations (A.5) or even (A.10, A.11) is still non-linear and therefore complicated to deal with.

Presence of an Additional Commuting Killing Field

Now, we will introduce an additional space-like hypersurface orthogonal Killing vector field σ^a with norm denoted by R. Moreover, we will assume that the two Killing vector fields commute, i.e., $[\xi, \sigma]^a = 0$. Recall that with one Killing field we projected from ${}^4M$ to a 3-d manifold M. Now with the additional Killing field, there is a projection from M to a 2-d manifold ${}^2M$, that corresponds to the manifold of orbits of the two Killing vector fields. Furthermore there is a one-to-one correspondence between tensors in M (or ${}^4M$) and tensors in ${}^2M$. In particular there exits an induced metric h_{ab} on the manifold ${}^2M$ of orbits of the Killing vector fields given by:

$$h_{ab} = g_{ab} - \frac{1}{R^2}\sigma_a\sigma_b, \tag{A.14}$$

Moreover, restricting to 2-manifolds ${}^2M$ with topology R or $S^1 \times R$, h_{ab} is conformally flat, i.e., $h_{ab} = e^{\gamma}\overset{\circ}{h}_{ab}$, where $\overset{\circ}{h}_{ab}$ is a flat metric. Hence, from (A.14) the 3-metric can be written as:

$$g_{ab} = e^{\gamma}\overset{\circ}{h}_{ab} + \frac{1}{R^2}\sigma_a\sigma_b. \tag{A.15}$$

If we rewrite the Klein-Gordon equation for the scalar field given by (A.11) using (A.15), we obtain the following result:

$$g^{ab}\nabla_a\nabla_b\psi \equiv (e^{-\gamma})\overset{\circ}{g}^{ab}\,\overset{\circ}{\nabla}_a\overset{\circ}{\nabla}_b\,\psi = 0, \tag{A.16}$$

where $\overset{\circ}{\nabla}$ is the derivative operator with respect to $\overset{\circ}{g}_{ab}$, and

$$\overset{\circ}{g}_{ab} = \overset{\circ}{h}_{ab} + \frac{1}{R^2}\sigma_a\sigma_b. \tag{A.17}$$

Moreover, one can show that the condition for the system to decouple, or equivalently, the metric $\overset{\circ}{g}_{ab}$ to be flat is that the norm of the Killing vector, i.e. R, has to satisfy the following equations on the 2-manifolds ${}^2M$:

$$\partial_a^2 \ln R + (\partial_a \ln R)^2 = 0 \tag{A.18}$$

$$\partial_a\partial_b \ln R + (\partial_a \ln R)(\partial_b \ln R) = 0 \tag{A.19}$$

where ∂_a is the ordinary derivative operator with respect to the flat metric $\overset{\circ}{h}_{ab}$. This is a remarkable simplification. It tells us that for an appropriate choice of the function R the set of equations (A.10,A.11), or equivalently, (A.10,A.16) decouples, i.e., one first solves (A.16) for a scalar field propagating in a flat background $\overset{\circ}{g}_{ab}$ and then substitute the result in (A.10) to obtain the gravity part.

Recapitulating, we have shown that the 4-d vacuum Einstein's equations with two commuting hypersuface orthogonal space-like Killing fields, can be reduced to a massless scalar field propagating in flat background such that the metric is a derived quantity.

As we see this system is mathematically very simple. Moreover for an appropriate choice of the function R, one can show that the symmetries effectively erase all the non-linearities of general relativity. But, we are still left with an infinity number of degrees of freedom, (one per space-time point). Therefore it is an interesting system to investigate non-trivial issues of quantum gravity.

Vita

Monica Pierri-Galvao attended the Universidade do Rio de Janeiro, Brazil, where she received the B.S. degree, Magna Cum Laude, in Physics in January, 1988. She received the M.S. degree in physics from the Universidade de Sao Paulo, Brazil, in June 1991. Then, in the United States she received her M.S. degree in physics from Syracuse University in May 1993. She enrolled in the Ph.D. program at The Pennsylvania State University in August 1993.

Since 1986 she has been awarded fellowships. From 1986 to 1988 she had fellowship from the Universidade Federal do Rio de Janeiro to perform undergraduate research. From 1988 to 1991, the award was from CNPq, a Brazilian government agency, for research at the graduate level. From 1991 to 1995, CAPES, a Brazilian government agency, provided the financial assistance for research during her first four years of Ph.D.. For the 1995-1996 academic year, she has a graduate and research assistantship from The Pennsylvania State University to make research and teach.

In 1994 The United States Achievement Academy has conferred on her the honor of All-American Scholar.

She has written the following scientific articles: 'BRST Quantization of Relativistic Particles with Extended Supersymmetries', (with V. Rivelles), *Phys. Lett. B 251, 421 (1990)*; 'Comments on BRST Quantization of the Extended Supersymmetric Spinning Particle', (with V. Rivelles), *Phys. Rev. D 43, 2054 (1991)*; 'Gravity and Geometric Phase', (with A. Corichi), *Phys. Rev. D 51, 5870 (1995)*; 'Probing Quantum General

Relativity through Exactly Soluble Midi-Superspaces I', (with A. Ashtekar), in preparation; 'Probing Quantum General Relativity through Exactly Soluble Midi-Superspaces II', (with A. Ashtekar), in preparation.