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A COMBINATORIAL APPROACH TO QUANTUM GAUGE THEORIES AND QUANTUM GRAVITY

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Abstract

A combinatorial approach to treat diffeomorphism invariant gauge theories is developed. Among the theories encompassed by the framework are general relativity, and extensions to Yang-Mills fields (with or without fermions) coupled to gravity. Continuous treatments of field theories generally lead to ultraviolet divergences. These divergences are avoided by discrete approaches; however, often these approaches introduce an extra background dependence which is not compatible with the spirit of manifest diffeomorphism invariance. The combinatorial approach proposed here enjoys the characteristic finiteness of discrete frameworks, while not introducing extra background structure.

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Chapter 1

Introduction

The question what is the purpose of quantum gravity? does not have a clear cut answer. There are many reasons that make a quantum theory of gravity necessary. Even issues not connected a priori to the dynamics of the gravitational field or to the nature of spacetime have been related to quantum gravity. Questions like why the masses of the elementary particles have their particular values, why the cosmological constant is so small, why is that the entropy of the universe grows, or what is the fundamental nature of quantum measurements are not addressed by standard physical theories, but people have suggested that quantum gravity may answer them. Different people have invoked quantum gravity to solve different mysteries and to cure different illnesses of other theories. In order to understand any approach to quantum gravity, one first has to know what are the questions it aims to answer.

The motivations behind this work on quantum gravity are two-fold; they come from the two roles played by classical spacetime. First, it is the arena where the rest of the interactions happen. Second, it is a dynamical entity whose evolution is described by general relativity (GR).

Considering smooth spacetime spacetime as an arena for physics at small scales may be the cause of the ultraviolet problems of quantum field theory (QFT). In the case of renormalizable theories, the low energy regime is not sensitive to the details of the microscopic scales. Hence, a "quantum spacetime" which agrees with the spacetime that we know at macroscopic scales, but which is radically different at the microscale may be compatible with the known physics. In this respect, the situation may be analogous to that in the study of waves. Sound propagation was first studied ignoring the complexities of the molecular structure of the arena (materials) where it takes place. At the Planck scale ($10^{-32} cm$), or smaller scales, gravitational effects cannot be ignored; a framework that describes the dynamics of spacetime and quantum fields living on spacetime must be developed. One of the strongest motivations for quantum gravity is the expectation that an understanding of quantum spacetime can solve the problem of the ultraviolet divergences in QFT. Another peculiarity of spacetime as an arena for physics is the generic occurrence of singularities predicted by GR; classical spacetime is a good arena for physics in most regions, but it has inevitable breakdowns.

From the perspective of spacetime as a dynamical entity, there are several questions leading to quantum gravity. One may ask the following questions: If classical mechanics is generally regarded as an approximation to quantum mechanics, why should spacetime, a dynamical entity, remain classical?, or, why shouldn't general relativity accept a quantum formulation as the rest of the theories describing fundamental interactions? There are other reasons to develop a theory of quantum gravity, but I will consider these as my primary motivations.

These motivations claim for a new framework; it should provide the quantum dynamics of spacetime *together with* a notion of how the rest of the fundamental interactions are placed in the new arena. First of all the new framework must reproduce the known physics. More precisely, in an appropriate classical/macroscopic limit, the dynamics of the effective gravitational field should be governed by GR and the effective dynamics of other fields living in spacetime should be described by QFT in curved spacetimes. According to the motivations the new quantum framework must provide an arena which leads to a theory of quantum fields that is free of ultraviolet divergences.

This thesis is devoted to constructing such a framework; it is based on three of my articles on quantum gravity and quantum gauge theory. I will not attempt to review the status or history of the field (for recent reviews see [1]); I will simply present a particular approach to quantum gravity and gauge theories, show how some of its initial problems were solved and discuss possible solutions for some of its remaining problems.

To develop a new framework that solves our motivational problems and meets our requirements, a strategy will be proposed. In the remainder of this introduction I will outline the strategy and extract the assumptions that come with it. The care in stating the foundations has the aim of letting us distinguish between the derived and the assumed at later stages in the study of the models developed.

The ingredient which holds the strategy together is the ability to treat gravity and all the other fundamental interactions as gauge fields. When GR is written as a theory of connections it shares the phase space with Yang-Mills theory. The only difference is that, apart from the Gauss law, there are other constraints. Using this common language at the hart of the framework will allow us to have a natural incorporation of other interactions in our models for quantum gravity.

It is standard now to study gauge theories numerically by means of lattice regularization. Lattice gauge theory replaces the gauge theory on a smooth spacetime with the gauge theory on a lattice. In this way a field theory is modeled by a system with finitely many degrees of freedom. For a non-infinitesimal lattice spacing, the predictions are finite; there are no divergences in lattice gauge theory. I will use a variation of lattice gauge theory where the lattice is not embedded in a manifold and where the connectivity does not need to be that of a square lattice. I will discretize the phase space of GR as a lattice gauge theory. Then, I will have to implement a discrete analog of the constraints that define the GR phase space. After the discretization, one may attempt to quantize the phase space and the extra constraints according to the usual quantization of constrained Hamiltonian systems with finitely many degrees of freedom.

All the steps can be treated by standard techniques except for finding the discrete analog of the constraints. The problem is that Dirac quantization works only for systems with first-class constraints. The constraints in GR are first-class, but a naive discretization does not preserve this property. Since this point is difficult, a more elaborate strategy is needed [2]. Roughly speaking, the idea consists of using a topological field theory called BF theory as a bridge towards quantization. BF theory is readily quantizable, and it is also defined as a constrained system on the Yang-Mills phase space [3]. General relativity and BF theory share the phase space, and their actions are equal when the B field is constrained to be the exterior product of two tetrad fields. These constraints can be written in terms of the B field and will be called *geometricity conditions* for reasons that will be clear later. When the geometricity conditions are dropped the theory simplifies considerably. The strategy has two steps. One first ignores the geometricity conditions and finds a discrete version of the BF theory. After the first step is carried over, one finds a discretization of the geometricity conditions and looks for the special symmetries of the discrete BF theory that preserve the geometricity conditions. Then one declares these special symmetries the *physical symmetries* of the discrete model for GR. One can follow this strategy at the classical level or after quantization. At least in the case of Euclidean gravity, where the gauge group is compact, quantization (of the phase space, the constraint $F = 0$ [that defines BF theory] and the geometricity conditions) is straight forward. Finding the group of physical symmetries is the key problem. In chapter 2, I follow the strategy in the classical case and study the proposal for restricted initial data; that is, I study initial data which lies in a neighborhood of the surface defined by $F = 0$. Since a model with finitely many degrees of freedom is used to describe a system with local degrees of freedom, only partial success can be expected. Developing a model able to cast all the degrees of freedom of a field theory will be left for later stages of this work (see chapter 4).

From the strategy that I just described we can proceed to extract its fundamental assumptions. Some of these assumptions are either explicitly stated or evident, but some of the more important assumptions embedded in the strategy may not be easily recognized. In my view, there are six fundamental assumptions; each of these assumptions is discussed in a separate paragraph below. Knowledge of the fundamental assumptions will be essential in a later analysis of the main results and basic difficulties of the models developed.

The most important implicit assumption concerns **diffeomorphism invariance**. I mentioned the role of spacetime as a dynamical entity and its role in providing the arena where the rest of the interactions occur. When a theory that includes the arena itself is formulated in the continuum one obtains a diffeomorphism invariant theory. Such theories –vacuum GR or GR coupled to other gauge theories– come from the idealization of a universe where only the interactions explained by them are present. As such, they are simplified models for a unified theory. At first glance, the lattice formulation introduces an extra background dependence which seems to contradict this important consideration. In chapter 4 I will show how one can discard this extra background structure, but first I will give a preliminary treatment in which an undesirable background structure is present.

Throughout the strategy I have assumed that the theory of quantum spacetime follows the framework of a **usual quantum theory**. More precisely, the strategy uses Dirac quantization to derive a constrained quantum system from a classical Hamiltonian constrained system. In particular, this assumes that quantum gravity is a generally covariant quantum system that can be cast as an ordinary quantum constrained system.

The previous assumption reminds us of an even stronger assumption: by using a quantization method I am assuming that **GR is a fundamental theory**; moreover, I am assuming that the aspects of GR that I captured in my discretization are somehow fundamental. Here we may be committing one of the following two mistakes: Firstly, the discretization may not capture the true fundamental features of GR. Secondly, there may not be fundamental degrees of freedom to capture in GR; i.e. GR, like the theory of sound propagation, may be an effective theory.

Inspired by the ultraviolet problem, I am assuming explicitly that a **fundamentally discrete** system gives rise to **spacetime**. The approach does not rest on a background manifold structure; furthermore, it provides an arena where quantum gauge fields can be easily incorporated. Usually discrete models are considered approximations to a fundamental smooth model; in these cases, the gauge symmetries have to be implemented only in the limit that defines the fundamental, smooth model. Because our discrete models are considered to be fundamental, the constraints generating the gauge symmetries must be first-class. The framework developed in chapter 2 proposes a class of models, one for every lattice discretization of space. A continuous limit that preserves most of the characteristics of the discrete system will be employed at a later stage (see chapters 3 and 4). An alternative is resigning to have a single model and selecting the consequences that are independent of the lattice as the only predictions of the framework. The smooth spacetime described by GR is not fundamental; it is only recovered in a macroscopic/semiclassical limit. For a sufficiently fine lattice a macroscopic/semiclassical behavior of the discrete system should recover the results of GR and quantum field theory in curved spacetimes up to a certain degree of accuracy.

The construction of the quantum models uses lattice gauge theory of compact gauge groups; with such a starting point it should not be a surprise to find **kinematical flux quantization**. If flux quantization remains after the continuum limit and after the constraints are considered, it will be a remarkable result. Flux quantization, in the purely gravitational case, leads to the quantization of areas and volumes. This kinematical result of loop quantum gravity [7] is a direct consequence of using a compact gauge group and considering the Wilson loops (the typical operators of lattice gauge theory) as the basis of the quantization. From the lattice point of view, the nontrivial result is that after the continuum limit is taken, and after dealing with diffeomorphism invariance, the character of quantum geometry is preserved [7] (see chapter 4).

Another important assumption is that **space and time are fundamentally different**. The discrete framework induces an asymmetry by discretizing space and not spacetime. This asymmetry is intrinsic to the Hamiltonian framework and, after quantization, its consequences are similar for the continuum and the discrete cases. In the canonical framework, classical spacetime is reconstructed from any orbit generated by the constraints. Each point in the orbit gives a metric and a connection to a topological three-manifold. Quantum mechanically the reconstruction of spacetime is not guaranteed outside of the limit in which the notion of orbits is recovered; that is, spacetime in canonical quantum gravity is considered as fundamental as orbits are in particle mechanics. In particular, the causal structure is supposed to be a macroscopic/semiclassical effect; horizons are not perfect walls impenetrable to information, and trapped surfaces may not be completely trapped.

In our models for quantum gravity, spacetime is assumed to be a macroscopic/semiclassical concept, and the issue of whether singularities occur, and how generic they are, is a dynamical question. Being more aware of the characteristics of quantum spacetime that are assumed rather than derived we may start developing our models and search for consequences of our assumptions.

This thesis is composed of two kinds of chapters: technical chapters and discussion chapters. In the discussion chapters I set aside all technical details to focus on essential features; there, I relate the main results and struggles to the assumptions behind the models proposed in the technical chapters. With this knowledge we will be able to determine which characteristics of our theory for quantum gravity would persist if we modify one or two of the assumptions. For instance, a covariant variation of the strategy described above, and developed throughout this thesis, is now being pursued [8, 9]. There, one attempts to drop the assumption that makes space and time fundamentally different while keeping all the other assumptions. This strategy produces a new class of theories generically known as spin foam models. In the final chapter I discuss these models and translate some of the analysis presented in the previous chapters to the framework of the spin foam models.

In chapter 2 I present the class of models whose strategy and assumptions were discussed in this introduction. Chapter 3 relates the successes and difficulties encountered in chapter 2 with the basic assumptions and uses this knowledge to modify some of the assumptions. These observations lead to the work presented in chapter 4, where the continuum limit of the lattice framework is implemented. The extra background structure introduced by the triangulation is removed; a simple combinatorial framework emerges –the continuum limit defines the piecewise linear category of loop quantization. A comparison with previous formulations of loop quantization motivates the work presented in chapter 5. There I present a refined treatment of diffeomorphism invariance in loop quantization; following this treatment, loop quantization acquires a combinatorial character. In the final chapter again the relation between successes, difficulties, and basic assumptions is used to motivate some new proposals.

Chapter 2

From a topological model to lattice gravity

This chapter is based on an article titled "Topological Lattice Gravity Using Self-Dual Variables" published in *Classical and Quantum Gravity* **13**: 2617-2634,1996.

This chapter presents a classical connection-dynamic model closely related to $3+1$ Regge calculus [10]. From this classical toy model one can get hints to solve some problems that may occur in future lattice regularization of the constraints of quantum general relativity. The model describes flat space-times as the evolution of a three-dimensional simplicial lattice. It is based on a $SO(3,1)$ lattice gauge theory where every cell has four neighbors. In addition, the variables of the theory are required to satisfy some "geometricity conditions". Once the geometricity conditions are fulfilled, the variables of the lattice gauge theory specify the geometry of a three-dimensional piecewise linear space that generates space-time as it evolves. In terms of self-dual variables the model for Regge calculus acquires an Ashtekar-like description. After changing to self-dual variables the geometricity conditions take the form of the reality conditions of Ashtekar's GR. Spatial and time-like translations that preserve the geometricity conditions are generated by constraints; the local part of these symmetry generators resembles the constraints of general relativity written in terms of self-dual variables. The phase space variables of the model and the geometricity conditions of the lattice are closely related to the ones given by Immirzi in [11]; see also [12].

The origin of this model is a $2+1$ lattice theory formulated by Waelbroeck [13], but a closer relative is the extension of the $2+1$ theory: a lattice $B \wedge F$ theory in $3+1$ dimensions (Waelbroeck and Zapata [2]). In the lattice $B \wedge F$ (LBF) case the geometricity conditions fixed a "geometrical gauge", a precise statement of what the geometricity conditions mean in the LBF case is stated in section 2.3. Once in the geometrical gauge the $B \wedge F$ symmetry generators take the form of the 4d-translation generators of the vertices of the lattice; in the geometrical gauge, one can recognize that the lattice $B \wedge F$ theory has no degrees of freedom associated with the lattice vertices or cells, and that the theory describes flat spacetimes.

In this chapter I introduce two results that help in the construction of a bridge between lattice $B \wedge F$ (LBF) and lattice gravity. First, self-dual variables in the model allow the geometricity conditions, that in LBF were ordinary second-class constraints, to be treated as reality conditions. Secondly, the model's symmetry group is effectively smaller than that of LBF and the symmetry generators do not restrict the lattice to be flat making viable an extension of the model to a theory with local degrees of freedom. Heuristically, one could expect that these degrees of freedom are precisely the gravitational degrees of freedom because they come from imposing a constraint that changes the $B \wedge F$ action for the Hilbert-Palatini action. A vector and a scalar constraint (per lattice cell) replace the four dimensional covariant generators of vertex translations of

LBF. Remarkably, the local part of these new constraints has the form of the Ashtekar's constraints which, as explained in sec. 2.3, implies that the continuum limit of the model is Ashtekar's formulation of GR. An important aspect of the work presented in this chapter is the notation. The mentioned relation between geometricity and reality conditions and the indication that the local part of the constraints are Ashtekar-like, are highly clarified after the introduction of an "affine" notation for the lattice [14]. The affine notation indicates "space directions" in a manner natural for the discreteness of the lattice, while resembling the notation used in the continuum. The "affine" notation simplifies the difficult task of translating physically meaningful expressions from the continuum to the lattice [15], thus, providing a useful tool for $3+1$ Regge calculus.

Once the relation between the model and canonical continuum gravity has been realized, it is natural to wonder about the quantization strategy. One would like to adapt the original version of Ashtekar's quantization program [16] to lattice gravity. To this end one would select the physical Hilbert space following Dirac's prescription and then fix the inner product to make former real quantities Hermitian operators. However, in a study on the quantization of Regge calculus [17] Immirzi pointed out that the plan of choosing the inner product according to the quantum reality conditions fails when following the conventions derived from canonical gravity. The plan's failure is nothing but another manifestation of the parallelism between the particular lattice approach to quantum gravity followed in this chapter and the approach of Ashtekar and collaborators for continuum gravity. In the context of continuum gravity Thiemann [18] introduced a generalized Wick transform to implement the quantum reality conditions. Some implications of importing Thiemann's strategy to lattice gravity are discussed in the concluding section.

The organization of the chapter is the following. Section 2.1 describes the framework of the lattice theory. It presents the affine notation, a review of the lattice $B\wedge F$ theory, and introduces self-dual variables for the lattice. Section 2.2 contains a derivation of the geometricity conditions and its expression as reality conditions. In section 2.3, the constraints of the model are introduced, as well as the induced symmetries and their algebra. The possibility of extending this model to space-times with curvature is thoroughly discussed in the concluding section.

2.1 Framework

2.1.1 Affine Notation

A convenient tool for translating expressions from the continuum to a lattice framework is the affine notation [14]. In a n-dimensional simplicial lattice Σ, the analog of its tangent bundle is a collection of n-dimensional vector spaces (one attached to every cell of the lattice). In each of these vector spaces, the $n+1$ intrinsically defined bivectors related to the boundary faces of the simplex and the natural volume element select $n+1$ one-forms. These intrinsically defined one-forms $(\omega^j)_a$ ($j=1,...,n+1$ label the one-forms, and a is an abstract Minkowski index) can play the role of an affine basis. For any one-form σ_a

$$\sigma_a = (\omega^j)_a \sigma_j \tag{2.1}$$

where

$$\sum_j (\omega^j)_a = 0 \quad , \quad \sum_j \sigma_j = 0 \quad , \tag{2.2}$$

the first condition holds because the lattice is formed by closed cells and the second condition guarantees the uniqueness of the affine components σ_j. One can construct a dual basis of vectors $(e_j)^a$ at each cell from the condition

$$(\omega^j)_a (e_k)^a = \hat{\delta}^j{}_k \tag{2.3}$$

where

$$\hat{\delta}^j{}_k = \delta^j{}_k - \hat{n}^j \hat{n}_k \quad , \quad \hat{n}^j = \hat{n}_j = \frac{1}{\sqrt{n+1}} \tag{2.4}$$

in particular, for a three dimensional space

$$\hat{n}^j = \hat{n}_j = \frac{1}{2} \quad , \quad \hat{\delta}^j{}_j = \frac{3}{4} \quad , \quad \hat{\delta}^j{}_{k \neq j} = -\frac{1}{4}. \tag{2.5}$$

One can see that the projector $\hat{\delta}$ satisfies $\hat{\delta}^j{}_k \hat{\delta}^k{}_l = \hat{\delta}^j{}_l =$ and $\hat{\delta}^j{}_j = n$. A basis for tensors of higher order can be constructed directly from these two. If one labels the lattice cells by Greek letters $\alpha, \beta, \ldots$ in such a way that cell α has as neighbors $\beta, \gamma, \ldots$, then the directions of the affine basis of the vector space of cell α can be labeled by its neighbors $j = \beta, \gamma, \ldots$.

To any p-form in a manifold, with components $\sigma(x)_{j_1, \ldots, j_p}$, one assigns a lattice counterpart $\sigma(\alpha)_{j_1, \ldots, j_p}$ (where (α) plays the role of the base point (x)). The lattice p-form can be regarded as a p-cochain, that is, a linear function that assigns real numbers to the p-chains of the lattice (e.g. a 2-cochain assigns numbers to the faces of the lattice). I should emphasize that $\sigma(\alpha)_{j=\beta} \equiv \sigma(\alpha)_\beta$ should be regarded as the lattice counterpart of e.g. $\sigma(x)_{j=1} \equiv \sigma(x)_1$; therefore, the Einstein summation convention should not be applied for Greek indices appearing in the right. Using the affine notation, one can easily find the discrete counterpart of the $B \wedge F$ action.

2.1.2 Lattice $\mathbf{B \wedge F}$ Theory

This subsection mirrors part of [2]; however, it is reviewed here using the affine notation for the convenience of the reader (the notation and conventions of this paper follow Peldán in [19], where he gives a formulation of continuum GR using $SO(3,1)$ as internal group).

The $B \wedge F$ theory, as Horowitz formulated it [3], starts from a modified Palatini action

$$S_{B\wedge F} = \int_{\mathcal{M}} B_A \wedge F^A \tag{2.6}$$

where the internal group is $SO(3,1)$ and the indices A can be written as $A = [ab]$, $a, b = 0, 1, 2, 3$. In this modified Palatini action the constraint on the $so(3,1)$ valued two-form

$$B_A := B_{[ab]} = \varepsilon_{abcd}\, e^c \wedge e^d \tag{2.7}$$

has been dropped, making gravity and $B \wedge F$ theory different theories.

A space-like discrete version of $B \wedge F$ theory can be formulated from the discrete counterpart of the $3+1$ split of the $B \wedge F$ action [3]:

$$S[B,A]_{B\wedge F} = 3\int dx^0 \int_{\Sigma} (\dot{A}^A_{[i} B_{jk]A} - F^A_{[ij} B_{k]0A} + A^A_0 D_{[i} B_{jk]A}) dx^i dx^j dx^k \tag{2.8}$$

The lattice counterpart of the last expression should be considered as the starting point of this lattice formalism [2]

$$S[E,M] = 3\int dx^0 \sum_{\alpha} (4\mathbf{E}(\alpha)^j \cdot \dot{\mathbf{A}}(\alpha)_j - \mathbf{P}(\alpha)_{jk} \cdot \mathbf{E}(\alpha)^{jk} + \mathbf{A}(\alpha)_0 \cdot \mathbf{J}(\alpha)) \tag{2.9}$$

where $E(\alpha)^j_A := \frac{1}{8} B(\alpha)_{klA} \hat{\varepsilon}^{jkl}$, $\hat{\varepsilon}^{jkl} := \varepsilon^{jklm} \hat{n}_m$, and the action is a functional of the variables $E(\alpha)^\beta_A$, $M(\alpha)_{\beta A}^{\ \ B}$ attached to every face (α, β) of the lattice. The curvature form was replaced by $P(\alpha)^A_{jk} = F(\alpha)^A_{jk} + O(F^3)$, where $P(\alpha)^A_{j=\beta k=\gamma} := \frac{1}{4} f^{AB}_{\ \ C} W(\alpha)_{\beta\gamma B}^{\ \ \ C}$, and $W(\alpha)_{\beta\gamma B}^{\ \ \ C} := (M(\alpha)_\beta M(\beta)_\mu \ldots M(\gamma)_\alpha)_B^{\ C}$ is the holonomy around the lattice link of cell α where faces β and γ intersect. The three-form of the Gauss law term in the continuum action is replaced by the integral of $\mathbf{B}$ over the boundary of a lattice cell $\mathbf{J}(\alpha)$. In the $3+1$ split $A(\alpha)^A_0$, and $E(\alpha)^{jk}_A := B(\alpha)_{l0A} \hat{\varepsilon}^{jkl}$ are Lagrange multipliers.

In a discrete scenario, the role of a connection is better played by matrices that define parallel transport along non-infinitesimal paths. Thus, the connection $A(\alpha)^C_j$ that appears in the Lagrangian is regarded as a secondary quantity defined in terms of the matrix that parallel transports to the reference frame at cell α, from its neighbor in direction (j), by

$$\exp(A(\alpha)^C_j f_{CA}^{\ \ B}) := M(\alpha)_{jA}^{\ \ B} \tag{2.10}$$

In the adjoint representation the structure constants of $so(3,1)$ and the generators of the group are related by

$$(T_A)_B^{\ C} = f_{AB}^{\ \ C} = f_{[ab][cd]}^{\ \ \ \ [ef]} = -\delta_{[ab]}^{\ \ r}{}_s \delta_{[cd]}^{\ \ s}{}_t \delta^{[ef]t}{}_r \tag{2.11}$$

and the Lie algebra indices are raised and lowered with the Cartan metric $g_{AB} = -\frac{1}{4} f_{AC}^{\ \ D} f_{BD}^{\ \ C}$.

To write the Lagrangian explicitly in terms of the parallel transport matrices, one manipulates formally the kinetic term[1] [13]

$$
\begin{aligned}
\mathbf{fA} &= \ln \mathbf{M} && (2.12)\\
\dot{\mathbf{A}} &= \frac{1}{4}\mathbf{fM}^{-1}\dot{\mathbf{M}} && (2.13)\\
L &= \sum_{\alpha}(4\mathbf{E}(\alpha)^{j} \cdot \dot{\mathbf{A}}(\alpha)_{j} - \mathbf{P}(\alpha)_{jk} \cdot \mathbf{E}(\alpha)^{jk} + \mathbf{A}(\alpha)_{0} \cdot \mathbf{J}(\alpha)) && (2.14)\\
&= \sum_{\alpha\beta} E(\alpha)^{\beta}_{A} f_{B}^{\ CA} M(\beta)_{\alpha D}^{\ \ B} \dot{M}(\alpha)_{\beta C}^{\ \ D}\\
&+ \sum_{\alpha}(-\mathbf{P}(\alpha)_{jk} \cdot \mathbf{E}(\alpha)^{jk} + \mathbf{A}(\alpha)_{0} \cdot \mathbf{J}(\alpha)) && (2.15)
\end{aligned}
$$

where in the $(\alpha\beta)$ sum there is a term for (α, β) and a term for (β, α) if cells α , β share a face. Notice that for this Lagrangian (2.15) the variables $\mathbf{E}(\alpha)^{\beta}, \mathbf{E}(\beta)^{\alpha}, \mathbf{M}(\alpha)_{\beta}$, and $\mathbf{M}(\beta)_{\alpha}$ are all independent. To relate these variables to a lattice, one has to impose the relations

$$
\begin{aligned}
E(\alpha)^{\beta}_{A} &= -M(\alpha)_{\beta A}^{\ \ B} E(\beta)^{\alpha}_{B} && (2.16)\\
M(\alpha)_{\beta A}^{\ \ C} M(\beta)_{\alpha C}^{\ \ B} &= \delta_{A}^{\ B} && (2.17)\\
M(\alpha)_{\beta A}^{\ \ C} M(\alpha)_{\beta\ C}^{\ B} &= \delta_{A}^{\ B} && (2.18)
\end{aligned}
$$

which form a second-class set with the momentum constraints coming from the action. Through the Dirac procedure, one gets [13] the result first derived in the context of lattice gravity by Renteln and Smolin [20]

$$
\begin{aligned}
\{E(\alpha)^{\beta}_{A}, E(\alpha)^{\beta}_{B}\} &= f_{AB}^{\ \ D} E(\alpha)^{\beta}_{D} && (2.19)\\
\{E(\alpha)^{\beta}_{A}, M(\alpha)_{\beta B}^{\ \ C}\} &= f_{AB}^{\ \ D} M(\alpha)_{\beta D}^{\ \ C} && (2.20)\\
\{E(\alpha)^{\beta}_{A}, M(\beta)_{\alpha B}^{\ \ C}\} &= f_{A\ D}^{\ C} M(\beta)_{\alpha B}^{\ \ D} \quad . && (2.21)
\end{aligned}
$$

Now $\mathbf{M}(\alpha)_{j}$ can be considered a parallel transport matrix and $\mathbf{E}(\alpha)^{j=\beta}$ a variable related to the boundary between cells α, β, because the relations (2.16)-(2.18) are identities for the Poisson brackets (2.19)-(2.21).

[1] Equation (2.13) is strictly correct only for an Abelian group, since it neglects the ordering ambiguity of the two matrices. However, there are only two ways to write the kinetic term (2.15) considering that indices can be contracted only if they live in the same frame. One shows the equivalence between the other possibility and (2.15) integrating by parts.

From the Lagrangian (2.8), one also obtains the Gauss law and the flatness constraints

$$J(\alpha)_A = \sum_j E(\alpha)^j_A \approx 0 \tag{2.22}$$

$$P(\alpha)^A_{jk} = \frac{1}{4} f^{AB}_{\ \ C} W(\alpha)_{jkB}^{\ \ \ C} = \frac{1}{4} f^{AB}_{\ \ C} (\mathbf{M}(\alpha)_j \mathbf{M}(\beta)_l \ldots \mathbf{M}(n)_i)_B^{\ C} \approx 0 \tag{2.23}$$

If the geometricity conditions presented in the next section are satisfied, the previous conditions can be interpreted as the requirements that the lattice cells close and that the parallel transport around a lattice link is the identity map. The Gauss law constraint generates gauge transformations

$$\{E(\alpha)^j_A, J(\alpha)_B\} = f_{AB}^{\ \ D} E(\alpha)^j_D \tag{2.24}$$

$$\{M(\alpha)_{jA}^{\ \ C}, J(\alpha)_B\} = f_{AB}^{\ \ D} M(\alpha)_{jD}^{\ \ C} \tag{2.25}$$

and the flatness constraint generates "translations" of $\mathbf{E}$

$$\{E(\alpha)^j_A, P(\alpha)^B_{jk}\} = \delta_A^{\ B} + O(P) \quad . \tag{2.26}$$

where in $O(P)$ I group a collection of terms of first and higher order in the curvature. Along the paper I am going to keep track of terms that vanish in this model where the lattice is flat, in order to be able to discuss the issue of extending the model to a theory for general lattices.

A remarkable feature of the Poisson algebra (2.19)-(2.21), and hence of the constraints (2.22), (2.23), is that under a decomposition of the variables into their self-dual and antiself-dual parts, the whole theory splits into two identical parts related by complex conjugation.

2.1.3 Self-Dual Variables

In $SO(3,1)$ apart from the Cartan metric g_{AB}, there is another invariant symmetric bilinear form $g^*_{AB} = g^*_{[ab][cd]} := \varepsilon_{[ab][cd]}$. Its invariance follows directly from the invariance of the four-volume element under Lorentz transformations. This metric is used to define duality in the Lie algebra

$$V^*_A = g^{*B}_A V_B \quad , \quad g^{*B}_A := g^*_{AC} g^{CB} \quad . \tag{2.27}$$

The Lorentzian signature of the Cartan metric implies $g^{*B}_A g^{*C}_B = -\delta_A^{\ C}$. Therefore, to split them *real* Lorentz Lie algebra into its self-dual $^{(+)}$ and anti-self-dual $^{(-)}$ components, the projectors[2] involve complex numbers.

$$V_A^{(\pm)} := \delta_A^{(\pm)B} V_B \;=\; \frac{1}{2}(\delta_A^B \mp i g^{*B}_A) V_B \tag{2.28}$$

$$g^{*B}_A V_B^{(\pm)} \;=\; \pm i V_A^{(\pm)} \quad . \tag{2.29}$$

The images of the self-dual and antiself-dual projectors are complementary orthogonal subspaces of $so(3,1;C)$. Also, the following formulas containing the structure constants hold

$$\delta_A^{(\pm)B} f_{BCD} = \delta_A^{(\pm)B}\delta_C^{(\pm)E} f_{BED} \;=\; \delta_A^{(\pm)B}\delta_C^{(\pm)E}\delta_D^{(\pm)F} f_{BEF} := f^{(\pm)}_{ACD} \tag{2.30}$$

$$\delta_A^{(+)B}\delta_C^{(-)E} f_{BED} \;=\; 0 \tag{2.31}$$

$$f^{(+)}_{ABC} + f^{(-)}_{ABC} \;=\; f_{ABC} \quad . \tag{2.32}$$

In terms of self and antiself-dual variables

$$E(\alpha)^{(\pm)j}_A \;:=\; \delta_A^{(\pm)C} E(\alpha)^j_C \tag{2.33}$$

$$M(\alpha)^{(\pm)B}_{jA} \;:=\; \delta_A^{(+)C}\delta_D^{(\pm)B} M(\alpha)_{jC}^{\ \ D} = \delta_A^{(\pm)C} M(\alpha)^{(\pm)B}_{jC} \quad , \tag{2.34}$$

the Poisson algebra is

$$\{E(\alpha)^{(\pm)\beta}_A, E(\alpha)^{(\pm)\beta}_B\} \;=\; f_{AB}^{\ \ D} E(\alpha)^{(\pm)\beta}_D = \pm f^{(\pm)\ D}_{AB} E(\alpha)^{(\pm)\beta}_D \tag{2.35}$$

$$\{E(\alpha)^{(\pm)\beta}_A, M(\alpha)^{(\pm)C}_{\beta\ B}\} \;=\; f_{AB}^{\ \ D} M(\alpha)^{(\pm)C}_{\beta\ D} = \pm f^{(\pm)\ D}_{AB} M(\alpha)^{(\pm)C}_{\beta\ D} \tag{2.36}$$

$$\{E(\alpha)^{(\pm)\beta}_A, M(\beta)^{(\pm)C}_{\alpha\ B}\} \;=\; f_{A\ D}^{\ C} M(\beta)^{(\pm)D}_{\alpha\ B} = \pm f^{(\pm)C}_{A\ \ D} M(\beta)^{(\pm)D}_{\alpha\ B} \tag{2.37}$$

The Poisson brackets between self-dual and antiself-dual variables always vanish. Since the structure constants $f^{(+)C}_{AB}$ and $f^{(-)C}_{AB}$ are totally antisymmetric three tensors in three-dimensional (complex) spaces, they are proportional to the intrinsic volume element. In the basis suggested by the reality conditions of next section the proportionality constant for the self-dual part is $i\sqrt{2}$ and for the antiself-dual is $-i\sqrt{2}$. That is, the

[2] These are projectors of the complexified Lie algebra. Here, one first includes the real Lie algebra into the complex Lie algebra and then split it into its self and antiself-dual parts. The fact that the images of $\delta^{(\pm)}$ lie out side of the image of the real Lie algebra does not prevent the "split"; the only objection could be to call $\delta^{(\pm)}$ projectors.

Lie algebra $so(3,1)$ "splits" into two copies of the Lie algebra $so(3;C)$. Each of these $so(3;C)$ algebras contains all the information of $so(3,1)$

$$V_A = V_A^{(+)} + V_A^{(-)} = V_A^{(+)} + c.c. \quad . \tag{2.38}$$

An immediate but important consequence is that the symmetry generators also split, yielding two parallel theories.

It would have been possible to start with self-dual variables in the action; however, I decided against it in order to preserve the direct geometric interpretation of the variables. On the other hand, using self-dual variables one learns that the geometricity conditions are the lattice counterpart of the reality conditions of Ashtekar's GR.

2.2 Geometricity-Reality Conditions

The motivation for demanding geometricity conditions on the variables is to guarantee the existence of a one-to-one mapping between the space of simplicial lattices and the space of variables $\mathbf{E}$, $\mathbf{M}$ satisfying the geometricity conditions. Simultaneously, one gets a selection rule for the symmetry generators, ruling out the symmetries that do not preserve the geometricity conditions.

A set of variables $\mathbf{E}(\alpha)^j$ related to a face of a lattice of simplices is of the form

$$\begin{aligned} E(\alpha)^j_A = E(\alpha)^j_{[ab]} &= \frac{1}{2}\varepsilon_{abcd}\, l(\alpha,j_1)^c l(\alpha,j_2)^d \\ &= \frac{1}{2}\varepsilon_{abcd}\, l(\alpha,j_2)^c l(\alpha,j_3)^d = \frac{1}{2}\varepsilon_{abcd}\, l(\alpha,j_3)^c l(\alpha,j_1)^d \end{aligned} \tag{2.39}$$

where the *space-like* Minkowski vectors $\mathbf{l}(\alpha,j)$ are associated with the links of the face that is the frontier between the cell α and its neighbor in direction (j). Clearly, the link vectors satisfy the condition $\mathbf{l}(\alpha,j_1) + \mathbf{l}(\alpha,j_2) + \mathbf{l}(\alpha,j_3) = 0$. Since every link of a tetrahedron is shared by two of its faces, a relation of the form $\mathbf{l}(\alpha,j_2) = -\mathbf{l}(\alpha,k_1)$ holds for each link too.

The geometricity conditions (2.39) are equivalent to the lattice analog of the condition $B = e\wedge e$ that distinguishes gravity from $B\wedge F$ theory. In this sense, discarding the symmetries that do not preserve the geometricity conditions bring us an step closer to gravity. To avoid confusion between the B of $B \wedge F$ theory and the magnetic field of the curvature in the lattice in the lattice I will write $b = e \wedge e$, more precisely,

$$\begin{aligned} E(\alpha)^j_{[ab]} &\approx \frac{1}{8}\hat{\varepsilon}^{jkl} b(\alpha)_{kl[ab]} = \frac{1}{8}\hat{\varepsilon}^{jkl}\varepsilon_{abcd} e(\alpha)^c_k e(\alpha)^d_l \\ &= \frac{1}{32}\hat{\varepsilon}^{jkl}\varepsilon_{ab}{}^{cd}\hat{\varepsilon}_{kmn} l(\alpha)^{mn}_c \hat{\varepsilon}_{lpq} l(\alpha)^{pq}_d \end{aligned} \tag{2.40}$$

where $l(\alpha)^{jk\,c} = \hat{\varepsilon}^{jkl} e(\alpha)^c_l$. The weak equivalence sign indicates that I have used the constraint $J(\alpha)_A = \sum_j E(\alpha)^j_A \approx 0$. I decided to write the "affine triads" $e(\alpha)^a_j$ that appear just as an intermediate step between E's and l's to make contact with other works, and because these affine triads are the ones that indicate directions naturally in the lattice, and are going to be very helpful to write the constraints.

All the geometricity requirements (2.39) for cell (i) can be written purely in terms of the variables $\mathbf{E}(\alpha)^j$

$$q(\alpha)^{*jk} := g^{*AB} E(\alpha)^j_A E(\alpha)^k_B = 0 \tag{2.41}$$

or in terms of self-dual variables

$$i(E(\alpha)^{(+)jA} E(\alpha)^{(+)k}_A - c.c.) = -2\mathrm{Im}(E(\alpha)^{(+)jA} E(\alpha)^{(+)k}_A) = 0 \quad . \tag{2.42}$$

This first set of conditions guarantees the geometricity of each separate cell: for $j = k = \beta$ it requires that $\mathbf{E}(\alpha)^j$ represent the dual of the area bivector of the face (α, β) between cells α and β. In addition, for $j \neq k$, the condition is satisfied if the faces (α, j) and (α, k) of cell α intersect. Once conditions(2.42) are satisfied, the variables $\mathbf{E}(\alpha)^j$ characterize a tetrahedron that is contained in a *space-like* three-dimensional subspace of Minkowski space-time if

$$g^{AB} E(\alpha)^j_A E(\alpha)^j_B = 2\mathrm{Re}(g^{AB} E(\alpha)^{(+)j}_A E(\alpha)^{(+)j}_B) < 0 \quad . \tag{2.43}$$

The similarity between (2.42) and the condition which requires the spatial metric of Ashtekar's formulation of gravity to be real is remarkable considering that the geometricity conditions (2.41) were first proposed [2] in a context not related to Ashtekar's formulation of general relativity. Furthermore, inequality (2.43) has a continuum analog that demands the metric to be Lorentzian.

One also wants a covariant description in which parallel transport between neighboring faces is described by Lorentz matrices $M(\alpha)_{j\ a}^{\ b}$ [3]. The variables $\mathbf{l}$ and $\mathbf{M}$ are called the geometrical variables. After enforcing the first set of geometricity conditions (2.42), the geometrical variables are completely determined by the fundamental variables $\mathbf{E}, \mathbf{M}$. The complete set of geometricity conditions for the fundamental variables must imply that the geometrical variables of cell α and its neighbor β satisfy the compatibility conditions

$$\mathbf{l}(\alpha)^{\beta\gamma} = -\mathbf{M}(\alpha)_\beta \mathbf{l}(\beta)^{\alpha\phi} \tag{2.44}$$

where the faces defining $\mathbf{l}(\alpha)^{\beta\gamma}$ are $\mathbf{E}(\alpha)^\beta$, $\mathbf{E}(\alpha)^\gamma$, and the ones defining $\mathbf{l}(\beta)^{\alpha\phi}$ are $\mathbf{E}(\beta)^\alpha$, $\mathbf{E}(\beta)^\phi$.

Some of these requirements are contained in identity (2.16)

[3] I will use the same notation $M(\alpha)_{j\ a}^{\ b} = \exp(A(\alpha)^{cb}_j \eta_{ca})$ for these matrices, which act on Minkowski vectors, as for the previously defined $M(\alpha)_{jA}^{\ \ B} = \exp(A(\alpha)^C_j f_{CA}^{\ \ B})$ in the bivector representation. Both matrices are different representations of the same $SO(3,1)$ element.

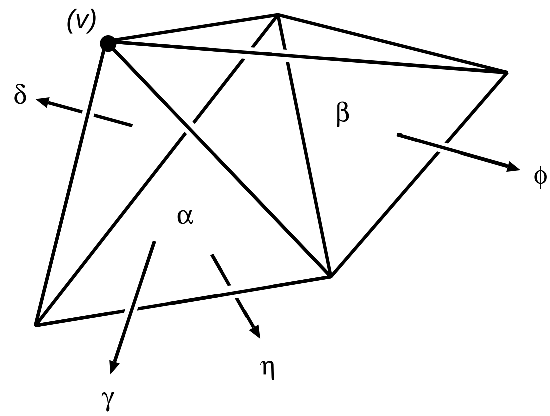

Fig. 2.1. Once the condition (2.44) is satisfied $\mathbf{l}(\alpha)^{\beta\gamma}$, that is defined by relation (refe=ll), equals $-\mathbf{M}(\alpha)_\beta \mathbf{l}(\beta)^{\alpha\phi}$, that is defined by the analog of (2.39) for tetrahedron β. Then the variables $\mathbf{E}$'s prescribe the geometry of the simplicial lattice while the $\mathbf{M}$'s give the connection.

$$\mathbf{E}(\alpha)^\beta = -\mathbf{M}(\alpha)_\beta \mathbf{E}(\beta)^\alpha \quad ; \tag{2.45}$$

however, other conditions exist. These new restrictions relate to the different ways in which $\mathbf{E}^*$ can be written as $\mathbf{l}\wedge\mathbf{l}$; in particular, there is one degree of freedom corresponding to the rotations within the plane defined by $\mathbf{E}^*$. The constraint on the connection matrices, which freezes this degree of freedom, imposes *zero torsion* on the lattice.

An expression of condition (2.44) in terms of the fundamental variables $\mathbf{E}, \mathbf{M}$ follows from its geometrical meaning (see fig. 1). The condition requires that links on the boundary between cells α and β be the same when defined using variables $\mathbf{E}$ from either cell. Consider $\mathbf{l}(\alpha)^{\beta\gamma}$ lying in the intersection of faces $\mathbf{E}(\alpha)^{*\beta}$ and $\mathbf{E}(\alpha)^{*\gamma}$ of cell α and $\mathbf{l}(\beta)^{\alpha\phi}$ lying in the intersection of two faces $\mathbf{E}(\beta)^{*\alpha}$ and $\mathbf{E}(\beta)^{*\phi}$ of cell β. Then equation (2.44) holds if the planes defined by $\mathbf{E}(\alpha)^{*\beta}$, $\mathbf{E}(\alpha)^{*\gamma}$ and $\mathbf{M}(\alpha)_\beta \mathbf{E}(\beta)^{*\phi}$ all intersect (2.46), (2.47), (2.48) and intersect along the same line (2.49):

$$g^{*AB} E(\alpha)^\beta_A E(\alpha)^\gamma_B = -2\mathrm{Im}(E(\alpha)^{(+)\beta B} E(\alpha)^{(+)\gamma}_B) = 0 \tag{2.46}$$

$$g^{*AB} E(\alpha)^\beta_A M(\alpha)_{\beta B}^{\ C} E(\beta)^\phi_C = -2\mathrm{Im}(E(\alpha)^{(+)\beta B} M(\alpha)^{(+)C}_{\beta B} E(\beta)^{(+)\phi}_C) = 0 \tag{2.47}$$

$$g^{*AB} E(\alpha)^\gamma_A M(\alpha)_{\beta B}^{\ C} E(\beta)^\phi_C = -2\mathrm{Im}(E(\alpha)^{(+)\gamma B} M(\alpha)^{(+)C}_{\beta B} E(\beta)^{(+)\phi}_C) = 0 \tag{2.48}$$

$$E(\alpha)^{*\underline{\beta}}_A f^{ABC} E(\alpha)^{*\gamma}_B M(\alpha)_{\underline{\beta C}}^{\ D} E(\beta)^{*\phi}_D$$

$$= 2\mathrm{Im}(E(\alpha)_A^{(+)\underline{\beta}} f^{(+)ABC} E(\alpha)_B^{(+)\gamma} M(\alpha)_{\underline{\beta}C}^{(+)D} E(\beta)_D^{(+)\phi}) = 0 \tag{2.49}$$

where there is no summation over the underlined index $\underline{\beta}$ of the last equation. Condition (2.46) was already present in the first set of geometricity conditions on cell α (2.42), and condition (2.47) is a consequence of the geometricity conditions for cell β and the identity (2.16). The second set of geometricity conditions is: one condition of the kind (2.48) and one condition of the kind (2.49) for the relation $\mathbf{l}(\alpha)^{\beta\gamma} = -\mathbf{M}(\alpha)_\beta \mathbf{l}(\beta)^{\alpha\phi}$ and other set of conditions for the relation $\mathbf{l}(\alpha)^{\beta\gamma} = -\mathbf{M}(\alpha)_\gamma \mathbf{l}(\gamma)^{\theta\alpha}$. Clearly, the relations arising from $\mathbf{l}(\alpha)^{\beta\gamma} = -\mathbf{M}(\alpha)_\beta \mathbf{l}(\beta)^{\alpha\phi}$ and the ones coming from $\mathbf{l}(\beta)^{\alpha\phi} = -\mathbf{M}(\beta)_\alpha \mathbf{l}(\alpha)^{\beta\gamma}$ are equivalent; then an straight forward exercise in linear algebra shows that the "symmetrized" relations

$$\begin{aligned} &\mathrm{Im}(E(\alpha)^{(+)\gamma B} M(\alpha)_{\beta B}^{(+)C} E(\beta)_C^{(+)\phi}) \\ &+\mathrm{Im}(E(\alpha)^{(+)\beta B} M(\alpha)_{\gamma B}^{(+)C} E(\gamma)_C^{(+)\theta}) = 0 \end{aligned} \tag{2.50}$$

$$\begin{aligned} &\mathrm{Im}(E(\alpha)_A^{(+)\underline{\beta}} f^{(+)ABC} E(\alpha)_B^{(+)\gamma} M(\alpha)_{\underline{\beta}C}^{(+)D} E(\beta)_D^{(+)\phi}) \\ &+\mathrm{Im}(E(\alpha)_A^{(+)\underline{\gamma}} f^{(+)ABC} E(\alpha)_B^{(+)\beta} M(\alpha)_{\underline{\gamma}C}^{(+)D} E(\gamma)_D^{(+)\theta}) = 0 \end{aligned} \tag{2.51}$$

imposed for every link of each cell are equivalent to relations (2.48), (2.49) imposed once for each of the three links of each face of every cell.

In a recent paper, Immirzi wrote the reality conditions (2.42) for the lattice; the motivation of his work is to obtain a consistent Ashtekar-like framework for the lattice [11]. This coincidence with Immirzi's formulas, that come from a rather different approach, is the lattice manifestation of a well known result of Capovilla, et al [5]. The mentioned result arrives to the Ashtekar formalism as the Hamiltonian version of a formulation of gravity based on two forms.

The set of geometricity conditions (2.42), (2.50), (2.51) on the fundamental variables form a *complete* set in the sense that they imply the existence and uniqueness of the geometrical variables (2.39) and that condition (2.44) holds. An expression for the link $l(\alpha)^{\gamma\delta\ a}$ of tetrahedron α where face $(\alpha\gamma)$ $(E(\alpha)_A^\gamma = E(\alpha)_{[ab]}^\gamma = \frac{1}{2}\varepsilon_{ab\ \ cd}\ l(\alpha)_c^{\gamma\delta} l(\alpha)_d^{\gamma\beta})$, and face $(\alpha\delta)$ $(E(\alpha)_A^\delta = E(\alpha)_{[ab]}^\delta = \frac{1}{2}\varepsilon_{ab}^{\ \ cd} l(\alpha)_c^{\delta\beta} l_d^{\delta\gamma})$ intersect is

$$l(\alpha)_a^{\gamma\delta} = \frac{1}{v(\alpha)} \varepsilon(\alpha)_{abc} \varepsilon(\alpha)^{bef} E(\alpha)_{ef}^{*\gamma} \varepsilon(\alpha)^{cgh} E(\alpha)_{gh}^{*\delta} \quad , \tag{2.52}$$

where the volume of the tetrahedron α is given by

$$v(\alpha)^2 = \frac{1}{9} \hat{\varepsilon}_{jkl} f^{ABC} E(\alpha)_A^{*j} E(\alpha)_B^{*k} E(\alpha)_C^{*l} \tag{2.53}$$

and the volume element for cell α

$$\varepsilon(\alpha)_{abc} = \frac{\phi(\alpha)_{abc}}{\sqrt{\frac{1}{6}\phi(\alpha)_{def}\phi(\alpha)^{def}}} \tag{2.54}$$

$$\phi(\alpha)_{abc} = \hat{\varepsilon}_{jkl}\varepsilon_{defg}E(\alpha)^{*j}{}_{[a}{}^{d}E(\alpha)^{*k}{}_{b}{}^{e}E(\alpha)^{*l}{}_{c]}{}^{f}x(\alpha)^{g} \tag{2.55}$$

does not depend on the choice of the transverse vector $x(\alpha)^g$. The geometricity conditions (2.42), (2.48), (2.49) are also *necessary*: any set of variables $\mathbf{E}$, that can be written as in (2.39) where the compatibility condition (2.44) holds, satisfies them.

From (2.44) one can see that if $\mathbf{W}(\alpha)_{\beta\gamma} := (\mathbf{M}(\alpha)_{\beta}M(\beta)_{\mu}\ldots M(\nu)_{\gamma}M(\gamma)_{\alpha})$ is the holonomy around $\mathbf{l}(\alpha)^{\beta\gamma}$, then a consequence of the geometricity conditions is

$$\mathbf{l}(\alpha)^{\beta\gamma} = \mathbf{W}(\alpha)_{\beta\gamma}\mathbf{l}(\alpha)^{\beta\gamma} \tag{2.56}$$

Because the geometricity conditions imply that $\mathbf{W}(\alpha)_{\beta\gamma}$ has an axis and that the axis cross the planes of $\mathbf{E}(\alpha)^{\beta *}$ and $\mathbf{E}(\alpha)^{\gamma *}$

$$P(\alpha)^{*A}_{\beta\gamma}P(\alpha)_{\beta\gamma A} = -2\mathrm{Im}(P(\alpha)^{(+)*A}_{\beta\gamma}P(\alpha)^{(+)}_{\beta\gamma A}) = 0 \tag{2.57}$$

$$P(\alpha)^{*A}_{\beta\gamma}E(\alpha)^{\beta}_{A} = -2\mathrm{Im}(P(\alpha)^{(+)*A}_{\beta\gamma}E(\alpha)^{(+)\beta}_{A}) = 0 \tag{2.58}$$

$$E(\alpha)^{\beta *}_{A}f^{AB}_{\ \ C}E(\alpha)^{\gamma}_{B}P(\alpha)^{C}_{\beta\gamma} = 2\mathrm{Im}(E(\alpha)^{(+)\beta *}_{A}f^{AB}_{\ \ C}E(\alpha)^{(+)\gamma}_{B}P(\alpha)^{(+)C}_{\beta\gamma}) = 0 \tag{2.59}$$

Evidently the geometricity conditions in the lattice are stronger than what one expected from experience in the continuum; in particular, the continuum counterpart of relations (2.58), (2.59) does not hold. This rigidity of the lattice makes some of the constraints trivial. I discuss this issue in the next two sections.

2.3 The Constraints

Not all the symmetries generated by the constraints $P \approx 0$ of lattice $B \wedge F$ theory (LBF) preserve the geometricity conditions. The largest subgroup that preserves them is the group of translation of vertices of the lattice. The generator of translations of vertex (v) of cell α, in the direction of $w(\alpha)^{(+)a}$, was introduced in [2]. In the notation of this paper it is $2\mathrm{Re}(w(\alpha)^{(+)a}T(\alpha,v)^{(+)}_{a})$; one can easily write it in self-dual variables using that $\delta^{(+)} = \frac{1}{2}(g - ig^*)$ and that $w(\alpha)^{(+)a}$ is real. $2\mathrm{Re}(w(\alpha)^{(+)a}T(\alpha,v)^{(+)}_{a})$ equals

$$\begin{aligned}\Big[\quad & w(\alpha)^{(+)a} \sum_{\{jk\}\to v} \frac{1}{2}\varepsilon_{abcd}l(\alpha)^{jk\ b}P(\alpha)^{(+)cd}_{jk} \\ + \quad & (M(\beta)_{\alpha}w(\alpha)^{(+)})^{a} \sum_{\{jk\}^{-}\to v} \frac{1}{2}\varepsilon_{abcd}l(\beta)^{jk\ b}P(\beta)^{(+)cd}_{jk} + \ldots \Big] + c.c.\end{aligned} \tag{2.60}$$

The summation written above runs over the links $l(\alpha)^{jk\ a}$ pointing in the direction of vertex (v). The terms of the summation have been split for convenience, according to the cell $\alpha, \beta, \ldots$ where the variables are expressed, but each link must be included only once in the summation. For this reason, the index of the second summation written is $\{jk\}^-$ indicating to sum only over the links not included in the previous summation.

One can easily prove that the action of $T(\alpha, v)_a := 2\mathrm{Re}(T(\alpha, v)_a^{(+)})$ on the variables E, that determine the geometricity of the lattice, is to generate translations of the vertex (v) (see fig. 1). First one sees that if the face σ of cell ρ does not contain vertex (v) the Poisson bracket of $T(\alpha, v)_a$ and $E(\rho)^\sigma$ is zero; and that in the case of a face that contains (v), like face β of cell α, the action of the generator $T(\alpha, v)_a$ on the variable $E(\alpha)_A^{*\beta} = E(\alpha)_{[ab]}^{*\beta} = l(\alpha)_{[a}^{\beta\gamma} l(\alpha)_{b]}^{\beta\delta} = l(\alpha)_{[a}^{\beta\delta} l(\alpha)_{b]}^{\beta\eta} = l(\alpha)_{[a}^{\beta\eta} l(\alpha)_{b]}^{\beta\gamma}$ has the same geometrical effect as a translation of the vertex (v), where faces β, γ, δ of cell α intersect.

$$\begin{aligned}
\{E(\alpha)_{ef}^{*\beta}, w(\alpha)_a^{(+)} T(\alpha, v)^a\} &= \frac{1}{4}\varepsilon_{cd}^{ab}\varepsilon_{ef}^{gh} w(\alpha)_a^{(+)} l(\alpha)_b^{\beta\gamma} \{E(\alpha)_{gh}^{\beta}, P(\alpha)_{\beta\gamma}^{cd}\} \\
&+ \frac{1}{4}\varepsilon_{cd}^{ab}\varepsilon_{ef}^{gh} w(\alpha)_a^{(+)} l(\alpha)_b^{\beta\delta} \{E(\alpha)_{gh}^{\beta}, P(\alpha)_{\beta\delta}^{cd}\} + O(P) \\
&= w(\alpha)_{[e}^{(+)} (l(\alpha)_{f]}^{\beta\gamma} - l(\alpha)_b^{\beta\delta}) + O(P) \\
&= w(\alpha)_{[e}^{(+)} l(\alpha)_{f]}^{\beta\eta} + O(P) \quad . \qquad (2.61)
\end{aligned}$$

An immediate consequence is that the geometricity conditions are preserved by the translation generators

$$\{q(\beta)^{*jk}, w(\alpha)^{(+)a} T(\alpha, v)_a\} = 0 + O(P) \quad . \qquad (2.62)$$

Two remarks are in order. First, even though it seems natural to consider $T(\alpha, v)_a$ as the symmetry generator but leave the full (complex) $T(\alpha, v)_a^{(+)} \approx 0$ as constraint, with the reality condition $w(\alpha)^{(+)a} \epsilon R$ and a real Hamiltonian, the model would not make sense without the geometricity conditions. The strategy of forging a consistent theory for lattice gravity even without geometricity conditions is discussed in the last section. Second, $\mathrm{Im}(T(\alpha, v)_a^{(+)}) = 0$ is automatically satisfied for lattices where the geometricity conditions are satisfied; thus, writing the constraints as $T(\alpha, v)_a^{(+)} \approx 0$ *is correct*, but only half of them are independent of the geometricity conditions (and the formula makes sense only for geometrical lattices).

The result of imposing the geometricity conditions and from the symmetry the generators of LBF choosing only the symmetries generated by $T(\alpha, v)_a$ is (GLBF) a description of the geometric sector of LBF [2]. More precisely:

- All the solutions of GLBF are solutions of LBF.
- The solutions of GLBF are the flat space-times $\Sigma \times R$ generated by a geometrical lattice (a lattice made of vertices, links, faces and cells) Σ during its evolution.

- There are some solutions of LBF with "global torsion" that do not admit any geometric representation [21].
- Both GLBF and LBF have *zero* local degrees of freedom. Both theories have only discrete, topological, degrees of freedom.
- GBF, the restriction of $B \wedge F$ to torsion free connections is equivalent to GLBF. The proof follows the procedure used by Waelbroeck to prove the equivalence of the lattice $2+1$ theory and the continuum theory [22]

A lattice theory for gravity must be geometrical (at least when restricted to flat space-times), and must possess local degrees of freedom to reproduce a theory with two degrees of freedom per point in its macroscopic limit. GLBF has the largest symmetry group that preserves the geometricity of the lattice, and it is a proper subgroup of the LBF symmetry group. These properties of GLBF may be used to find a well-defined theory. The constrained system whose only constraints are the symmetry generators of GLBF and the geometricity conditions may have a closed algebra, making it a well-defined theory. Moreover, if the geometricity conditions and the remaining curvature constraints do not make the connection flat, one could have a theory with local degrees of freedom. It is encouraging that, for sufficiently fine lattices, the number of geometricity conditions and remaining constraints are not enough to make the connection flat.

Two inputs are behind this proposed avenue towards lattice gravity. The first is a discrete version of BF theory, and the second is a discretization of the restriction making the two form field B come from a tetrad. In the continuum, when the B field of BF theory is restricted to come from a tetrad one obtains GR. I am trying to construct a discrete theory for gravity using this mechanism. To study this proposal I introduce a model with severe limitations, but that allows me to study some aspects of the discrete theories for gravity that may be obtained following the proposed strategy.

The model presented in this chapter studies the proposed theory in its restriction to flat connections as initial data, and its behavior as one departs from flat initial data (a precise definition of what I mean by *the model* will be given shortly). As the symmetry group is smaller than that of LBF one can interpret the model as the result of restricting a theory of lattice gravity to act on flat initial data. This hypothetical theory of lattice gravity would be a theory with local degrees of freedom, and part of the motivation of this work is to learn as much as possible about the hypothetical theory from its restriction to flat initial data and its behavior as one departs from initial data (for an extended discussion see the concluding section).

A problem will be to write the proposed constraints in a form that makes them recognizable; in particular, the use of self-dual variables and the appearance of reality conditions make one wonder if the constraints are related to Ashtekar's constraints. The key to answering this problem is the affine notation employed in this chapter. For the affine notation, cells are the lattice counterpart of points in the continuum. I will show that the translations of lattice cells constitute a proper subgroup of GLBF's symmetry group (the group of vertex translations). The generator of translations of cell α in the direction of $w(\alpha)^{(+)a}$ is simply the one that moves the four vertices of α by $w(\alpha)^{(+)a}$.

$$\begin{aligned} w(\alpha)^{(+)a}T(\alpha)_a \;&:=\; w(\alpha)^{(+)a}T(\alpha,v_1)_a + w(\alpha)^{(+)a}T(\alpha,v_2)_a \\ &+\; w(\alpha)^{(+)a}T(\alpha,v_3)_a + w(\alpha)^{(+)a}T(\alpha,v_4)_a \end{aligned} \tag{2.63}$$

The commutativity of cell translations inside the group of vertex translations is a simple consequence of the commutativity of vector addition. Hence the symmetry group of this model has dimension $4N_3$ (four times the number of lattice cells). In a lattice that admits local deformations that make it flat the number of vertices is (locally) bigger than the number of cells (see appendix A). As a result, the symmetry group of this model is a proper subgroup of that of GLBF. Considering the cell translations or the vertex translations as the basis for the model may not make a difference as far as the continuum limit is concerned. By choosing the cell translations, the symmetry generators will be recognizable, and the model will have local degrees of freedom even in the case of coarse lattices; for some coarse lattices the geometricity conditions and the vertex translation generators (seen as constraints) make the connection flat.

In the lattice the links have been described using internal Minkowski vector spaces that are related by parallel transport matrices; the same internal space has been used to point the directions of translation for the translation generators. These internal Minkowski spaces are just side products of the geometricity conditions and are not the internal spaces where the dynamical variables live. Because of this, it would be desirable to label the translation generators of the lattice using notions that are more compatible with the continuum. A general translation of a cell α can be specified by a *real* "lapse" $N(\alpha)^{(+)}$ and a *real* "shift" $N(\alpha)^{(+)j}$ $j=\beta,\gamma,\delta,\eta$ (using the affine notation described in sec. 2.1.1 $2\mathrm{Re}(N(\alpha)^{(+)j}\mathcal{H}(\alpha)^{(+)}_j + N(\alpha)^{(+)}\mathcal{H}(\alpha)^{(+)})$. The generator $\mathcal{H}(\alpha)_\beta := 2\mathrm{Re}(\mathcal{H}(\alpha)^{(+)}_{j=\beta})$ moves cell α in the direction of its neighbor β, and $\mathcal{H}(\alpha) := 2\mathrm{Re}(\mathcal{H}(\alpha)^{(+)})$ translates cell α in the direction orthogonal to the three dimensional space that contains it

$$\mathcal{H}(\alpha)^{(+)}_\beta := \frac{1}{16}(l(\alpha)^{\eta\gamma} + l(\alpha)^{\gamma\delta} + l(\alpha)^{\delta\eta})^a T(\alpha)^{(+)}_a \approx 0 \tag{2.64}$$

$$\mathcal{H}(\alpha)^{(+)} := \frac{1}{2} l(\alpha)^{\eta\gamma}_a l(\alpha)^{\gamma\delta}_b l(\alpha)^{\delta\eta}_c \varepsilon^{abcd} T^{(+)}_d \approx 0 \tag{2.65}$$

$$J(\alpha)^{(+)}_A = \sum_j E(\alpha)^{(+)j}_A \approx 0 \tag{2.66}$$

$$P(\alpha)^{(+)A}_{jk}(\text{initial}) = 0 \tag{2.67}$$

I wrote the lattice Gauss law, and the condition restricting to flat lattices as initial data in order to precise what I mean by *the model*. The model's phase space is that of LBF described by variables $E^{(+)}, M^{(+)}$ and Poisson bracket structure (2.35), (2.36), (2.37). It is subject to the geometricity conditions (2.42), (2.50), (2.51), and to constraints (translation and gauge generators) (2.64), (2.65), (2.66). Also the model's initial data is restricted to flat lattices (2.67). For the model presented here $\mathcal{H}(\alpha)^{(+)}_{\beta} = 0$, $\mathcal{H}(\alpha)^{(+)} = 0$ do not imply $P(\alpha)^{(+)A}_{jk} = 0$ even for a geometrical lattice; this opens the possibility of considering the model as the restriction of a hypothetical theory for non flat lattices to the case of initial data satisfying $P(\alpha)^{(+)A}_{jk}(\text{initial}) = 0$. The translation generators $\mathcal{H}(\alpha)^{h(+)}_{\beta} = 0$, $\mathcal{H}(\alpha)^{h(+)} = 0$ of the of the hypothetical lattice theory when restricted to flat lattices have the form (2.64), (2.65), i.e. $\mathcal{H}(\alpha)^{h(+)}_{\beta} = \mathcal{H}(\alpha)^{(+)}_{\beta} + O(P^2)$, $\mathcal{H}(\alpha)^{h(+)} = \mathcal{H}(\alpha)^{(+)} + O(P^2)$. The fact that the symmetry group of the model is smaller than that of GLBF means that the hypothetical lattice theory that reduces to the model for flat initial data is a theory with local degrees of freedom.

As in the case of GLBF, one should regard only $\mathcal{H}(\alpha)_j$, $\mathcal{H}(\alpha)$ as symmetry generators and consider all the $\mathcal{H}(\alpha)^{(+)}_{j} \approx 0$, $\mathcal{H}(\alpha)^{(+)} \approx 0$ as constraints. However, they are not independent within themselves[4], and their imaginary part is a direct consequence of the geometricity conditions.

Although it is difficult to calculate the algebra of the symmetry generators, there is a simple way to prove that their Poisson bracket vanishes weakly. The translation generators $\mathcal{H}(\alpha)_j$ and $\mathcal{H}(\alpha)$ are scalars and, therefore, their Poisson brackets with the gauge transformation generator $J^{(+)}_{(\alpha)A}$ vanish. These translation generators, ($\mathcal{H}(\alpha)_j$ and $\mathcal{H}(\alpha)$), span the same space as the former translation generators $T(\alpha)_a$. Introducing a bit of notation, $N(\alpha)^{(+)\mu}\mathcal{H}(\alpha)_\mu := N(\alpha)^{(+)j}\mathcal{H}(\alpha)_j + N(\alpha)^{(+)}\mathcal{H}(\alpha)$, the previous statement signifies that $T(\alpha)_a = C(a)_a^{\ \mu}\mathcal{H}(\alpha)_\mu$ where the matrix $C(a)_a^{\ \mu}$ has rank four. Then the Poisson brackets of T's and those of H's are related by

$$\begin{aligned}
\{T(\alpha)_a, T(\beta)_b\} &= C(\alpha)_a^{\ \mu} C(\beta)_b^{\ \nu}\{\mathcal{H}(\alpha)_\mu, \mathcal{H}(\beta)_\nu\} \\
&+ C(\alpha)_a^{\ \mu}\{\mathcal{H}(\alpha)_\mu, C(\beta)_b^{\ \nu}\}\mathcal{H}(\beta)_\nu \\
&+ \mathcal{H}(\alpha)_\mu\{C(\alpha)_a^{\ \mu}, \mathcal{H}(\beta)_\nu\}C(\beta)_b^{\ \nu} \\
&\approx C(\alpha)_a^{\ \mu} C(\beta)_b^{\ \nu}\{\mathcal{H}(\alpha)_\mu, \mathcal{H}(\beta)_\nu\} \quad .
\end{aligned} \tag{2.68}$$

An immediate consequence of the fact that the translation generators $T(\alpha)^{(+)}_{a}$ commute (in flat space-time) $\{T(\alpha)_a, T(\beta)_b\} = 0 + O(P^2)$ is that the Poisson brackets of the new form of the translation generators weakly vanish up to second order in the curvature

$$\{\mathcal{H}(\alpha)_\mu, \mathcal{H}(\beta)_\nu\} \approx 0 + O(P^2) \quad . \tag{2.69}$$

[4]In a three dimensional simplicial lattice there are $6(N_1 - N_0) = 6N_3$ independent curvature variables $P(\alpha)^{A}_{jk}$ because of the Bianchi identities; thus, the $8N_3$ projections of them ($\mathcal{H}(\alpha)^{(+)}_{j}$, $\mathcal{H}(\alpha)^{(+)}$) can not be independent.

An extremely interesting feature of the spatial translation constraints $\mathcal{H}(\alpha)^{(+)}_j$ and the time translation constraints $\mathcal{H}(\alpha)^{(+)}$ is that their local parts are algebraically identical to the diffeomorphism and Hamiltonian constraints of the Ashtekar formulation of general relativity. To define the local parts one requires

$$\{E(\alpha)^{(+)j}_A, \mathcal{H}(\alpha)^{(+)}_\mu\} = \{E(\alpha)^{(+)j}_A, \mathcal{H}(\alpha)^{(+)local}_\mu\} \tag{2.70}$$

$$\{M(\alpha)^{(+)B}_{jA}, \mathcal{H}(\alpha)^{(+)}_\mu\} = \{M(\alpha)^{(+)B}_{jA}, \mathcal{H}(\alpha)^{(+)local}_\mu\} \quad . \tag{2.71}$$

and that in the expression for $\mathcal{H}(\alpha)^{(+)local}_\mu$ only variables related to links of cell α appear. The link variables $l(\alpha)^{jk\ a}$ occur in pairs that recombine in the form of the variables of the theory $E(\alpha)^{(+)j}_A$ to yield

$$\mathcal{H}(\alpha)^{(+)local}_j = \frac{1}{2}\hat{\varepsilon}_{jkl}E(\alpha)^{(+)k}_A B(\alpha)^{(+)lA} - (\frac{1}{2}P(\alpha)^{(+)A}_{jk}\hat{n}^k)J(\alpha)^{(+)}_A \tag{2.72}$$

$$\mathcal{H}(\alpha)^{(+)local} = \frac{1}{4}\hat{\varepsilon}_{jkl}f^{AB}_{\ \ C}E(\alpha)^{(+)j}_A E(\alpha)^{(+)k}_B B(\alpha)^{(+)lC} \quad , \tag{2.73}$$

where the "magnetic field" is written as $B(\alpha)^{(+)jA} := \hat{\varepsilon}^{jkl}P(\alpha)^{(+)A}_{kl}$. The second term of $\mathcal{H}(\alpha)^{(+)local}_j$ generates Lorentz transformations and vanishes for flat space-times. The non-locality of the translation constraints $\mathcal{H}(\alpha)^{(+)}_\mu$ is very mild. The difference between $\mathcal{H}(\alpha)^{(+)local}_\mu$ and $\mathcal{H}(\alpha)^{(+)}_\mu$ is function only of variables of the lattice sharing vertices with cell α. One of the first basic distinctions between the lattice and the continuum[5] appears: the concept of neighborhood fundamentally differs. Two points in the continuum p and q can either be the same or be separated by open neighborhoods. In contrast, two cells in the lattice can either be the same, be immediate neighbors, or be separated by other cells. Clearly the category of immediate neighbors disappears during any acceptable continuum limit; therefore, in any acceptable (see last section for a discussion) continuum limit, only the local parts of the expressions are going to remain. Hence, the continuum limit of the translation generators is, precisely, Ashtekar's diffeomorphism and Hamiltonian constraints.

In order to summarize ideas and prepare the discussion, I am going to count the degrees of freedom of the hypothetical lattice theory resulting from an extension of the constraints of the model (2.64), (2.65) to first-class constraints $\mathcal{H}^h(\alpha)_j$, $\mathcal{H}^h(\alpha)$. Recall that the number of points, links, faces, and cells are denoted by N_0, N_1, N_2 and N_3 respectively, and also, for a lattice of tetrahedra, $N_2 = 2N_3$. The phase space variables are given by the $12N_2 = 24N_3$ numbers $E(\alpha)^{(+)j}_A$, $M(\alpha)^{(+)B}_{jA}$, that are subject to $6N_3$

[5] I am assuming that the continuum is an smooth manifold; obviously, excluding pathological topologies such as non-Hausdorff spaces.

closure constraints $J(\alpha)_A^{(+)} \approx 0$, and $3N_3$ vector constraints $\mathcal{H}^h(\alpha)_j$, and N_3 scalar constraints $\mathcal{H}^h(\alpha)$. These constraints are first-class and generate Lorentz transformations, spatial translations, and time translations respectively. Thus, the dimension of the reduced phase space, without taking into account the geometricity conditions, is

$$24N_3 - 2(6N_3 + 3N_3 + 1N_3) = 4N_3 \quad . \tag{2.74}$$

And the geometricity conditions reduce the number of degrees of freedom to at least half. The reason is that all the scalar information in the variables $E(\alpha)_A^{(+)j}$ of cell α is captured in the symmetric matrix $q(\alpha)^{(+)jk} = E(\alpha)_A^{(+)j} E(\alpha)_B^{(+)k} g^{AB}$; thus, the first set of geometricity conditions –restricting the matrix $q(\alpha)^{(+)jk}$ to be real– reduces the scalar information on $E(\alpha)_A^{(+)j}$ from 12 to 6 numbers. Some local degrees of freedom remain, because as proven in appendix A, the symmetry group of the theory is "locally smaller" than that of GLBF. For a discrete, *microscopic*, theory of gravity local degrees of freedom are essential, but to get the expected $2N_3$ is not. A discussion of this point and related issues is the topic of the last section.

2.4 Discussion

A classical lattice theory that describes space-time in $3+1$ form must be geometrical, i.e. a unique three-dimensional piecewise linear space must be assigned to any set of variables of the theory (in the present case $\{E, M\}$ satisfying the geometricity conditions, $\{l, \pi_l\}$ in the usual formulation of Regge Calculus). Also, a lattice theory with no local degrees of freedom cannot be related to gravity simply because any sensible continuum limit cannot convert a set of configurations that are equivalent in the lattice description into inequivalent geometries in the corresponding continuum theory. On one hand, the model presented here is geometrical; on the other hand, however, the model describes flat space-times. In this sense the model is not better than GLBF (the geometric sector of lattice $B \wedge F$ theory). There are two essential differences between the model and GLBF. First, GLBF is the geometric sector of a theory with first-class constraints that describes the geometric lattice $B \wedge F$ theory in the sense described in section 2.3. This is different from the model introduced in this chapter, that has a symmetry group whose flows commute only if the initial data is a flat lattice. Second, the constraints of the model do not force the lattice curvature to be zero; therefore one can consider the model as the restriction of a *hypothetical theory*, that includes non-flat lattices, restricted to the case of flat initial data. The reason to call the theory "hypothetical" is not that its existence is in doubt; for instance, the method to find symplectic coordinates [23] can be used to get an extension of the translation generators $T(\alpha, v)_a$, in an open neighborhood of the submanifold $P = 0$ where their flows commute, to momentum coordinate functions that with some configuration coordinate functions form a set of symplectic coordinates. Then, the extensions $T^h(\alpha, v)_a$ of the translation generators of the model are the constraints of the hypothetical theory. The enormous freedom in the choice of coordinates makes this method more of an existence statement than a constructive process: that is the reason to call the extension h-theory.

In this chapter an avenue towards finding a theory for lattice gravity, and particular candidate for the h-theory were proposed. Since the geometricity conditions are defined in the whole phase space, one should require the symmetry generators to respect them to all orders. This restriction still leaves enormous freedom for the h-theory. In spite of that, our strategy suggests natural candidate for the generators. These generators where shown to respect the geometricity conditions up to first order in the curvature and to have a closed algebra closes up to second order in the curvature. Unfortunately, it was not determined whether the symmetry generators respect the geometricity conditions to all orders or if their algebra closes exactly. Only in the case in which they do not behave properly at all orders would one add higher order corrections in the curvature.

Along the chapter I presented four results concerning *any* h-lattice theory that reduces to the model when restricted to flat initial data.

1. *There are local degrees of freedom in the h- theory.* This follows from the fact that the symmetry group of the model is a proper subgroup of that of GLBF (see appendix A).

2. *The continuum limit has to be a macroscopic limit.* If the continuum limit were taken by simply shrinking the lengths of the lattice links to zero, and by identifying the cells (or vertices) of the lattice with points of the continuum, the resulting continuum theory would have less degrees of freedom than gravity because the h-theory has less than two degrees of freedom per lattice cell. Thus, to be consistent one must regard this theory as a microscopic theory.

3. *Regarding the macroscopic limit of the h-theory and Ashtekar's formulation of general relativity.* Although a serious study of the macroscopic limit is yet to be performed, two facts indicate that the macroscopic limit of the h-theory and general relativity formulated in terms of the new variables are related. First, the local part of the translation generators of the model is algebraically identical to Ashtekar's constraints, and the reality conditions have the geometricity-reality conditions as lattice counterpart. Second, an appropriate procedure to take the macroscopic limit could be through refinements of the (dual) lattice and a prescription for projecting down the the variables from the refined lattice to the original lattice. Along these refinements the phase space of the lattice becomes bigger and given a non-vanishing macroscopic curvature it is possible to reach it as the limit of lattices where the curvature in every link goes to zero. These special "smooth configurations" would be the ones that define the spatial manifold Σ and for them the constraints of the h-theory become the translation generators of the model. Furthermore, in the macroscopic limit the concept of immediate neighboring cells is lost and, hence, only the local part of the translation generators is relevant. This local part is the one that is algebraically identical to Ashtekar's constraints.

4. *The h-theory is condemned to remain classical.* Immirzi's observation [17] that the quantum reality conditions a nd the algebra of the area bivectors is incompatible makes the theory well defined only classically (see discussion below).

Most of the effort behind the work presented in this chapter was directed towards a slightly different goal than the one achieved. The relation between the geometricity conditions and the reality conditions (when using self-dual variables) was the motivation to structure a model for lattice gravity that precisely mirrored Ashtekar's formulation of GR. In particular, a self-consistent model with symmetry generators whose flows commute for flat initial data ($P(initial) = 0$), that resembled Ashtekar's seemed feasible. A summary of the results, using as proposed constraints $\mathcal{H}(\alpha)_{\mu}^{(+)local}$ (2.72,2.73), is the following:

- $\{\mathcal{H}(\alpha)_{\mu}^{local}, \mathcal{H}(\beta)_{\nu}^{local}\} \approx 0 + O(P^2)$ *except* in the case of α and β being neighbors sharing only a link. In this case $\{\mathcal{H}(\alpha)_{\mu}^{local}, \mathcal{H}(\beta)_{\nu}^{local}\} \approx 0 + O(P)$, which means that the symmetries generated by the constraints do not commute even for flat space-times.
- The "symmetry generators" $\mathcal{H}(\alpha)_{\mu}^{local}$ do not respect the geometricity of neighboring cells even for flat space-times.
- On the other hand, the generators

$$\begin{aligned}
\mathcal{H}(\alpha)_{j}^{local} &= 2\mathrm{Re}(\frac{1}{2}\hat{\varepsilon}_{jkl}E(\alpha)_{A}^{(+)k}B(\alpha)^{(+)lA} - (\frac{1}{2}P(\alpha)_{jk}^{(+)A}\hat{n}^{k})J(\alpha)_{A}^{(+)}) \\
&= \hat{\delta}_{j}^{\ k}P(\alpha)_{kl}^{A}E(\alpha)_{A}^{l} \qquad (2.75)\\
\mathcal{H}(\alpha)^{local} &= 2\mathrm{Re}(\frac{1}{4}\hat{\varepsilon}_{jkl}f^{AB}_{\ \ C}E(\alpha)_{A}^{(+)j}E(\alpha)_{B}^{(+)k}B(\alpha)^{(+)C}) \\
&= f^{AB}_{\ \ C}E(\alpha)_{A}^{j}E(\alpha)_{B}^{k}P(\alpha)_{jk}^{C} \qquad (2.76)
\end{aligned}$$

  do generate the translations expected for variables $E(\alpha)_{A}^{(+)j}$ *of the same cell.*

The conclusion is that the constraints $\mathcal{H}(\alpha)_{\mu}^{(+)local}$ are not correct even at first-order in the curvature; but they are the local part of constraints that are correct at first order. This attempt deviates from the path that uses the lattice $B \wedge F$ theory as a firm ground from which one can extract a richer theory. Using only the local parts of the expressions written in this paper leads to a discrete theory closely related to a discretization of general relativity using Ashtekar's new variables proposed recently by Boström, Miller and Smolin [24]. The translation generators introduced in this paper should be considered as a refinement of those proposed in [24], because they differ only in their non-local part that is necessary for them to have commuting flows (for initial data satisfying $P = 0$). In this sense, it is the first time that a connection oriented formulation of $3+1$ lattice gravity has symmetry generators that are correct up to first order. Thus, this model is of potential interest to develop numerical codes. As shown in section 2.3 these non-local terms have an entirely discrete origin and, therefore, their absentia is natural in a theory derived directly from a discrete analog of a continuum action. Unfortunately, only the local part is easily written in terms of the dynamical variables E, M; the non-local terms are naturally written using the geometrical variables

l, M. The problem with straightforward substitution using the formulas giving the links $l(\alpha)^{a}_{jk}$ in terms of the variables $E(\alpha)^{(+)j}_{A}$ (2.52) is that the 3-volume element $\varepsilon(\alpha)_{abc}$ for each tetrahedron α is a complicated function of the E's. A solution to this problem could be modifying the model to have $SO(3)$ as internal group and then inherit the natural volume element of the Lie algebra. The counterpart of this approach in the continuum has no complicated reality conditions and is linked to Lorentzian gravity via the generalized Wick transform [18, 25] while keeping the Hamiltonian constraint algebraically simple. To import this approach to the lattice means trading immediate geometrical interpretation for algebraic simplicity. Direct geometric interpretation of the lattice theory should not be regarded as a first priority, because a theory of classical gravity with the correct number of degrees of freedom can be recovered only as the macroscopic limit of the lattice theory anyway.

In an attempt to organize ideas for future investigations, a program towards a quantum theory based on this kind of lattice gravity has been outlined. The consequences up to now have been only to state several well known facts in the language used in this chapter. As mentioned earlier, the lattice theory must be regarded as a microscopic theory to achieve a continuum limit with the correct number of degrees of freedom. A particularly appealing strategy is to do both, the connection to continuum gravity and quantization, simultaneously; such a task is not an utopia, the projective techniques developed by Ashtekar et al and Baez [26] were designed for this purpose. An adaptation of the mentioned strategy for quantization has already been used for abstract lattices by Loll [27]. Immirzi showed that implementation of the quantum reality conditions that result from using $SL(2,C)$ or $SO(3,1)$ as internal groups is inconsistent; rather than a final word this observation should be considered as an other factor in favor of adopting Thiemann's strategy of solving the Lorentzian reality conditions via the generalized Wick transform.

In the quantization program the issue of constructing a regularization of the constraints is a central one; provide hints to avoid future problems, like the presence of anomalies, was one of the motivations for working on this model. An extended discussion of the dynamics in quantum gravity is the subject of the last chapter of this thesis.

Chapter 3

Retrospective and prelude

In this chapter I will look in retrospect to the lattice models presented in the previous chapter. First, the positive aspects and the further developments that rest on similar quantization strategies will be summarized. Then, the difficulties encountered in the previous chapter will be related with the fundamental assumptions behind the lattice models. This analysis will serve as the motivation to modifying some aspects of the strategy, leading to the work presented in chapters 4 and 5.

For three-dimensional gravity the strategy produces a satisfactory quantum model for spacetime. In this case there are no geometricity conditions, and the theory is greatly simplified. Waelbroeck formulated $2 + 1$ gravity as a lattice gauge theory [13]; his formulation is the lattice version of Witten's $2 + 1$ gravity. In collaboration with A. Corichi, I introduced a lattice version of Ashtekar's constraints [28]. If degenerate triangles (triangles with zero area) are present, the lattice curvature may not vanish, but in non-degenerate cases this formulation is equivalent to Waelbroeck's theory. Even in the presence of point masses or degenerate metrics, which are sectors with local degrees of freedom, the constraints are first-class. However, one must accept that 2+1 gravity's local simplicity is behind the success.

In the last chapter I presented a strategy which used BF theory as an intermediate step in finding a discretization of GR; a new trend that uses a covariant version of this strategy is now in vogue. The models arising from the mentioned strategy is referred as *spin foam* models [8, 9]. I will briefly describe the spin foam models and the significance of the $3 + 1$ lattice models. An extended discussion of this topic is given in chapter 6.

Using BF theory as an stepping stone to reach quantum gravity becomes even more appealing from the four-dimensional perspective. In this respect, a four-dimensional framework would be useful to the extent in which it leads to a well-defined expression for the exponential of the action. Fortunately, the quantum partition function of BF theory is known: it is given in purely combinatorial fashion by the Crane-Yetter-Ooguri model [29]. Thus, the main obstacle in obtaining a partition function for quantum gravity is a quantization of the four-dimensional geometricity conditions. In the continuum and using self-dual variables these constraints are known as the Capovilla-Dell-Jacobson constraints [5]. The first systematic covariant approach in this direction is due to Reisenberger [8], and Barrett, Crane and Baez have more recent proposals [9].

The discrete approach of Reisenberger follows from a simplicial action for Euclidean general relativity [8]. It is not difficult to see that the kinematics that follow from his action is compatible with the Euclidean version of the kinematics of chapter 2. Moreover, both lattice frameworks would agree in their treatment of BF theory. It is also possible to relate the local geometricity conditions (2.41) and the classical simplicity conditions of Barrett and Crane [9]. In turn, the classical form of the simplicity conditions are equivalent to Reisenberger's discretization of the Capovilla-Dell-Jacobson constraints.

An important feature of the covariant formulation is locality. In the covariant approach, the variables of neighboring four-simplices are connected only by sharing the connection in the boundary. That is, the connection is the only boundary data [8]. This gives a very appealing locality to the action and the field equations; unfortunately, locality is lost by going to the canonical formulation. Once the relation between the covariant and the canonical approaches is understood, it is easy to see that non-localities in the canonical approach are unavoidable.

The significance of the canonical lattice framework for the spin foam models is two-fold: First, a Hilbert space where the spin foam models prescribe the transition amplitudes is constructed by the canonical lattice framework (see chapter 4). Second, the canonical framework provides information about the nature of the geometricity conditions; the study presented in the previous chapter suggests a way to prove that that the quantization of the conditions by Barrett and Crane must produce a framework equivalent to Reisenberger's (see chapter 6).

In the introduction I stated that diffeomorphism invariant theories come from the idealization of a universe where only the interactions explained by them are present; as such, they are toy models for a unified theory. From this point of view, the extra background dependence introduced by our lattice regularization is undesirable. In addition, the work presented in the last chapter does not determine if the algebra of the symmetry generators of the model closes. These weaknesses of the lattice framework will be resurrected in a single step. From the work developed in chapter 2 and other works [11, 27] one can speculate that the lattice framework and the loop framework should converge in a certain way. This convergence would help the lattice approach filter out its extra background structure and would provide a systematic regularization procedure for loop quantization. I began working on the problems of diffeomorphism invariance and the continuum limit in the lattice, and solved them using ideas borrowed from loop quantization. The combinatorial and piecewise linear (PL) categories of loop quantization (chapter 4) were developed following this path.

Even before the work on the PL category was completed, the relation between the lattice framework and loop quantization produced two independent results.

The first result was obtained in collaboration with Alejandro Corichi [28]. We proved that working with abstract lattices (as opposed to embedded lattices), does not automatically solve the diffeomorphism constraint in the quantum theory. We used an exact lattice formulation of $2+1$ gravity to show that imposing the diffeomorphism constraint is imperative in obtaining a physically acceptable quantum theory.

The second work was a collaboration with Abhay Ashtekar and Alejandro Corichi [30]. One can easily see that the relation (2.16) written in the last chapter

$$E(\alpha)_A^{\beta} = -M(\alpha)_{\beta A}^{\ \ B} E(\beta)_B^{\alpha} \tag{3.1}$$

and the symplectic structure of the lattice models (2.19, 2.20)

$$\{E(\alpha)_A^{\beta}, E(\alpha)_B^{\beta}\} \quad = \quad f_{AB}^{\ \ D} E(\alpha)_D^{\beta} \tag{3.2}$$

$$\{E(\alpha)^{\beta}_{A}, M(\alpha)_{\beta B}^{\ \ C}\} \quad = \quad f_{AB}^{\ \ D} M(\alpha)_{\beta D}^{\ \ C} \tag{3.3}$$

are not independent; they are related by the Jacobi identity. In other words, one can not ask the E variables to act as left (or right) invariant vector fields and to have commutative actions simultaneously. Since algebraically identical relations lie at the heart of loop quantization, one is led to discover a non-commutativity in the quantum geometry defined by loop quantization. Then it was noticed the that area operators of surfaces that intersect do not commute. This feature is surprising because in the classical continuum theory the density weighted triads Poisson commute. We reviewed the quantization procedure, which originally assumed the commutativity of the triads, and analyzed the origin of the non-commutativity in detail. In particular, we showed that there is no anomaly in the quantization and indicated why the uncertainties associated with this non-commutativity become negligible in the semi-classical regime.

In taking the continuum limit the kinematics of the lattice framework has been modified fundamentally; the new model does not have finitely many degrees of freedom. This seems to be a drawback of the new strategy because divergences in the resulting quantum field theories may arise. However, the geometry of our model continues to have a discrete character. Even after the continuum limit has been taken, the geometric operators have a discrete spectrum (at the kinematical level). In the final picture, a region of finite dimension may have finitely many degrees of freedom. Another important feature of the kinematics of loop quantization is that the Euclidean regime of quantum gravity is simpler to treat; in particular, the Hamiltonian constraint of Lorentzian general relativity is much more complicated than the Euclidean one.

Loop quantization's geometrical character is suited for dealing with spatial diffeomorphism invariance. A representation of the diffeomorphism group in the Hilbert space of loop quantization is naturally constructed (see chapters 4, 5). However, one can show that there is no generator of spatial diffeomorphisms; only finite diffeomorphisms can be represented in the kinematical Hilbert space constructed by the loop framework. Any regularization of the diffeomorphism constraint is hopeless in generating diffeomorphisms. Therefore, adopting only the kinematics of loop quantization and keeping the original strategy for the dynamics, where diffeomorphism invariance was supposed to be achieved through constraints, is inconsistent. However, the essence of the strategy for dynamics is not in conflict with loop quantization's kinematics because the geometricity conditions do extend from the lattice to the loop approach. This is what makes the spin foam models attractive; by using BF theory and the geometricity conditions, they provide a manifestly 4d covariant dynamics, while keeping the kinematical structure of loop quantum gravity. An extended discussion of spin foam models will be presented in chapter 6.

If we adopted the dynamics of loop quantization, the strategy of our lattice framework would change completely. In particular, we would adopt a strategy where the constraints of GR are treated in a fundamentally different manner. In spite of not having generators of spatial diffeomorphisms, the loop framework leaves the Hamiltonian constraint to be treated as a usual constraint. First one regularizes it, and then, one finds its kernel. BF theory is no longer an intermediate step in the quantization of general relativity. Not having diffeomorphism generators, but having a Hamiltonian constraint, means that even after having dealt with all the constraints, one does not have a manageable representation of the constraint algebra. This is problematic because one cannot verify whether or not there is an anomaly. An extended discussion of loop quantum gravity's dynamics is given in chapter 6.

Chapter 4

Combinatorial quantum gauge theories

This chapter is based on an article tilted "A Combinatorial Approach to Diffeomorphism Invariant Quantum Gauge Theories" published in the *Journal of Mathematical Physics* **38**(11): 5663-5680, 1997.

Quantum gauge theories can be described using the holonomies along the edges of a regular lattice as basic configuration observables. This idea, pioneered by Mandelstam and followed by Wilson [31], is now the basis of the modern lattice gauge theory. In diffeomorphism invariant gauge theories (like gravity using Ashtekar variables [16] or Yang-Mills coupled to gravity) , the use of Wilson loops as primary observables of the theory led to the discovery of an interesting relation between quantum gauge theories and knot theory [32].

Twenty years after the early works, the notion of Wilson loops was extended and serves as a rigorous foundation of quantum gauge field theory [33]. The modern approach to *loop quantization* rests on the following idea: Begin by considering "the family of all the possible lattice gauge theories" defined on graphs whose edges are embedded in the base space. Then use a projective structure to organize the repeated information from graphs that share edges. For a manageable theory, the precise definition of "the family of all the possible lattice gauge theories" had to avoid situations where two different edges intersect each other an infinite number of times. The first solution to this problem [26] led to the framework referred as the analytic category of loop quantization; by restricting the set of allowed graphs Γ_ω to contain only graphs with piecewise analytic edges, one acquires a controllable theory. In the analytic category the diffeomorphisms are restricted to be analytic accordingly. After a subtle analysis, it was possible to sacrifice part of the simplicity of the results of the analytic case and extend the theory to the smooth category [34].

While the foundations were solidifying, the theory also produced its first kinematical results for quantum gravity (the canonical quantization of gravity expressed in terms of Ashtekar variables). Regularized expressions for operators measuring the area of surfaces and volume of regions were developed [7]. These operators were also diagonalized and its eigenvectors were found to be labeled by spin networks (one-dimensional objects). In other words, a picture of polymer-like geometry arises from quantum gravity [35]. The usual smooth geometry of physical space is considered a semiclassical/macroscopic effect. However, the foundations behind this polymer-like picture of quantum geometry require space to be an analytic manifold. This peculiar situation was the main motivation for the work presented in this chapter.

In this chapter we present two quantum models: the combinatorial and the piecewise linear (PL) categories. The intention is to keep a simple framework that minimizes background structure and is suited to a polymer-like geometry, but that can still recover the classical macroscopic theory. Both models are based on the projective techniques used for the analytic and smooth categories; again, the difference relies on the family of graphs Γ considered and the corresponding "diffeomorphisms."

In the the piecewise linear category we fix a piecewise linear structure in the space manifold to specify the elements of the family of graphs Γ_{PL} that define the Hilbert space. A piecewise linear structure on a manifold Σ can be specified by a division of the manifold into cells with a fixed affine structure (flat connection). Also it can be specified by a triangulation, that is, a fixed homeomorphism $\varphi : \Sigma \to \Sigma_0$ where $\Sigma_0 \subset R^{2n+1}$ is a n-dimensional polyhedron with a fixed decomposition into simplices. An element of Γ_{PL} is a graph whose edges are piecewise linear according to the fixed PL structure. This seems to be far from a background-free situation, but a PL structure is much weaker than an analytic structure; the same PL structure can be specified by any refinement of the original triangulation. Furthermore, we will prove that in three (or less) dimensions different choices of PL structures yield unitarily equivalent representations of the algebra of physical observables. This result is of particular interest for $3+1$ ($2+1$ or $1+1$) quantum models of pure gravity or of gravity coupled to Yang-Mills fields. To avoid confusion, we stress that the piecewise linear spaces used in this approach are not directly related to the ones used in Regge calculus. In simplified theories of gravity, like $2+1$ gravity and BF theory, the lattice dual to the one induced by one of our piecewise linear spaces can be successfully related to a Regge lattice [13, 2] (see chapter 2). On the other hand, our approach contains a treatment based in cubic lattices as a particular case; the difference with the usual lattice gauge theory is that the continuum limit is taken by considering every lattice instead of just one.

The manifestly combinatorial model has two main ingredients: simplicial complexes that describe geometry in combinatorial fashion, and a refinement mechanism that makes it capable to describe field theories. If we use a simplicial complex as the starting point of our combinatorial approach, the resulting model would be appropriate to describe topological field theories, but we want to generate a model for gauge theories with local degrees of freedom. A way to achieve this goal is to replace physical space (the base space) with a sequence of simplicial complexes $K_0, K_1, \ldots$ that are finer and finer. Our combinatorial model for quantum gauge theory is based in the family of graphs defined using our combinatorial representation of space.

Even though the PL and the combinatorial categories are closely related, the resulting kinematical Hilbert spaces $\mathcal{H}_{\mathrm{kin}_{\mathrm{PL}}}$ and $\mathcal{H}_{\mathrm{kin}_{\mathrm{C}}}$ are dramatically different. While the combinatorial Hilbert space $\mathcal{H}_{\mathrm{kin}_{\mathrm{C}}}$ is separable (admits a countable basis), $\mathcal{H}_{\mathrm{kin}_{\mathrm{PL}}}$ (like the Hilbert space constructed from the analytic category) is much bigger.

Physically, what we need is a Hilbert space to represent physical (gauge and "diffeomorphism" invariant) observables; such Hilbert space can be constructed by "averaging" the states of the kinematic Hilbert space to produce physical states. An encouraging result is that the two models produce unitarily equivalent representations of the algebra of physical observables in the naturally isomorphic separable Hilbert spaces $\mathcal{H}_{\mathrm{diff}_{\mathrm{PL}}}, \mathcal{H}_{\mathrm{diff}_{\mathrm{C}}}$. Separability in the combinatorial case is no surprise, and that both spaces of physical states (PL and combinatorial) are isomorphic follows from the fact that every knot-class of piecewise linear graphs has a representative that fits in our combinatorial representation of space.

Two aspects of the loop approach to gauge theory are enhanced in its combinatorial version. On the mathematical-physics side, other approaches to quantum gravity coming from topological quantum field theory [29] are much closer to the combinatorial category than they are to the analytic or smooth categories. On the practical side, the loop approach to quantum gauge theory is at least as attractive; a powerful computational technique comes built into this approach. Given any state in the Hilbert space of the continuum we can express it, to any desired accuracy, as a finite linear combination of states that come from the Hilbert space of a lattice gauge theory. Therefore, the matrix elements of every bounded operator can be computed, to any desired accuracy, in the Hilbert space of a lattice gauge theory. In this respect, the combinatorial picture presented in this chapter is favored because it is best suited for a computer implementation.

We organize this chapter as follows. Section 4.1 reviews the general procedure to construct the kinematical Hilbert space in the continuum starting from a family of lattice gauge theories. Then, in section 4.2, we carry out the procedure in the combinatorial and PL frameworks. In section 4.3, we construct the Hilbert space of "diffeomorphism" invariant states. We treat separately the PL and combinatorial categories. Then we prove that the combinatorial and PL frameworks provide unitarily equivalent representations of the algebra of physical observables. We also prove that the mentioned algebra of physical observables is independent of the background PL structure when the dimension of the space manifold is three or less. A summary, an analysis of some problems from the combinatorial perspective and a comparison with the analytic category are the subjects of the concluding section.

4.1 The Continuum Limit

From quantum gauge theory in the lattice we can define the continuum theory via the projective limit.This section reviews the necessary techniques. A connection on a principal bundle is characterized by the group element that it assigns to every possible path in the base space. Historically, this simple observation led to treat the set of holonomies for all the loops of the base space as the basic configuration observables to be promoted to operators.

Now we start the construction of a kinematical Hilbert space for quantum gauge theories. To avoid extra complications, we only treat cases with a compact base space Σ and we restrict our attention to trivializable principal bundles over Σ with the gauge group G being a compact connected Lie group. For convenience, we start with a fixed trivialization. In the modern approach (Baez, Ashtekar et al [26]) the concept of paths or loops has been extended to that of graphs $\gamma \subset \Sigma$ whose edges, in contrast with their predecessors, are allowed to intersect.

A *graph* γ is, by definition, a *finite* set E_γ of oriented edges and a set V_γ of vertices satisfying the following conditions:

- $e \in E_\gamma$ implies $e^{-1} \in E_\gamma$.

- The vertex set is the set of boundary points of the edges.
- The intersection set of two different edges $e_1, e_2 \in E_\gamma$ $(e_1 \neq e_2, e_1 \neq e_2^{-1})$ is a subset of the vertex set.

Generally an edge $e \in E_\gamma$, is considered to be an equivalence class of not self-intersecting curves, under orientation preserving reparametrizations. Formally, $e := [e'(I) \subset \Sigma]$ such that $e'(I) \approx I$, where we denoted the unit interval by $I = [0,1]$. Composition of edges e, f is defined if they intersect only at the final point of the initial edge and the initial point of the final edge $e'(I) \cap f'(I) = e'(1) = f'(0)$. Then the composition is defined by $f \circ e := [f' \circ e'(I)]$; and given an edge $e := [e']$ the edge defined by paths with the opposite orientation is denoted by $e^{-1} := [e'^{-1}]$.

The idea of considering "every possible path" in the base space to construct the space of generalized connections has to be made precise. Different choices in the class of edges that form the family of graphs considered lead to the different categories – analytic, smooth, PL and combinatorial– of this general approach to diffeomorphism invariant quantum gauge theories. We denote a generic family of graphs by Γ, and the analytic, smooth and combinatoric families by $\Gamma_\omega, \Gamma_\infty$, $\Gamma_{\rm PL}$ and Γ_C.

A connection on a graph assigns a group element to each of the $2N_1$ graph's edges. Therefore, we can identify the space of connections $\mathcal{A}_\gamma$ of graph γ with G^{N_1}. An element $A \in \mathcal{A}_\gamma$ is represented by $(A(e_1), A(e_1^{-1}) = A(e_1)^{-1}, \ldots, A(e_{N_1}), A(e_{N_1}^{-1}) = A(e_{N_1})^{-1})$, where $A(e_i) \in G$.

The collection of the spaces $\mathcal{A}_\gamma$ for every graph $\gamma \in \Gamma$ gives an *over-complete* description of the space of generalized connections in the category specified by Γ. For example, Γ_ω determines the analytic category and Γ_C specifies the combinatorial category.

It is possible to organize all the repeated information by means of a projective structure. We say that graph γ is a refinement of graph γ' $(\gamma \geq \gamma')$ if the edges of γ' are "contained" in edges of γ; more precisely, if $e \in \gamma'$ then either $e = e_1$ or $e = e_1 \circ \ldots \circ e_n$ for some $e_1, \ldots, e_n \in \gamma$. Given any two graphs related by refinement $\gamma \geq \gamma'$ there is a projection $p_{\gamma' \gamma} : \mathcal{A}_\gamma \to \mathcal{A}_{\gamma'}$

$$(A(e_1), A(e_2), \ldots, A(e_{N_1})) \xrightarrow{p_{\gamma'\gamma}} (A'(e_1) = A(e_2)A(e_1), A'(e_2), \ldots, A'(e_{N_1})) \tag{4.1}$$

where $e = e_1 \circ e_2$, $e \in \gamma'$, $e_1, e_2 \in \gamma$.

The projection map and the refinement relation have two properties that will allow us to define $\overline{\mathcal{A}}$ as "the space of connections of the finest lattice." First, we can easily check that $p_{\gamma \gamma'} \circ p_{\gamma' \gamma''} = p_{\gamma \gamma''}$. Second, equipped with the refinement relation "$\geq$", the set Γ is a partially ordered, directed set; i.e. for all γ, γ' and γ'' in Γ we have:

$$\gamma \geq \gamma\,; \quad \gamma \geq \gamma' \text{ and } \gamma' \geq \gamma \Rightarrow \gamma = \gamma'; \;\; \gamma \geq \gamma' \text{ and } \gamma' \geq \gamma'' \Rightarrow \gamma \geq \gamma''; \tag{4.2}$$

and, given any $\gamma', \gamma'' \in \Gamma$, there exists $\gamma \in \Gamma$ such that

$$\gamma \geq \gamma' \quad \text{and} \quad \gamma \geq \gamma''. \tag{4.3}$$

This last property, that Γ is directed, is the only non trivial property; it will be proved for the PL and the combinatorial categories in the next section. The *projective limit* of the spaces of connections of all graphs yields the space of *generalized connections* $\overline{\mathcal{A}}$

$$\overline{\mathcal{A}} := \left\{ (A_\gamma)_{\gamma\in\Gamma} \in \prod_{\gamma\in\Gamma} \mathcal{A}_\gamma \;\;:\;\; \gamma' \geq \gamma \Rightarrow p_{\gamma\gamma'} A_{\gamma'} = A_\gamma \right\}. \tag{4.4}$$

That is, the projective limit is contained in the cartesian product of the spaces of connections of all graphs in Γ, subject to the consistency conditions stated above. There is a canonical projection p_γ from the space $\overline{\mathcal{A}}$ to the spaces $\mathcal{A}_\gamma$ given by,

$$p_\gamma \;\;:\; \overline{\mathcal{A}} \to \mathcal{A}_\gamma, \quad p_\gamma((A_{\gamma'})_{\gamma'\in\Gamma}) := A_\gamma. \tag{4.5}$$

With this projection, functions f_γ defined on the space $\mathcal{A}_\gamma$ can be pulled-back to $\mathrm{Fun}(\overline{\mathcal{A}})$. Such functions are called *cylindrical functions*. The sup norm

$$||f||_\infty = \sup_{A\in\mathcal{A}_\gamma} |f(A)| \tag{4.6}$$

can be used to complete the space of cylindrical functions. As result we get the Abelian C^* algebra usually denoted by $\mathrm{Cyl}(\overline{\mathcal{A}})$; to simplify the notation, in the rest of the chapter we will denote this algebra by $\mathrm{Cyl}_\bigcirc$, where $\bigcirc = \omega, \infty, \mathrm{PL}, C$ labels the family of graphs defining the space of cylindrical functions considered.

The uniform generalized measure $\mu_0 : \mathrm{Cyl}_\bigcirc \to C$, sometimes called the Ashtekar-Lewandowski measure [36], is induced in $\mathcal{A}$ by the uniform (Haar) measure on the spaces $\mathcal{A}_\gamma = G^{N_1}$. Other gauge invariant measures are available; when they are diffeomorphism invariant they induce "generalized knot invariants" (see [37]). Finally, we define the kinematical Hilbert space to be the completion of $\mathrm{Cyl}(\mathcal{A})$ on the norm induced by the (strictly positive) generalized measure μ_0

$$\mathcal{H}_{\mathrm{kin}} := L^2(\overline{\mathcal{A}}, d\mu_0). \tag{4.7}$$

This construction yields a cyclic representation of the algebra of cylindrical functions, the so called *connection representation*. Given a function defined on a lattice γ, for example the trace of the holonomy T_α along a loop α contained in γ, the corresponding operator $\hat{T}_\alpha$ will act by multiplication on states $\Psi_\gamma \in \mathcal{H}_{\mathrm{kin}}$:

$$(\hat{T}_\alpha \cdot \Psi_\gamma)(\bar{A}) := T_\alpha(\bar{A})\Psi_\gamma(\bar{A}). \tag{4.8}$$

A complete set of Hermitian momentum operators on the Hilbert space $L^2(G_e, d\mu_{\mathrm{Haar}})$ of a graph with a single edge e come from the left $L_e(f)$ and right invariant $R_e(f)$ vector fields on G_e as labeled by $f \in \mathrm{Lie}(G_e)$. These momentum operators are compatible with the projective structure [38]; thus, the set of momentum operators

$$X_{\alpha,e}(f) = \begin{cases} L_e(f) & \text{if edge } e \text{ goes out of vertex } \alpha \\ -R_e(f) & \text{if edge } e \text{ comes into vertex } \alpha \end{cases} \tag{4.9}$$

is a complete set of Hermitian momentum operators on $\mathcal{H}_{\rm kin}$ when we use the generalized measure μ_0. In regularized expressions of operators involving the triad, the place of the triad is taken by the vector fields X; therefore, the measure μ_0 incorporates the reality conditions induced by the classical gauge theory being quantized.

Our main goal is to construct a Hilbert space where we can represent the algebra of physical (gauge and diffeomorphism invariant) observables. As it is customary we will proceed in steps; in this section we deal with the issue of gauge invariance and in the next with that of diffeomorphism invariance. If we had chosen to generate the space of states invariant under both symmetries simultaneously we would arrive at the same result.

A finite gauge transformation takes the holonomy A_{e_1} to $g(\alpha)A_{e_1}g(\beta)^{-1}$ (where edge e_1 goes from vertex α to vertex β). Then a quantum gauge transformation is given by the unitary transformation

$$G(g)\Psi_\gamma(A_{e_1},\dots A_{e_n}) := \Psi_\gamma(g(\alpha)A_{e_1}g(\beta)^{-1},\dots g(\mu)A_{e_n}g(\nu)^{-1}) \tag{4.10}$$

Gauge transformations are just generalizations of right and left translations in the group. This implies that they are generated by left and right invariant vector fields. Given a graph γ, $C_\alpha(f)$ generates gauge transformations at vertex α. Therefore gauge invariance of $\Psi_\gamma = \Psi_\gamma(A_{e_1},\dots A_{e_n})$ at vertex α means that it lies in the kernel of the Gauss constraint

$$C_\alpha(f)\cdot\Psi_\gamma := \sum_{e\to\alpha} X^I_e\cdot\Psi_\gamma = 0 \quad , \tag{4.11}$$

where the sum is taken over all the edges e that start at vertex α. Because it is a real linear combination of the momentum operators (4.9), the Gauss constraint is essentially self-adjoint on $\mathcal{H}_{\rm kin}$.

We could construct the space of connections modulo gauge transformations of a graph $\mathcal{A}_\gamma/\mathcal{G}_\gamma$. Then, using the same projective machinery, we could construct the Hilbert space $L^2(\overline{\mathcal{A}/\mathcal{G}}, d\nu_0)$. It is easy to see that the space of gauge invariant functions of $L^2(\overline{\mathcal{A}}, d\mu_0)$ is naturally isomorphic to $\mathcal{H}'_{\rm kin} = L^2(\overline{\mathcal{A}/\mathcal{G}}, d\nu_0)$ if the measure ν_0 is the one induced by μ_0. The space $\mathcal{H}'_{\rm kin}$ of gauge invariant functions is spanned by spin network states. Spin network states are cylindrical functions $S_{\vec{\gamma},j(e),c(v)}(A)$ labeled by an *oriented graph* (a graph γ plus a choice of either $e\in E_\gamma$ or $e^{-1}\in E_\gamma$, for each edge in γ constitute an oriented graph $\vec{\gamma}$) whose edges and vertices are colored. The "colors" $j(e)$ on the edges $e\in E_\gamma$ assign a non trivial irreducible representation of the gauge group to the edges. And the "colors" $c(v)$ on the vertices $v\in V_\gamma$ assign to each vertex a gauge invariant contractor (intertwining operator) that has indices in the representations determined by the colored edges that meet at the vertex. The spin network states are defined by

$$S_{\vec{\gamma},j(e),c(v)}(A) = \prod_{e\in E_{\vec{\gamma}}} \pi_{j(e)}[A(e)]\cdot\prod_{v\in V_{\vec{\gamma}}} c(v) \quad , \tag{4.12}$$

where '$\cdot$' stands for contraction of all the indices of the matrices attached to the edges with the indices of the intertwiners attached to the vertices. In the inner product that the uniform measure μ_0 induces in $\mathcal{H}'_{\text{kin}}$ two spin network states are orthogonal if they are not labeled by the same (unoriented) graph or if their edge's colors are different. For calculational purposes it is convenient to choose an orthonormal basis for $\mathcal{H}'_{\text{kin}}$ by normalized spin network states with special labels of the intertwining operators assigned to the vertices; see [39].

4.2 PL and combinatorial categories

In this section we construct two quantum models using the general framework outlined above. First the family of piecewise linear (PL) graphs is introduced. Then we prove that it is a partially ordered, directed set. As a result, the algebra of functions of the connection defined by the PL graphs has a cyclic representation in the Hilbert space $\mathcal{H}_{\text{kin}_{\text{PL}}}$. The second subsection briefly reviews some elements of combinatorial topology while constructing the family of combinatorial graphs. In this case, the resulting algebra of functions is represented in the separable Hilbert space $\mathcal{H}_{\text{kin}_{\text{C}}}$. While at this level the two quantum models yield completely different Hilbert spaces, in the section 4.3 we will prove that the corresponding spaces of "diffeomorphism" invariant states are naturally isomorphic.

4.2.1 The PL category

To specify the elements of the family of graphs Γ_{PL} that define the Hilbert space of the PL category we need a fixed piecewise linear structure on space Σ. A piecewise linear structure on a manifold Σ can be specified by a division of the manifold into cells with a fixed affine structure (flat connection). Also it can be specified by a triangulation, that is, a fixed homeomorphism $\varphi : \Sigma \to \Sigma_0$ where Σ_0 is a n-dimensional polyhedron with a fixed decomposition into simplices. To be more explicit, we can use the fact that every n-dimensional polyhedron can be embedded in R^{2n+1} and consider from the beginning $\Sigma_0 \subset R^{2n+1}$. Then Σ_0 can be decomposed into a collection of convex cells (geometrical simplices). A *geometric simplex* in R^{2n+1} is simply the convex region defined by its set of vertices $\{\mathbf{s}_0, \ldots, \mathbf{s}_k\}$, $\mathbf{s}_i \in R^{2n+1}$

$$\Delta(\{\mathbf{s}_0, \ldots, \mathbf{s}_k\}) = \{\mathbf{s} = \sum_{i=0}^{k} t_i \mathbf{s}_i\} \tag{4.13}$$

where $t_i \in [0,1]$ and $\sum_{i=0}^{k} t_i = 1$. The triangulation of Σ_0 fixes an affine structure in its cells, namely, a PL structure. Using the local affine coordinate systems t_i, we can decide which curves are straight lines inside any cell. Then a piecewise linear curve in Σ_0 is a curve that is straight inside every cell except for a *finite* set of points; in this set of points and in the points where it crosses the boundaries of the cells the curve bends, but is continuous.

A piecewise linear graph $\gamma \in \Gamma_{\mathrm{PL}}$ is a graph (according to the definition given in the previous section) such that every edge $e \in E_\gamma$ is piecewise linear.

In the previous section we gave a natural partial order ("refinement relation", $\geq$) for any family of graphs. Our task is now to prove that the partially ordered set Γ_{PL} is a projective family; once we prove this property, the general procedure outlined in the previous section gives us the Hilbert space of the PL category.

The only non-trivial property to prove is that the family of graphs Γ_{PL} is directed. For instance, according to the definition of a graph given in last section, the family of all the graphs with piecewise smooth edges is not directed. In this case, two edges of different graphs can intersect an infinite number of times; such two graphs would only accept a common refinement with an infinite number of edges, that according to our definition is not a graph.

We will construct a graph γ_3 that refines two given graphs γ_1 and γ_2.

A trivial property of PL edges lies in the heart of our construction; due to its importance, it is stated as a lemma.

LEMMA 4.1. *Given two edges of different graphs $e_1 \in \gamma_1$ and $e_2 \in \gamma_2$, we know that $e_1 \cap e_2$ has* finitely many *connected components. These connected components are either isolated points or piecewise linear segments.*

Now we start our construction. First we note that every graph γ is refined by a graph γ' constructed from γ simply by adding a finite number of vertices $v \in V'$ in the interior of its edges (and by splitting the edges in the points where a new vertex sits).

Because of lemma 4.1, we know that given two graphs $\gamma_1, \gamma_2 \in \Gamma_{\mathrm{PL}}$ we can refine each of them trivially by adding *finitely many* new vertices to form the graphs $\gamma_1' \geq \gamma_1, \gamma_2' \geq \gamma_2$ that satisfy the following property. Every edge $e_1 \in E_{\gamma_1'}$ falls into one of the three categories given bellow:

- e_1 does not intersect any edge of γ_2'.
- e_1 is also an edge of γ_2'; $e_1 \in E_{\gamma_2'}$.
- e_1 intersects an edge e_2 of γ_2' at vertices (one or two) of both graphs $e_1 \cap e_2 \subset V_{\gamma_1'}$, $e_1 \cap e_2 \subset V_{\gamma_2'}$.

A direct consequence of these properties is the following:

LEMMA 4.2. *The graph γ_3 defined by $E_{\gamma_3} = E_{\gamma_1'} \cup E_{\gamma_2'}$ and $V_{\gamma_3} = V_{\gamma_1'} \cup V_{\gamma_2'}$ is a refinement of γ_1' and γ_2'. By the properties of the partial ordering relation it follows that γ_3 is also a refinement of the original graphs $\gamma_3 \geq \gamma_1, \gamma_3 \geq \gamma_2$; thus the family of piecewise linear graphs Γ_{PL} is a projective family.*

In the light of lemma 4.2, the rest of the construction is a simple application of the general framework described in the previous section. There is a canonical projection

p_γ from the space of generalized connections $\overline{\mathcal{A}}_{\mathrm{PL}}$ to the spaces of connections $\mathcal{A}_\gamma$ on graphs $\gamma \in \Gamma_{\mathrm{PL}}$ given by,

$$p_\gamma \;:\; \overline{\mathcal{A}}_{\mathrm{PL}} \to \mathcal{A}_\gamma, \quad p_\gamma((A_{\gamma'})_{\gamma' \in \Gamma_{\mathrm{PL}}}) := A_\gamma. \tag{4.14}$$

This projective structure is the main ingredient that yields the Hilbert space of the connection representation in the PL category. Below we state our result concisely.

THEOREM 4.1. *The completion (in the sup norm) of the family of functions* $p^*_\gamma f_\gamma(\bar{A})$*, defined by graphs* $\gamma \in \Gamma_{\mathrm{PL}}$*, is an Abelian* C^* *algebra* $\mathrm{Cyl}_{\mathrm{PL}}$*. A cyclic representation of* $\mathrm{Cyl}_{\mathrm{PL}}$ *is provided by the Hilbert space*

$$\mathcal{H}_{\mathrm{kin}_{\mathrm{PL}}} := L^2(\overline{\mathcal{A}}_{\mathrm{PL}}, d\mu_0). \tag{4.15}$$

that results after completing $\mathrm{Cyl}_{\mathrm{PL}}$ *in the norm provided by the Ashtekar-Lewandowski measure* μ_0.

In the manner described in the previous section we can also consider the space of gauge invariant states and obtain $\mathcal{H}'_{\mathrm{kin}_{\mathrm{PL}}}$ that is is spanned by spin network states labeled by piecewise linear graphs.

4.2.2 The combinatorial category

In this subsection we introduce the family of combinatorial graphs that leads to a manifestly combinatorial approach to quantum gauge theory. The construction of combinatorial graphs uses as a corner stone the same stone that serves as the combinatorial foundation of topology. Thus, our construction provides a quantum/combinatorial model for physical space, the space where physical processes take place.

Simplicial complexes appear first as the combinatorial means of capturing the topological information of a topological space X. By definition, a *simplicial complex* K is a set of finite sets closed under formation of subsets, formally:

$$x \in K \,\mathrm{and}\, y \subset x \;\Rightarrow\; y \in K \quad . \tag{4.16}$$

A member of a simplicial complex $x \in K$ is called an n-simplex if it has $n+1$ elements; n is the dimension of x. Generically, the set of which all simplices are subsets is called the vertex set and denoted by Λ. Some examples of simplicial complexes are given in figure 4.1

Given an open cover $\mathcal{U}(\Lambda) = \{U_\lambda : \lambda \in \Lambda\}$ of a topological space X the information about the relative position of the open sets $U_1, U_2, ... \in \mathcal{U}(\Lambda)$ is the combinatorial information that the *nerve* $K(\Lambda)$ of $\mathcal{U}(\Lambda)$ casts. $K(\Lambda)$ is the simplicial complex formed by all finite subsets of Λ such that

$$\bigcap_{\lambda \in \Lambda} U_\lambda \neq \emptyset \tag{4.17}$$

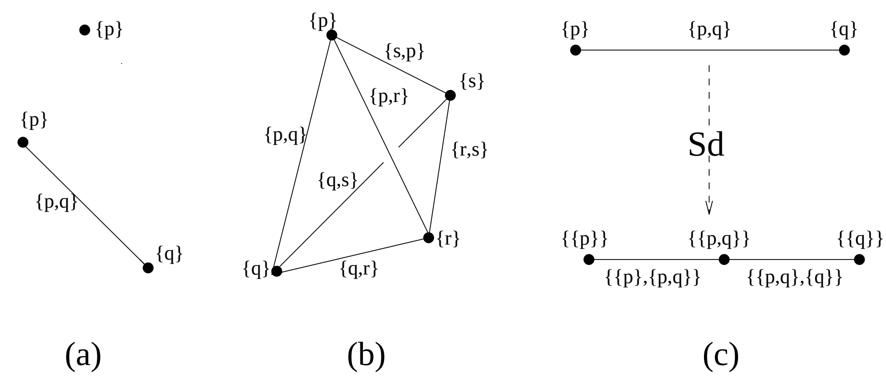

Fig. 4.1. a) Geometrical representations of a zero dimensional simplex $x = \{p\}$ and a one dimensional simplex $x = \{p, q\}$. *The simplices are the sets*; in the figures, what we draw are the geometric realizations Δ_x of the abstract simplices x.

Fig. 4.2. b) A two dimensional complex is a set of simplices of dimension smaller or equal to two. In this case the complex $K = \{\ \{p\}, \{q\}, \{r\}, \{s\}, \{p,q\}, \{q,r\}, \{r,p\}, \{s,p\}, \{s,q\}, \{s,r\}, \{p,q,r\}, \{p,q,s\}, \{q,r,s\}, \{r,p,s\}\}$ represents a sphere S^2. Figure (b) is the geometric realization $\|K^1\|$ of the one dimensional subcomplex of K given by $K^1 = \{\{p\},\{q\},\{r\},\{s\},\ \{p,q\},\{q,r\},\{r,p\},\{s,p\},\{s,q\},\{s,r\}$.

Fig. 4.3. c) The vertices of the barycentric subdivision $Sd(K)$ are the simplices of K. For example $Sd(K) = \{\{\{p\}\},\{\{p,q\}\},\{\{q\}\},\{\{p\},\{p,q\}\},\{\{p,q\},\{q\}\}\}$ for $K = \{\{p\},\{q\},\{p,q\}\}$.

Using the information encoded in the $K(\Lambda)$ one can often recover the topological space X. More precisely, every open cover $\mathcal{U}(\Lambda)$ of X admits a refinement $\mathcal{U}'(\Lambda')$ such that the geometric realization (to be defined bellow) of its nerve is homeomorphic to X, $|K(\Lambda')| \approx X$. This is the sense in which simplicial complexes constitute a combinatorial foundation of topology.

A simplicial complex stores topological information combinatorially, but the same information can be encoded in a geometric fashion (see [40]). The *geometric realization* $|K|$ of a simplicial complex $K = K(\Lambda)$, is the subset of R^{Λ} given by $|K| := \bigcup_{x \in K} \Delta_x$ where Δ_x is a geometrical simplex represented as a segment of a plane of codimension one, embedded in R^x; more precisely,

$$\Delta_x := \left\{ \mathbf{s} := (\mathbf{s}_\lambda : \lambda \in \xi) \in \mathbf{I}^{\mathbf{x}} : \sum_{\lambda \in \mathbf{x}} \mathbf{s}_\lambda = \mathbf{1} \right\} \tag{4.18}$$

where $I = [0, 1]$ is the unit interval. The topology of $|K|$ is determined by declaring all its geometrical simplices Δ_x to be closed sets.

Our main purpose is to find a combinatorial analog of a generalized connection. We need to find the appropriate concept of the space of all combinatorial graphs; then a generalized connection will be an assignment of group elements to the edges of the graphs. We could fix a simplicial complex K to represent the base space and consider that a combinatorial graph is a one-dimensional subcomplex $\gamma \subset K$. The resulting model would properly describe topological field theories, but we want to generate a model for gauge theories with local degrees of freedom. To achieve our goal, we replace physical space (the base space) with a sequence of simplicial complexes $K_0, K_1, \ldots$ that are finer and finer.

The concept of barycentric subdivision give us the option of generating finer and finer simplicial complexes. Given a simplicial complex K its *barycentric subdivision* $Sd(K)$ is defined as the simplicial complex constructed by assigning a vertex to every simplex of K, $\Lambda = K$. Then, the simplices of $Sd(K)$ are the finite subsets $X \subset \Lambda$ that satisfy

$$x, y \in X \Rightarrow x \subset y \text{ or } y \subset x \tag{4.19}$$

A geometric representation of the operation barycentric subdivision Sd is given in figure 1.

Our approach to quantum gauge theory replaces the base space Σ with a sequence of simplicial complexes $\{K, Sd(K), \ldots, Sd^n(K), \ldots\}$ such that $|K| \approx \Sigma_0$, where Σ_0 is a compact Hausdorff three dimensional manifold. This concept of space leads to the definition of combinatorial graphs.

A *combinatorial graph* $\gamma \in \Gamma_C$ is simply a graph, according to the definition given in the previous section, where the set of vertices V_γ and the set of edges E_γ are restricted to be subsets of the set of points $V(K)$ and the set of oriented paths $E(K)$.

In the combinatorial representation of space, a point $p \in V(K)$ is represented by an equivalence class of sequences of the kind

$$\{p_n, p_{n+1} = Sd(p_n), p_{n+2} = Sd^2(p_n), \ldots\} \tag{4.20}$$

of zero-dimensional simplices $p_n \in Sd^n(K)$, $p_{n+1} \in Sd^{n+1}(K)$, etc. Notably one single element of the sequence determines the whole sequence. Two sequences $\{p_n, p_{n+1} = Sd(p_n), \ldots\}$ $\{q_m, q_{m+1} = Sd(q_m), \ldots\}$ are equivalent if all their elements coincide, $p_s = q_s \in Sd^s(K)$ for all $s \geq \max(n, m)$.

The definition of oriented paths follows the same idea, but is a little more involved. First we will define paths, then oriented paths, and composition of oriented paths. A path $e \in P(K)$ is an equivalence class of sequences $\{e_n, e_{n+1} = Sd(e_n), \ldots\}$ of one dimensional subcomplexes $e_n \subset Sd^n(K)$ such that the geometric realizations of its elements are homeomorphic to the unit interval $|e_n| \approx I$. Again, two sequences $\{e_n, e_{n+1} = Sd(e_n), \ldots\}, \{f_m, \ldots\}$ are equivalent if all their elements coincide $e_s = f_s \in Sd^s(K)$ for all $s \geq \max(n, m)$.

An oriented path $e \in E(K)$ is a path $e' \in P(K)$ and a sequence of relations that order the vertices[1] of each of the one-dimensional subcomplexes e'_n in the path. We denote the initial point of a path by $e(0) \in V(K)$ and it is defined by the class of the sequence of initial vertices $e(0) = [\{e_n(0), e_{n+1}(0) = Sd(e_n(0)), \ldots\}] \in V(K)$; the final point of a combinatorial path is denoted by $e(1) \in V(K)$. Composition of two oriented paths $e, f \in E(K)$ is possible if they intersect only at the final point of the initial path and the initial point of the final path $[\{e_n \cap f_n, e_{n+1} \cap f_{n+1}, \ldots\}] = e(1) = f(0)$; it is denoted by $f \circ e \in E(K)$ and is defined by

$$f \circ e = [\{(f \circ e)_n = f_n \cup e_n, (f \circ e)_{n+1} = Sd((f \circ e)_n), \ldots\}] \qquad (4.21)$$

and the obvious sequence of ordering relations.

Given an oriented path $e \in E(K)$ its inverse $e^{-1} \in E(K)$ is defined by the same path $e' \in P(K)$ and the opposite orientation. Notice that the composition relation is not defined for e and e^{-1}; it is possible to define combinatorial curves that behave like usual curves, but it is not necessary for the purpose of the work presented in this chapter.

Once the set of edges E is endowed with the composition operation, the rest of our construction is almost a simple application of the general framework reviewed in the previous section. The only gap to be filled is proving that the family of combinatorial graphs Γ_C is directed.

To prove the directedness in the PL case we used the finiteness property stated in lemma 4.1; an adapted statement of this same property holds trivially in the combinatorial case.

LEMMA 4.3. *The intersection of two one dimensional subcomplexes $e_n, f_n \subset Sd^n(K)$, defining the paths $e, f \in P(K)$ respectively, has* finitely many *connected components. These connected components are either isolated zero-dimensional simplices or one-dimensional subcomplexes homeomorphic to the unit interval. That is,*

[1] Here the term vertex refers to a zero-dimensional simplex in the one of the one-dimensional subcomplexes e_n in the path e. It should not be confused with a vertex $v \in V_\gamma$ of a combinatorial graph.

$$e_n \cap f_n = (\bigcup_{i=1}^{N} p(i)_n) \bigcup (\bigcup_{j=1}^{M} g(j)_n) \tag{4.22}$$

where $p(i)_n \subset Sd^n(K)$ *is a zero-dimensional simplex and* $I \approx g(i)_n \subset Sd^n(K)$. *In addition,* $p(i)_n \cap p(j)_n = p(i)_n \cap g(j)_n = g(i)_n \cap g(j)_n = \emptyset$ *for all* $i \neq j$.

By defining the appropriate notion of union and intersection of classes of sequences we can state the result as

$$e \cap f = (\bigcup_{i=1}^{N} p(i)) \bigcup (\bigcup_{j=1}^{M} g(j)) \tag{4.23}$$

where $p(i) \in V(K)$, $g(j) \in P(K)$, *and* $p(i) \cap p(j) = p(i) \cap g(j) = g(i) \cap g(j) = \emptyset$ *for all* $i \neq j$.

Therefore, the construction of a graph $\gamma_3 \in \Gamma_C$ that refines two given graphs $\gamma_1, \gamma_2 \in \Gamma_C$ is just an adaptation of the construction given for the piecewise linear case.

Using lemma 4.3 it is easy to prove that given two graphs $\gamma_1, \gamma_2 \in \Gamma_C$ we can refine each of them trivially by adding *finitely many* new vertices; forming graphs $\gamma_1^{\prime} \geq \gamma_1, \gamma_2^{\prime} \geq \gamma_2$ such that every edge $e_1 \in E_{\gamma_1^{\prime}}$ falls in one of the three categories (4.2.1), (4.2.1), (4.2.1) itemized in the previous subsection.

From the previous construction the following lemma is evident.

LEMMA 4.4. *Let* γ_3 *be the graph defined by*
$V_{\gamma_3} := V_{\gamma_1^{\prime}} \cup V_{\gamma_2^{\prime}} \subset V(K)$ *and* $E_{\gamma_3} := E_{\gamma_1^{\prime}} \cup E_{\gamma_2^{\prime}} \subset E(K)$.

γ_3 *is a refinement of* $\gamma_1^{\prime}$ *and* $\gamma_2^{\prime}$. *By the properties of the partial ordering relation it follows that* γ_3 *is also a refinement of the original graphs* $\gamma_3 \geq \gamma_1, \gamma_3 \geq \gamma_2$; *thus the family of combinatorial graphs* Γ_C *is a projective family.*

Following the general framework described in the previous section we will complete the construction of our combinatorial/quantum model for gauge theory. There is a canonical projection p_γ from the space of generalized connections $\overline{\mathcal{A}}_C$ to the spaces of connections $\mathcal{A}_\gamma$ on graphs $\gamma \in \Gamma_C$ given by,

$$p_\gamma \ : \overline{\mathcal{A}}_C \to \mathcal{A}_\gamma, \quad p_\gamma((A_{\gamma^{\prime}})_{\gamma^{\prime} \in \Gamma_C}) := A_\gamma. \tag{4.24}$$

These projections are the key ingredient that yields the Hilbert space of the connection representation in the combinatorial category. Below we state our result concisely.

THEOREM 4.2. *The completion (in the sup norm) of the family of functions* $p_\gamma^* f_\gamma(\bar{A})$, *defined by graphs* $\gamma \in \Gamma_C$, *is an Abelian* C^* *algebra* Cyl_C. *A cyclic representation of* Cyl_C *is provided by the Hilbert space*

$$\mathcal{H}_{\mathrm{kin_C}} := L^2(\overline{\mathcal{A}}_C, d\mu_0). \tag{4.25}$$

that results after completing Cyl_C *in the norm provided by the Ashtekar-Lewandowski measure* μ_0.

As described in the previous section we can consider the space of gauge invariant states and get $\mathcal{H}'_{\mathrm{kin_C}}$ that is is spanned by spin network states labeled by combinatorial graphs.

The constructions, given in this and the previous subsection, of the Hilbert spaces for the piecewise linear and the combinatorial categories were similar. Despite the parallelism, the resulting Hilbert spaces are completely different. A property that marks the difference is the size of these Hilbert spaces.

THEOREM 4.3. *The Hilbert space $\mathcal{H}'_{\mathrm{kin_C}}$ is separable.*

Proof – We will prove that the spin network basis is countable in the combinatorial case.

We did not describe precisely the spin network basis, but we stated that two spin network states $S^1_{\vec{\gamma},j(e),c(v)}(A)$, $S^2_{\vec{\delta},j(e),c(v)}(A)$ are orthogonal if $\gamma \neq \delta$ or if their edge's colors are different.

Let $L_{\gamma,j(e)}$ be the space spanned by all the spin network states with labels $\vec{\gamma}, j(e)$. Our task is to determine a bound for $n = \dim(L_{\gamma,j(e)})$. We know that n is less than the number of labels that we would get by assigning not one integer but three integers to the graphs edges. The first integer $j(e)$ labels the irreducible representation assigned to e, and the other two $m_L(e), m_R(e)$ determine basis vectors in the vector space selected by $j(e)$. With these basis vectors sitting at both ends of every edge we can label any set of (generally non gauge invariant) contractors for the vertices.

Thus, the spin network basis is countable if the set of finite subsets of

$$E(K) \times \mathbf{N} \tag{4.26}$$

is countable. Then to prove the theorem we just have to show that the set $E(K)$ is countable, which in turn reduces to prove that the set of paths $P(K)$ is countable.

A path $e \in P(K)$ is determined by a sequence of one-dimensional subcomplexes that are all related by barycentric subdivision. Therefore, a path $e \in P(K)$ can be specified by just one one-dimensional subcomplex of an appropriate $Sd^n(K)$. A particular one-dimensional subcomplex can be described by specifying which of the one-dimensional and zero-dimensional simplices belong to it. We can use the set $\{0,1\}$ to specify which simplex belong or does not belong to a particular subcomlpex.

Therefore, there is an onto map

$$M : \bigcup_{n=1}^{\infty} Sd^n(K) \times \{0,1\} \to P(K) \tag{4.27}$$

since a countable union of finite sets is countable and each $Sd^n(K)$ is finite, we have proved that $P(K)$ is countable. $\diamondsuit$

4.3 Physical observables and physical states

In this section we construct the Hilbert space of physical states of our model for quantum gauge theory; there we can represent the algebra of physical (gauge and "diffeomorphism" invariant) observables. Our quantization procedure follows the same steps as in the analytic category; that is, it follows (a refined version of) the algebraic quantization program [4, 41]. When we deal with theories with extra constraints, like gravity, we need to solve these extra constraints to find the space of physical states.

Since the issue of "diffeomorphism" invariance acquires quite different faces in the PL and combinatorial categories, we tackle it first for the PL category. Then we find the space of physical states of the combinatorial category and prove that it is separable and isomorphic to the space of physical states of the PL category.

4.3.1 "Diffeomorphism" invariance in the PL category

Any operator can be defined by specifying its action on the space of cylindrical functions Cyl and then using continuity to extend it to the whole Hilbert space $\mathcal{H}_{\text{kin}}$. This is what we did to define the unitary operators induced by the gauge symmetry and it is what we will do in this section to define quantum "diffeomorphisms."

Our piecewise linear framework is based on the family of graphs Γ_{PL} selected by a fixed piecewise linear structure in Σ. Therefore, the role of "diffeomorphisms" is played by *piecewise linear homeomorphisms.* It is important to note that the space of such maps can be defined as

$$\text{Hom}_{\text{PL}}(\Sigma) := \{h \in \text{Hom}(\Sigma) : h(\Gamma_{\text{PL}}) = \Gamma_{\text{PL}}\} \quad . \tag{4.28}$$

The unitary operator $\hat{U}_h : \mathcal{H}_{\text{kin}_{\text{PL}}} \to \mathcal{H}_{\text{kin}_{\text{PL}}}$ induced by a piecewise linear homeomorphism h is determined by its action on cylindrical functions

$$\hat{U}_h \cdot \Psi_\gamma(\bar{A}) := \Psi_{h^{-1}(\gamma)}(\bar{A}). \tag{4.29}$$

In contrast with our treatment of gauge invariance, the space of diffeomorphism invariant states is not the kernel of any Hermitian operator; the reason is that the one-dimensional subgroups of the diffeomorphism group induce one-parameter families of unitary transformations that are not strongly continuous in our Hilbert space [26]. Another important difference is that the space of "diffeomorphism" invariant states cannot be made a subspace of the Hilbert space $\mathcal{H}_{\text{kin}_{\text{PL}}}$, the solutions are true distributions, i.e., they lie in a subspace of the topological dual of Cyl_{PL}.

A distribution $\bar{\phi} \in \text{Cyl}^*_{\text{PL}}$ is "diffeomorphism" invariant if

$$\bar{\phi}[\hat{U}_h \circ \psi] = \bar{\phi}[\psi] \quad \forall \;\; h \in \text{Hom}_{\text{PL}}(\Sigma) \quad \text{and} \quad \psi \in \text{Cyl}_{\text{PL}}. \tag{4.30}$$

Such distributions are constructed by "averaging" over the group $\text{Hom}_{\text{PL}}(\Sigma)$. The infinite size of $\text{Hom}_{\text{PL}}(\Sigma)$ makes a precise definition of the group average procedure very subtle. Here we follow the procedure used for the analytic category [26].

An inner product for the space of solutions is given by the same formula that defines the group averaging; therefore, a summation over all the elements of $\text{Hom}_{\text{PL}}(\Sigma)$ would yield states with infinite norm. In this sense, prescribing an adequate definition for the averaging over the group $\text{Hom}_{\text{PL}}(\Sigma)$ involves "renormalization." The issue is resolved once the following two observations have been made: First, the inner product between two states based on graphs $\gamma, \delta \in \Gamma_C$ must be zero unless there is a homeomorphism $h_0 \in \mathrm{Hom}_{\mathrm{PL}}(\Sigma)$ such that $\gamma = h_0\delta$. Second, our construction of generalized connections assigns group elements to unparametrized edges. Therefore, two homeomorphisms that restricted to a graph γ are equal except for a reparametrization of the edges of γ should be counted only once in our construction of group averaging of states based on graph γ. Thus, we define a map $\eta : \mathrm{Cyl}_{\mathrm{PL}} \to \mathrm{Cyl}^*_{\mathrm{PL}}$ that transforms any given gauge invariant cylindrical function into a "diffeomorphism" invariant distribution. We define η acting on spin network states, then by *antilinearity* we can extend its action to any gauge invariant cylindrical function. Averaging a spin network state $S_{\vec{\gamma},j(e),c(v)}$ produces a s-knot state $s_{[\vec{\gamma}],j(e),c(v)} = \eta(S_{\vec{\gamma},j(e),c(v)}) \in \mathrm{Cyl}^*_{\mathrm{PL}}$ defined by

$$\begin{aligned} s_{[\vec{\gamma}],j(e),c(v)}[S_{\vec{\delta}',j(e),c(v)}] \;&:=\; \delta_{[\gamma][\delta]}\, a([\gamma]) \\ &\cdot \sum_{[h]\in \mathrm{GS}(\gamma)} \langle S_{U_{h\cdot \vec{h}_0}\gamma,j(e),c(v)} | S_{\vec{\delta}',j(e),c(v)} \rangle \end{aligned} \tag{4.31}$$

where $\delta_{[\gamma][\delta]}$ is non vanishing only if there is a homeomorphism $h_0 \in \mathrm{Hom}_{\mathrm{PL}}(\Sigma)$ that maps γ to δ, $a([\gamma])$ is a normalization parameter, and $h \in \mathrm{Hom}_{\mathrm{PL}}(\Sigma)$ is any element in the class of $[h] \in \mathrm{GS}(\gamma)$. The discrete group $\mathrm{GS}(\gamma)$ is the group of symmetries of γ; i.e. elements of $\mathrm{GS}(\gamma)$ are maps between the edges of γ. The group can be constructed from subgroups of $\mathrm{Hom}_{\mathrm{PL}}(\Sigma)$ as follows: $\mathrm{GS}(\gamma) = \mathrm{Iso}(\gamma)/\mathrm{TA}(\gamma)$ where $\mathrm{Iso}(\gamma)$ is the subgroup of $\mathrm{Hom}_{\mathrm{PL}}(\Sigma)$ that maps γ to itself, and the elements of $\mathrm{TA}(\gamma)$ are the ones that preserve all the edges of γ separately.

The Hilbert space of physical states $\mathcal{H}_{\mathrm{diff}_{\mathrm{PL}}}$ is obtained after completing the space spanned by the s-knot states $\eta(\mathrm{Cyl}_{\mathrm{PL}})$ in the norm provided by the inner product defined by

$$(F, G) = (\eta(f), \eta(g)) := G[f] \quad . \tag{4.32}$$

Define the algebra $\mathcal{A}'_{\mathrm{diff}_{\mathrm{PL}}}$ to be the algebra of operators on $\mathcal{H}_{\mathrm{kin}_{\mathrm{PL}}}$ satisfying the following two properties: First, for $O \in \mathcal{A}'_{\mathrm{diff}_{\mathrm{PL}}}$, both O and $O^\dagger$ are defined on $\mathrm{Cyl}_{\mathrm{PL}}$ and map $\mathrm{Cyl}_{\mathrm{PL}}$ to itself. Second, both O and $O^\dagger$ are representable in $\mathcal{H}_{\mathrm{diff}_{\mathrm{PL}}}$ by means of

$$r_{PL}(\hat{O})F = r_{PL}(\hat{O})\eta(f) := \eta(\hat{O}f) \quad . \tag{4.33}$$

$\mathcal{A}_{\mathrm{diff}_{\mathrm{PL}}}$ is the analog of the algebra of weak "observables." Different weak observables can be weakly equivalent; in the same way, many operators of $\mathcal{A}'_{\mathrm{diff}_{\mathrm{PL}}}$ are represented by the same operator in $\mathcal{H}_{\mathrm{diff}_{\mathrm{PL}}}$. For example, $r_{PL}(\hat{U}_h) = r_{PL}(1) = 1$. We can define the algebra of classes of operators of $\mathcal{A}'_{\mathrm{diff}_{\mathrm{PL}}}$ that are represented by the same operator in $\mathcal{H}_{\mathrm{diff}_{\mathrm{PL}}}$; this algebra is faithfully represented in $\mathcal{H}_{\mathrm{diff}_{\mathrm{PL}}}$ and is called the algebra of physical operators $\mathcal{A}_{\mathrm{diff}_{\mathrm{PL}}}$ [41]. Even more, it is easy to prove that every operator on $\mathcal{H}_{\mathrm{diff}_{\mathrm{PL}}}$ is in the image of $r_{PL}(\mathcal{A}_{\mathrm{diff}_{\mathrm{PL}}})$. The algebra of strong observables (Hermitian operators invariant under gauge transformations and "diffeomorphisms") sits inside of $\mathcal{A}_{\text{diff}_{\text{PL}}}$ (with the commutator as product); then it is representable in $\mathcal{H}_{\text{diff}_{\text{PL}}}$ faithfully.

Since (4.33) maps any observable to a Hermitian operator in $\mathcal{H}_{\text{diff}_{\text{PL}}}$, this representation implements the reality conditions. In particular (when the space manifold is three dimensional and the gauge group is $SU(2)$), the construction provides a "quantum Husain-Kuchař model" [42], that has local degrees of freedom [26].

An interesting feature of the quantum Husain-Kuchař model (and of any other diffeomorphism invariant quantum gauge theory defined over a compact manifold Σ with $\dim(\Sigma) = 1, 2, 3$ following our general framework) is that the choice of background structure is not reflected in the resulting quantum theory. To be precise, fix a piecewise linear structure PL_0 on Σ and construct the algebra of physical operators $\mathcal{A}_{\text{diff}_{\text{PL}_0}}$ (acting on $\mathcal{H}_{\text{diff}_{\text{PL}_0}}$) that it induces. Given another piecewise linear structure PL_1 on Σ and a piecewise linear homeomorphism connecting both PL structures $h_1 : \Sigma_{PL_0} \to \Sigma_{PL_1}$, we get a representation of $\mathcal{A}_{\text{diff}_{\text{PL}_0}}$ in $\mathcal{H}_{\text{diff}_{\text{PL}_1}}$ by $r_{PL_1}(O) = \hat{U}_{h_1}^{-1} O \hat{U}_{h_1}$. In fact, $r_{PL_1} : \mathcal{A}_{\text{diff}_{\text{PL}_0}} \to \mathcal{A}_{\text{diff}_{\text{PL}_1}}$ is onto and it is independent of h. Thus we can label the operators of $\mathcal{A}_{\text{diff}_{\text{PL}_1}}$ by the elements of $\mathcal{A}_{\text{diff}_{\text{PL}_0}}$. Using $\mathcal{A}_{\text{diff}_{\text{PL}_0}}$ as a fiducial abstract algebra, the independence of the background PL structure on Σ may be stated as follows.

THEOREM 4.4. *Any piecewise linear structure PL_1 on a fixed manifold Σ of dimension* $\dim(\Sigma) = 1, 2, 3$ *defines a representation $r_{PL_1}(\mathcal{A}_{\text{diff}_{\text{PL}_0}})$ of $\mathcal{A}_{\text{diff}_{\text{PL}_0}}$. This representation is independent of the piecewise linear structure, in the sense that, given any two piecewise linear structures PL_1 and PL_2 on Σ, the representations $r_{PL_1}(\mathcal{A}_{\text{diff}_{\text{PL}_0}})$ and $r_{PL_2}(\mathcal{A}_{\text{diff}_{\text{PL}_0}})$ are unitarily equivalent.*

Proof – In dimensions $\dim(\Sigma) = 1, 2, 3$ it is known [43] that any two PL structures PL_i and PL_0 are related by a piecewise linear homeomorphism $h_i : \Sigma_{PL_0} \to \Sigma_{PL_i}$. This implies that $r_{PL_i}(\mathcal{A}_{\text{diff}_{\text{PL}_0}})$ defined above is a representation of $\mathcal{A}_{\text{diff}_{\text{PL}_0}}$. That the representations induced by PL_1 and PL_2 are equivalent is trivial. $U_{h_2^{-1} \circ h_1} : \mathcal{H}_{\text{diff}_{\text{PL}_1}} \to \mathcal{H}_{\text{diff}_{\text{PL}_2}}$; that $U_{h_2^{-1} \circ h_1}$ is the required unitary map and it induces an algebra isomorphism. $\diamondsuit$

4.3.2 Physical observables and physical states in the combinatorial category

Now our task is to find the analog of knot-classes of combinatorial graphs. In section 4.2 we reviewed how is that a simplicial complex K encodes combinatorially topological information, and how this information can be displayed in its geometric realization $|K|$. Then, to decide whether or not two combinatorial graphs $\gamma, \delta \in \Gamma_C$ belong to the same knot-class we are going to display them in the same space and compare them.

To this end, we fix the sequence of piecewise linear maps

$$M_n : |Sd^n(K)| \to |K| \tag{4.34}$$

defined by successive application of the canonical map $M_1 : |Sd(K)| \to |K|$ that maps the vertices of $|Sd(K)|$ to the baricenter of the corresponding simplex in $|K|$. Then, we map every every representative $\{\gamma_n, \gamma_{n+1} = Sd(\gamma_n), \ldots\}$ of the combinatorial graph γ in to a sequence

$$\{M_n(|\gamma_n|), M_{n+1}(|\gamma_{n+1}|) = M_n(|\gamma_n|), \ldots\} \tag{4.35}$$

that assigns the same geometric graph $|\gamma| := M_n(|\gamma_n|)$ to every integer. Using these maps we are going to define that the combinatorial graphs $\gamma, \delta \in \Gamma_C$ are "diffeomorphic" if the their corresponding geometrical graphs $|\gamma|, |\delta|$ are related by a piecewise linear homeomorphism.

One method in implementing the above idea is to use the sequence of maps M_n to induce a map that links the kinematical Hilbert spaces of the combinatorial and PL categories. The map $M : \mathrm{Cyl}_C \to \mathrm{Cyl}_{\mathrm{PL}}$ is defined by

$$M(f_\gamma) := f_{M_n(|\gamma_n|)} = f_{|\gamma|} \quad . \tag{4.36}$$

Now the map $\eta : \mathrm{Cyl}_{\mathrm{PL}}(\overline{\mathcal{A}/\mathcal{G}}) \to \mathrm{Cyl}^*_{\mathrm{PL}}(\overline{\mathcal{A}/\mathcal{G}})$ induces a new map $\eta_C : \mathrm{Cyl}_C(\overline{\mathcal{A}/\mathcal{G}}) \to \mathrm{Cyl}^*_C(\overline{\mathcal{A}/\mathcal{G}})$

$$\eta_C := M^* \circ \eta \circ M : \mathrm{Cyl}_C \to \mathrm{Cyl}^*_C \tag{4.37}$$

that produces "diffeomorphism" invariant distributions in the combinatorial category. Again, we characterize the averaging map by the s-knot states $s_{[\vec{\gamma}]_C, j(e), c(v)} \in \mathrm{Cyl}^*_C$ induced by the combinatorial spin network states $S_{\vec{\gamma}, j(e), c(v)}$

$$\begin{aligned} s_{[\vec{\gamma}]_C, j(e), c(v)}[S_{\vec{\delta}', j(e), c(v)}] &= \eta_C(S_{\vec{\gamma}, j(e), c(v)})[S_{\vec{\delta}', j(e), c(v)}] \\ &:= s_{[|\vec{\gamma}|], j(e), c(v)}[S_{|\vec{\delta}|', j(e), c(v)}] \end{aligned} \tag{4.38}$$

As follows from the above formula, the label $[\vec{\gamma}]_C$ of the s-knot states is an equivalence class of oriented combinatorial graphs, where $\vec{\gamma}$ and $\vec{\delta}$ are considered equivalent if there is $h \in \mathrm{Hom}_{\mathrm{PL}}(|K|)$ such that $h(|\vec{\gamma}|) = |\vec{\delta}|$.

Just as in the PL case, *the Hilbert space of physical states* $\mathcal{H}_{\mathrm{diff}_C}$ is obtained after completing the space spanned by the s-knot states $\eta_C(\mathrm{Cyl}_C(\overline{\mathcal{A}/\mathcal{G}}))$ in the norm provided by the inner product defined by

$$(F, G) = (\eta_C(f), \eta_C(g)) := G[f] \quad . \tag{4.39}$$

It may seem odd that we are constructing the space of "diffeomorphism" invariant states without a family of unitary maps called "diffeomorphisms". The reason for this peculiarity is behind the very beginning of our construction. We chose to represent space combinatorially with a sequence generated by the simplicial complex K, and we did not consider the sequence generated by other complex, say L, even if it had the same topological information $|K| \approx |L|$. If we had done that, we would have ended with a kinematical Hilbert space that would be made of two copies of the one that we defined here, and these two copies would be linked by "diffeomorphisms". What we did was to construct every thing above the minimal kinematical Hilbert space. A relevant question is if by shrinking the kinematical Hilbert space we also shrank the space of physical states. Below, we will prove that this is not the case.

Now we state two important characteristics of the spaces of physical states of the combinatorial and PL models.

First, we constructed the space $\mathcal{H}_{\mathrm{diff}_{\mathrm{C}}}$ using the map η_C; the same map can be restricted to give an onto map from the spin network basis of $\mathcal{H}'_{\mathrm{kin}_{\mathrm{C}}}$ to the basis of $\mathcal{H}_{\mathrm{diff}_{\mathrm{C}}}$. Since the kinematical Hilbert space is separable, we have the following physically interesting result.

THEOREM 4.5. *The Hilbert space $\mathcal{H}_{\mathrm{diff}_{\mathrm{C}}}$ is separable.*

Second, the map $M^* : \mathrm{Cyl}^*_{\mathrm{PL}} \to \mathrm{Cyl}^*_C$ can be extended by continuity to link the spaces of physical states of the PL and combinatorial categories. Using this map we can compare these two spaces.

THEOREM 4.6. *The spaces of physical states in the PL and combinatorial categories are naturally isomorphic, $\mathcal{H}_{\mathrm{diff}_{\mathrm{PL}}} \approx \mathcal{H}_{\mathrm{diff}_{\mathrm{C}}}$.*

Proof – If $\gamma_{\vec{\mathrm{PL}}} = |\vec{\gamma}|$ then M^* identifies the s-knot states that they generate by averaging, in other words, $M^*(s_{[|\gamma_{\vec{\mathrm{PL}}}|],j(e),c(v)}) = s_{[\vec{\gamma}]_C,j(e),c(v)}$. From the definition of the inner products and the definition of the combinatorial s-knot states it follows immediately that M^* is an isometry.

Since the spaces of physical states were constructed by completing the vector spaces spanned by the s-knot states, the theorem is a consequence of the following lemma.

LEMMA 4.5. *In any knot-class of PL oriented graphs $[\gamma_{\vec{PL}}]$ there is at least one representative that comes from the geometric representation of a combinatorial oriented graph $|\vec{\gamma}| \in [\gamma_{\vec{PL}}]$.*

The lemma is proved in appendix B; this concludes our proof $\diamondsuit$.

Now we proceed to construct a representation of the algebra of physical operators in the combinatorial category. As in the PL category, we define the algebra $\mathcal{A}'_{\mathrm{diff}_{\mathrm{C}}}$ to be the algebra of operators on $\mathcal{H}_{\mathrm{kin}_{\mathrm{C}}}$ that satisfy the following two conditions: First, for $O \in \mathcal{A}'_{\mathrm{diff}_{\mathrm{C}}}$, both O and $O^\dagger$ are defined on $\mathrm{Cyl}_{\mathrm{C}}$ and map $\mathrm{Cyl}_{\mathrm{C}}$ to itself. Second, both O and $O^\dagger$ are representable in $\mathcal{H}_{\mathrm{diff}_{\mathrm{C}}}$ by means of

$$r_C(\hat{O})F = r_C(\hat{O})\eta_C(f) := \eta_C(\hat{O}f) \quad . \tag{4.40}$$

We are interested in the algebra of classes of operators of $\mathcal{A}'_{\mathrm{diff}_{\mathrm{C}}}$ that are represented by the same operator in $\mathcal{H}_{\mathrm{diff}_{\mathrm{C}}}$; this algebra is faithfully represented in $\mathcal{H}_{\mathrm{diff}_{\mathrm{C}}}$ and is called the algebra of physical operators $\mathcal{A}_{\mathrm{diff}_{\mathrm{C}}}$ [41]. In contrast with the PL case, in the combinatorial framework the "diffeomorphism group" does not have a natural action; for this reason the notion of strong observables can not be intrinsically defined. However, it is easy to prove that in the PL case the subset of $\mathcal{A}_{\mathrm{diff}_{\mathrm{PL}}}$ consisting of Hermitian operators is, in fact, the algebra of strong observables (with the commutator as product). Therefore, in the combinatorial category we can regard the algebra of Hermitian operators in $\mathcal{A}_{\mathrm{diff}_{\mathrm{C}}}$ as the algebra of strong observables; this algebra is naturally represented in $\mathcal{H}_{\mathrm{diff}_{\mathrm{C}}}$.

Since (4.40) maps any observable to a Hermitian operator in $\mathcal{H}_{\mathrm{diff}_{\mathrm{C}}}$, this representation implements the reality conditions. In particular (when the space manifold is three dimensional and the gauge group is $SU(2)$), the construction provides another "quantum Husain-Kuchař model" [42]. A natural question is whether the PL and combinatorial models are physically equivalent or not. We saw that the algebra $\mathcal{A}_{\mathrm{diff}_{\mathrm{C(K)}}}$ is represented in $\mathcal{H}_{\mathrm{diff}_{\mathrm{C}}(\mathrm{K})}$ by $r_{C(K)}$; it is also natural to give the representation $d_K(\mathcal{A}_{\mathrm{diff}_{\mathrm{C(K)}}})$ on $\mathcal{H}_{\mathrm{diff}_{\mathrm{PL(|K|)}}}$ by $d_K(\hat{O})F_{PL} = d_K(\hat{O})(\eta \circ M f_C) := \eta \circ M(\hat{O} f_C)$. This two representations are identified by the isomorphism exhibited in (4.6), more precisely:

THEOREM 4.7. *The representations* $r_{C(K)}(\mathcal{A}_{\mathrm{diff}_{\mathrm{C(K)}}})$ *on* $\mathcal{H}_{\mathrm{diff}_{\mathrm{C}}(\mathrm{K})}$ *and* $d_K(\mathcal{A}_{\mathrm{diff}_{\mathrm{C(K)}}})$ *on* $\mathcal{H}_{\mathrm{diff}_{\mathrm{PL(|K|)}}}$ *of the algebra* $\mathcal{A}_{\mathrm{diff}_{\mathrm{C}}(\mathrm{K})}$ *are unitarily equivalent. In addition if* $\dim(\Sigma) = 1, 2, 3$ *this algebra does not depend on* K *but only on the topology of* $|K| \approx \Sigma$*; the combinatorial and PL frameworks (based on the choice of the Ashtekar-Lewandowski measure* μ_0 *on* $\mathcal{H}_{\mathrm{kin}}$*) provide unitarily equivalent representations of the abstract algebra* $\mathcal{A}_{\mathrm{diff}_{\Sigma}}$.

Proof – The unitary equivalence of $r_{C(K)}(\mathcal{A}_{\mathrm{diff}_{\mathrm{C(K)}}})$ and $d_K(\mathcal{A}_{\mathrm{diff}_{\mathrm{C(K)}}})$ is given by the unitary map $M^* : \mathcal{H}_{\mathrm{diff}_{\mathrm{PL(|K|)}}} \to \mathcal{H}_{\mathrm{diff}_{\mathrm{C}}(\mathrm{K})}$.

$d_K(\mathcal{A}_{\mathrm{diff}_{\mathrm{C(K)}}})$ maps $\mathcal{A}_{\mathrm{diff}_{\mathrm{C}}(\mathrm{K})}$ onto the algebra of operators on $\mathcal{H}_{\mathrm{diff}_{\mathrm{PL(|K|)}}}$ and the representation is faithful; the same thing happens for the combinatorial model based on a different simplicial complex L. From theorem (4.4) we know that if $\dim(\Sigma) = 1, 2, 3$ for any two simplicial complexes K, L such that $|K| \approx |L| \approx \Sigma$ the Hilbert spaces $\mathcal{H}_{\mathrm{diff}_{\mathrm{PL(|K|)}}}$, $\mathcal{H}_{\mathrm{diff}_{\mathrm{PL(|L|)}}}$ and the algebras of operators on them are *identified* (unambiguously) by a unitary map. Since $d_K(\mathcal{A}_{\mathrm{diff}_{\mathrm{C(K)}}})$, $d_L(\mathcal{A}_{\mathrm{diff}_{\mathrm{C(L)}}})$, $d_K(\mathcal{A}_{\mathrm{diff}_{\mathrm{PL(|K|)}}})$ and $d_L(\mathcal{A}_{\mathrm{diff}_{\mathrm{PL(|L|)}}})$ label the operators on $\mathcal{H}_{\mathrm{diff}_{\mathrm{PL}(\Sigma)}}$, there is an unambiguous invertible map *identifying* these algebras. Thus the family of all these equivalent algebras may be regarded as the abstract algebra $\mathcal{A}_{\mathrm{diff}_{\Sigma}}$ and the combinatorial and PL frameworks are procedures that yield unitarily equivalent representations of this abstract algebra. $\diamondsuit$

From the theorems it follows that the PL and combinatorial frameworks are physically equivalent. They yield representations of the algebra of physical observables in separable Hilbert spaces; hence, maintaining the usual interpretation of quantum field theory [44].

4.4 Discussion and Comparison

In this chapter we have presented two models for "diffeomorphism" invariant quantum gauge field theories. We proved that the two models represented the algebra of physical observables in separable Hilbert spaces $\mathcal{H}_{\text{diff}_{\text{PL}}}$ and $\mathcal{H}_{\text{diff}_{\text{C}}}$; furthermore, we proved that the two models where physically equivalent in the sense that they gave rise to unitarily equivalent representations of the algebra of physical observables. The equivalence of the two models is a good feature, but we may still ask if by choosing a different background structure (like a different PL structure for our base space manifold) we could have arrived at a physically different model. In fact, this problem has been thoroughly studied (see for example [43]). For example, in dimensions $\dim(\Sigma) = 1, 2, 3$ any two PL structures, like any two differential structures, of a fixed topological manifold Σ are known to be equivalent in the sense that they are related by a PL homeomorphism (diffeomorphism). Then, if the base space is three dimensional (like in canonical quantum gravity) all the different choices of background structure would yield unitarily equivalent representations of the algebra of physical (gauge and diffeomorphism invariant) observables (the unitary map given by a quantum "diffeomorphism").

Our quantum models are not equivalent to the ones created in the analytic category [26]; for instance, in the analytic category the physical Hilbert space is not separable. The reason for this size difference is not that the family of piecewise analytic graphs is too big; the kinematic Hilbert space of the PL category is also not separable. In the following chapter we show that the concept of knot-classes that should be used in the piecewise analytic category is with respect to the group of maps defined by

$$\text{Pdiff}_{\omega}(\Sigma) := \{h \in \text{Hom}(\Sigma) : h(\Gamma_{\omega}) = \Gamma_{\omega}\} \quad . \tag{4.41}$$

In appendix B we show how to adapt the proof of lemma 4.5 to show that every (modified) knot-class of piecewise analytic graphs has a representative induced by a combinatorial graph. Then, theorem 4.6 and theorem 4.7 have analogs proving that the the Hilbert space of physical states of the piecewise analytic category is also separable and that the representation of the algebra of physical observables given by the piecewise analytic category is unitarily equivalent to the one provided by the combinatorial framework.

We can expect (the author does) that a more satisfactory understanding of field theory may arise from this combinatorial picture of quantum geometry. The bridge between three-dimensional quantum geometry and a smooth macroscopic space-time is the missing ingredient to complete this picture of quantum field theory. Three unsolved problems prevent us from building this bridge. Dynamics in quantum gravity is only partially understood [45, 46]. The emergence of a four-dimensional picture from solutions to the constraints has just begun to be explored [47]. And the statistical mechanics needed to find the semi-classical/macroscopic behavior of the theory of quantum geometry is also at its developing stage [48]. For an extended discussion the dynamics of loop quantization see chapter 6.

Chapter 5

Combinatorial space from loop quantum gravity

After the PL category of loop quantization was developed (chapter 4), the differences between the PL and the analytic categories of loop quantization were intriguing. In particular, it was not a priori clear why one had a combinatorial nature while the other was only "almost combinatorial." In other words, it was not clear why the mechanism that yields a separable Hilbert space for the PL category did not work in the analytic category. This comparison motivated me to revisit the analytic category of loop quantization; the outcome was the article (to be published in *General Relativity and Gravitation*) on which this chapter is based.

In order to represent the algebra of observables one needs a space of invariant states. Since quantization involves completing the algebra of holonomy functions, the quantum gauge group is an appropriate completion of the classical gauge group. In the case of the internal gauge transformations one can solve the Gauss constraint after quantization, or give an intrinsically gauge invariant formulation. Both constructions agree if the quantum gauge group is taken to be a completion of the classical internal gauge group. For the diffeomorphism gauge group there is no intrinsically invariant construction; one can only solve the diffeomorphism constraint after quantization. In this chapter I argue that there is a natural candidate for the quantum gauge group, and it turns out to be a completion of the diffeomorphism group.

According to this refined treatment of diffeomorphism invariance an old expectation is realized. Namely, diffeomorphism invariance plays a double role. It forces one to consider an uncountable set of graphs to label the kinematical states of loop quantum gravity. However, it yields a representation of the algebra of observables (diffeomorphism invariant functions) in a *separable* Hilbert space spanned by states labeled by knot-classes of graphs. In contrast, Grot and Rovelli found that the space of invariant states of the previous formulation of loop quantization contains families of orthogonal states labeled by continuous parameters [49]. In the version of loop quantization that uses the completed diffeomorphism group, one can exhibit a countable basis of invariant states (the spin-knot basis). In fact, the completed diffeomorphism group simplifies the formalism, and the resulting quantum theory is equivalent to a model for diffeomorphism invariant gauge theories which replaces the space manifold with a manifestly combinatorial object (see chapter 4). Just as loop quantization conduces to a notion of quantum geometry with discrete areas of non-commutative nature, it also conduces to an intrinsically combinatorial picture of physical space.

In this chapter I revisit loop quantization emphasizing the issue of diffeomorphism invariance. For completeness, the kinematics of loop quantization is briefly reviewed in section 5.1. Internal and diffeomorphism gauge invariance of the classical and quantum theories are addressed in section 5.2–the main section of the chapter. In that section, a refined treatment of diffeomorphism invariance is presented, and its consequences are studied. A discussion section ends the chapter.

5.1 Kinematics of loop quantization

Recall that gravity, expressed in (real) Ashtekar-Barbero variables, is a Hamiltonian theory of connections that shares the phase space with $SU(2)$ Yang-Mills theory [50]. That is, the configuration variable is a connection A_a^i taking values in the Lie algebra of $SU(2)$, and the canonically conjugate momentum is a triad $\tilde{E}_i^a$ of densitized vector fields. In these variables the contravariant spatial metric is determined by $q^{ab} \det q = \tilde{E}_i^a \tilde{E}^{bi}$, which makes contact with the usual geometrodynamic treatment of general relativity. In this formulation, Einstein's equations are equivalent to a series of constraints: a set which generates diffeomorphisms on the Cauchy surface and constitutes a closed subalgebra of the constraint algebra, and a set of constraints generating motions transverse to the initial data surface.

If only the constraints that generate spatial diffeomorphisms are imposed and the Hamiltonian constraint is dropped, one gets a well-defined model to study diffeomorphism invariant theories of connections. This model is called the Husain-Kuchař model and can be derived from an action principle [42]; it shares the phase space, the gauge constraint and the diffeomorphism constraint with general relativity and has local degrees of freedom. More than a toy model, the Husain-Kuchař model provides an intermediate step in the quantization of general relativity; a quantization of the model requires to set up a kinematical framework and regularize and solve the gauge and diffeomorphism constraints. After a satisfactory quantum version of the model is developed, a quantization of general relativity amounts to the difficult tasks of regularizing and solving the Hamiltonian constraint and verifying that GR is recovered in the classical limit. This chapter is about the treatment of diffeomorphism invariance in the loop quantization framework; therefore it pertains to any diffeomorphism invariant theory of connections, in particular, to general relativity (possibly coupled to Yang-Mills fields) and the Husain-Kuchař model. For the sake of concreteness, the problems and the results are stated in reference to the the quantization of the Husain-Kuchař model. Issues like whether the algebra of the constraints is correct or if there is a classical limit in the theory resulting from Thiemann's Hamiltonian constraint [46] is matter of hot debate [51, 52]. Since the study includes diffeomorphism covariant functions and the Hamiltonian constraint is diffeomorphism covariant, the results presented in this chapter may shine some light on the difficult problem of regularizing the Hamiltonian constraint.

The cornerstone of loop quantization is the use of holonomies along loops as "coordinates on the classical configuration space" [32, 39]. For primary momentum functions one can use the triad (whose dual is a two form) smeared on surfaces [7, 30], or, in the manifestly gauge invariant treatment, a combination of holonomies and triads called the strip functions [53, 30]. In this chapter, the term *loop variables* is some times used as a collective name for the configuration and momentum variables described. This choice of variables is due to the symmetries of the theory; using them one can explicitly solve the gauge and diffeomorphism constraints of the quantum theory.

It was proven [54] that all the information about the connection is contained in the set of holonomies of the connection around every smooth path e

$$h_e(A) = \mathrm{Pexp}(\mathrm{i}\int_{\mathrm{e}} \tau_{\mathrm{i}} \mathrm{A}^{\mathrm{i}}_{\mathrm{a}} \mathrm{ds}^{\mathrm{a}}) \tag{5.1}$$

where $\tau_i = \frac{1}{2}\sigma_i$ are the $SU(2)$ generators [39]. The loop variables $h_e(A)$ are an overcomplete set of configuration functions that coordinatize the space of smooth connections $\mathcal{A}$ in the sense that two connections can always be differentiated by the loop variables. If only closed loops are used, the set of traces of the holonomies coordinatizes the space of connections modulo internal gauge transformations. Also, any two smooth triads can be differentiated by smearing the triads (two forms) over some surface. This property ensures that by keeping only functions of the loop variables as primary functions, and recovering every thing from them after quantization, no relevant information is omitted. Thus, at least in principle, any phase space function can be expressed in terms of functions of the loop variables. The holonomy functions are special because they form a subalgebra of the algebra of configuration functions; and this subalgebra is preserved by the primary momentum functions, the surface smeared triads. These important properties lie at the heart of loop quantization.

The classical algebra that is actually quantized is the algebra Cyl_0. A cylindrical function $f_\gamma(A) \in \mathrm{Cyl}_0$ is a function of the holonomies along the edges of the graph γ. With this definition, the product of two cylindrical functions is another cylindrical function if the edges of the two original graphs are contained in the set of edges of a bigger graph. To satisfy this condition, it was first proposed to consider only graphs with piecewise analytic edges [26]. Since among the cylindrical functions one has all the loop variables, it is clear that one can use the cylindrical functions as primary functions in the space of smooth[1] connections. After Cyl_0 is quantized the primary configuration functions become operators that act by multiplication, and the primary momentum functions (the surface smeared triads) become operators that act as derivative operators. Thus, loop quantization produces a regularized operator from any phase space function written in terms of the loop variables.

Cyl_0 is quantized by following a series of steps. First, completing it to form a C^* algebra Cyl. Second, represent the cylindrical functions and linear in momenta functions in $\mathcal{H}_{\mathrm{kin}} = L^2(\bar{\mathcal{A}}, \mu)$ (by multiplicative and derivative operators respectively), where $\bar{\mathcal{A}}$ is the spectrum of Cyl and μ is the Ashtekar-Lewandowski measure, which is selected by the reality conditions [26].

At a more operational level, the Hilbert space of gauge invariant states (under $SU(2)$ gauge transformations) is spanned by spin network states $|S\rangle$ [55]. A spin network S is labeled by a colored graph $\vec{\gamma}$ and represents the function of the holonomies along its edges given by

[1] I would loosely use the term smooth to mean real analytic; except in the last paragraphs of the chapter where I comment on the smooth (C^∞) category.

$$S_{\vec{\gamma},j(e),c(v)}(A) = \prod_{e\in E_{\vec{\gamma}}} \pi_{j(e)}[h_e(A)] \cdot \prod_{v\in V_{\vec{\gamma}}} c(v) \quad , \tag{5.2}$$

where the colors on the edges $j(e)$ are irreducible representations of $SU(2)$, and the vertices are labeled by gauge invariant contractors $c(v)$ that match all the indices (in the formula denoted by '$\cdot$') of the holonomies of the edges. An inner product in the space of gauge invariant states $L^2(\overline{\mathcal{A}/\mathcal{G}}, \mu)$ is given, alternatively, by the Ashtekar-Lewandowski measure [36, 26] or by recoupling theory [39, 56]. According to this inner product, two spin network states are orthogonal if their coloring or labeling graphs are different. Using a convenient set of contractors one can form an orthonormal basis with spin network states [56]

$$\langle S|S'\rangle = \delta_{SS'} \quad . \tag{5.3}$$

Non-gauge invariant spin network states are constructed by just dropping the gauge invariant contractors and the Ashtekar-Lewandowski measure induces an inner product in $\mathcal{H}_{\text{kin}}$.

5.2 Diffeomorphism invariance

Classical observables, gauge and diffeomorphism invariant functions, induce functions in the reduced phase space; loop quantization's objective is to produce a faithful representation of the algebra of observables. First the operators are regularized from their expressions as functions of the loop variables. The resulting operators are expected to be invariant under "quantum gauge transformations" and "quantum diffeomorphisms." Finally, from the algebra of invariant operators one induces (by dual action) a faithful representation on the space of diffeomorphism invariant states. Here, this process is followed, but special care is paid to the character acquired by diffeomorphism invariance after loop quantization.

In the description of the classical theory in terms of smooth fields there is a harmony between the space of smooth connections and the gauge group. As far as the internal gauge transformations, the internal gauge group may be characterized as the set of $SU(2)$-matrix valued functions g such that given a smooth connection $A \in \mathcal{A}$, the connection $g(A_a) = g^{-1}A_a g + g^{-1}\partial_a g$ is also smooth. Similarly, the diffeomorphism group can be characterized as the subgroup of the homeomorphism group composed by all the transformations which leave the space of smooth connections invariant

$$\text{Diff} = \{\phi \in Hom | \phi^*(A) \in \mathcal{A} \quad \text{for all} \quad A \in \mathcal{A}\} \tag{5.4}$$

This compatibility between configuration space and gauge group acquires a different form after loop quantization. Quantization takes the space of smooth connections and, by completing it, constructs the quantum configuration space $\bar{\mathcal{A}}$. A generalized connection $A \in \bar{\mathcal{A}}$ simply assigns group elements to piecewise analytic paths; that is, it acts as a connection which does not need to be smooth. Completing the configuration space requires adapting the gauge group also. The quantum internal gauge group $\bar{\mathcal{G}}$ is formed by the transformations acting at the end points of the paths,

$g(A)[e] = g^{-1}(e_0)g(A)[e]g(e_1)$. A quantum gauge transformation maps every generalized connection to another generalized connection. This group contains the classical internal gauge group, but it is not the classical gauge group. It is the completion of the group of smooth internal gauge transformations according to the operator norm. Most of the quantum gauge transformations would transform a smooth connection into a non-smooth connection.

In the diffeomorphism part of the gauge group a similar phenomena happens. The family of piecewise analytic graphs is left invariant by a bigger group than the group of smooth diffeomorphisms, but if one transforms a smooth connection using a non-smooth map one obtains a non-smooth connection. Again, because quantization involves completing the configuration space, the generalized connections are covariant with respect to a certain completion of the diffeomorphism group; $\phi^*(A)[e] := A[\phi(e)]$ is defined for a certain completion of the diffeomorphism group[2]. As a consequence, the primary configuration and momentum variables induce operators that are covariant with respect to the mentioned completion of the diffeomorphism group. Since every operator of the quantum theory is constructed from the primary configuration and momentum operators, this extended covariance becomes a feature of the quantum theory. Functions of the phase space with a geometrical label (like the holonomy functions, surface smeared triads, surface area functions, volume functions, etc) are diffeomorphism covariant, but operators coming from these functions with geometrical labels are naturally covariant under a certain completion of the diffeomorphism group. Note that the Hamiltonian constraint is diffeomorphism covariant and some of its regularizations have the mentioned extended covariance (comments on the Hamiltonian constraint are reserved for the discussion section).

More importantly, given the extended notion of covariance, it is necessary to review the notion of observable in the quantum theory. Observables (diffeomorphism invariant functions) naturally arise from covariant functions where the geometrical labels become dynamical. For example, area functions of surfaces specified by matter fields. If the fields specifying the geometrical labels also acquire the extended covariance, as they would if they are quantized using loop quantization, then the natural notion of an observable would be to be invariant under the mentioned completion of the diffeomorphism group.

To explain the details of the previous discussion let me show you how piecewise analytic diffeomorphisms come about. Consider the following situation. The Cauchy surface is R^3; an example of nonsmooth map is $\phi : R^3 \to R^3$ defined to be the identity above the $x - y$ plane and below the plane $x - y$ plane it is defined by $\phi(x, y, z) = (x, y + mz, z)$. This map is smooth above and below the $x - y$ plane but at the $x - y$ plane its derivative from above and its derivative from bellow do not match (in the

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[2]In the previous paragraph I defined $\bar{\mathcal{G}}$ algebraically. The algebraic relation came from the classical theory, but the definition of $\bar{\mathcal{G}}$ only involved quantum objects. I will show that this construction in the case of the diffeomorphism group yields $\bar{D}$. However, $\bar{\mathcal{G}}$ is the completion of $\mathcal{G}$ in the operator norm, and $\bar{D} \supset D$, but according to the operator norm $\bar{D}$ is a discrete group. Strictly speaking, $\bar{D}$ is an algebraic extension of the diffeomorphism group rather than a completion of it.

direction normal to the $x-y$ plane). One can see that ϕ maps some smooth loops to loops with kinks. Given any smooth connection $A \in \mathcal{A}$, one would like to say that the functions

$$h_l(\phi^*(A)) := h_{\phi(l)}(A) \quad . \tag{5.5}$$

are "covariantly" related to the loop coordinates of $A \in \mathcal{A}$, but the connection $A' = \phi^*(A)$ is not in the configuration space of the classical theory. However, in the quantum theory, the functions $h_{\phi(l)}(A)$ induce an operator that is as valid as the ones induced by the functions $h_{\pi(l)}(A)$ defined using any smooth map π. Hence the map ϕ is an object that will play a role in the quantum theory even though it did not define a canonical transformation in the classical theory. Classically, we cannot ask if the connections $A \in \mathcal{A}$ and $A' = \phi^*(A)$ are gauge related, but the quantum configuration space is the space of generalized connections, and $A \in \bar{\mathcal{A}}$ if and only if $\phi^*(A) \in \bar{\mathcal{A}}$.

Following the above example, a map $\phi : \Sigma \to \Sigma$, that maps any piecewise analytic graph to another, would map any generalized connection to another, and define a new loop operator from a given one.

A map $\phi : \Sigma \to \Sigma$ belongs to $\bar{D}$ iff for any piecewise analytic graph γ the new graph $\phi(\gamma)$ is also piecewise analytic.

Above I gave a description of $\bar{D}$ designed to show the natural role that it will play in the quantum theory, and to emphasize the parallelism between its definition and the definition of $\bar{\mathcal{G}}$. Alternatively, one can describe $\bar{D}$ as the group of piecewise analytic diffeomorphisms. In close analogy with the definition of a piecewise linear manifold (Regge lattice), a piecewise analytic manifold Σ is a topological manifold formed as a union of finitely many closed cells, each of which is an analytic manifold with boundary (these correspond to the higher dimensional simplices of the Regge lattice). Two of these cells may intersect only at their boundaries. A map $\phi : \Sigma_1 \to \Sigma_2$ is piecewise analytic if and only if there is a refinement of the cell decomposition of Σ_1 such that the restriction of ϕ to every cell is an analytic map. Clear examples of piecewise analytic manifolds (maps) are real-analytic manifolds (maps) and piecewise linear manifolds (maps).

Guidance from the classical theory tells us that the operators induced by $h_l(A)$ and $h_{\pi(l)}(A)$ for any smooth map π are gauge related. However, classically one can not say that the functions $h_l(A)$ and $h_l(\phi^*(A)) := h_{\phi(l)}(A)$ are gauge related since the non-smooth map ϕ does not define a canonical transformation because the connection $A' = \phi^*(A)$ is not in the configuration space of the classical theory, but the quantum states are functions of generalized connections $\mathrm{Cyl}(\bar{\mathcal{A}})$ and $A \in \bar{\mathcal{A}}$ if and only if $\phi^*(A) \in \bar{\mathcal{A}}$. Quantization involves completing the space of cylindrical functions to make it the C^* algebra $\mathrm{Cyl}(\bar{\mathcal{A}})$; to account for this enlargement of the configuration space, the internal gauge group is $\bar{\mathcal{G}}$ instead of $\mathcal{G}$. Smooth connections and generalized connections differ in more than their "internal degrees of freedom." Recall that in the smooth case $\phi^*(A)$ is defined only for smooth (analytic) maps, whereas in the case of generalized connections it is defined for any piecewise analytic map.

Because of these considerations, and since any piecewise analytic map ϕ can be obtained as a limit of smooth maps I will assume that the operators induced by $h_l(A)$ and $h_{\phi(l)}(A)$ are gauge related.

A quantum 'diffeomorphism' $\phi \in \bar{D}$ acts by shifting the labels of the spin networks by a diffeomorphism

$$U_\phi |S_{\vec{\gamma},j(e'),c(v')}\rangle := |S_{\phi(\vec{\gamma}),j(e'),c(v')}\rangle \quad . \tag{5.6}$$

Since the measure that defines the inner product is $\bar{D}$ invariant, the operator U_ϕ is unitary.

Before the significance of $\bar{D}$ was understood, it was noticed that the original regularization of the area and volume operators, and some versions of the Hamiltonian constraint, were not diffeomorphism covariant, but they were covariant under a bigger group. Later a version of the volume operator that was only covariant under smooth diffeomorphisms was developed and this version of the volume operator entered in the definition of Thieman's Hamiltonian constraint. Initially, it was believed that replacing the volume operator used by Thieman with the $\bar{D}$ covariant version would change the algebra of the constraints, but now it has been proven that it produces no changes [52].

Using the technique developed in [26], one solves the quantum diffeomorphism constraint by constructing the space of $\bar{D}$ invariant states $\mathcal{H}_{\text{diff}}$. It is spanned by s-knot states $\langle s|$, labeled by knot-classes of colored graphs, and defined by

$$\langle s_{[\vec{\gamma}],j(e),c(v)}|S'_{\vec{\eta},j(e'),c(v')}\rangle := a([\gamma])\delta_{[\gamma][\eta]} \sum_{[\phi]\in \mathrm{GS}(\gamma)} \langle S_{\vec{\gamma},j(e),c(v)}|U_{f\cdot\phi_0} S'_{\vec{\eta},j(e'),c(v')}\rangle \tag{5.7}$$

where $a([\gamma])$ is an undetermined normalization parameter, $\delta_{[\gamma][\eta]}$ is non vanishing only if there is a *piecewise analytic diffeomorphism* $\phi_0 \in \bar{D}$ that maps η to a graph γ that defines the knot-class $[\gamma]$, and $\phi \in \bar{D}$ is any element in the class of $[\phi] \in \mathrm{GS}(\gamma)$. The finite group $\mathrm{GS}(\gamma)$ is the group of symmetries of γ; in other words, the elements of $\mathrm{GS}(\gamma)$ are maps between the edges of γ (for a detailed explanation see chapter 4 or [26]).

The s-knot states are solutions of the diffeomorphism constraint because its action is invariant under quantum diffeomorphisms by construction. An inner product for $\mathcal{H}_{\text{diff}}$ is given simply by[3] [26]

$$\langle s_{[\vec{\gamma}],j(e),c(v)}|s'_{[\vec{\delta}],j(e'),c(v')}\rangle := \langle s'_{[\vec{\delta}],j(e'),c(v')}|S_{\vec{\gamma},j(e),c(v)}\rangle \tag{5.8}$$

The observables of the Husain-Kuchař model are naturally represented on $\mathcal{H}_{\text{diff}}$. If $\hat{O}$ is a "diffeomorphism" invariant Hermitian operator on the kinematical Hilbert space, $\tilde{O}: \mathcal{H}_{\text{diff}} \to \mathcal{H}_{\text{diff}}$ is defined by its dual action

$$(\langle s_{[\vec{\gamma}],j(e),c(v)}|\tilde{O})|S_{\vec{\gamma},j(e),c(v)}\rangle := \langle s_{[\vec{\gamma}],j(e),c(v)}|(\tilde{O}|S_{\vec{\gamma},j(e),c(v)}\rangle) \quad . \tag{5.9}$$

These are the foundations of the theory following from considering the extended notion of diffeomorphism covariance/invariance in loop quantization. In particular, they constitute a quantization of the Husain-Kuchař model [42], that has local degrees of freedom.

[3]Note that this inner product is determined only up to the unknown parameters $a([\gamma])$.

Here I will describe the properties of the quantum theory that are not shared by previous treatments of loop quantization. First, one should notice that $\mathcal{H}_{\text{diff}}$ is separable. The s-knot states are labeled by knot-classes of graphs $[\gamma]$ with respect to $\bar{D}$. Since the diffeomorphism group was replaced by a bigger group, the resulting knot-classes are much bigger and therefore there are very few of them; this is why separability arises. In contrast, states in the original treatment are labeled by continuous parameters parameterizing the knot-classes of graphs with higher valence vertices [49].

I sketch the proof of separability in the next few paragraphs. A mathematically rigorous proof can be found in appendix B.

Consider a three dimensional triangulated manifold $|K|$, which can be thought of as a three dimensional Regge lattice. Since the interior of the tetrahedrons of the lattice are flat, one can define the baricenter of any simplex (tetrahedron, face or link); by adding these points to the original lattice, and also adding new links and faces (see fig. 1), one constructs the finer lattice $|Sd(K)|$ called the *baricentric subdivision* of the original lattice $|K|$. One can do this subdivision again and again to get a sequence of lattices $\{|K|, |Sd(K)|, \ldots, |Sd^n(K)|, \ldots\}$. All these lattices are not disconnected, they are all subdivisions of $|K|$; using them, one defines a combinatorial graph γ_c to be a graph in $|K|$ all whose edges are links of some of the refined lattices $|Sd^n(K)|$. Also consider a *fixed* map $h : |K| \to \Sigma$ that maps every combinatorial graph γ_c to a piecewise analytic graph $h(\gamma_c)$ on Σ.

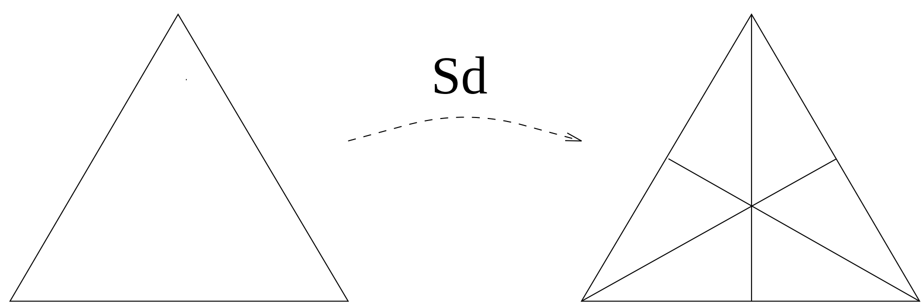

Fig. 5.1. A triangular face and its baricentric subdivision. Every link of $|K|$ is divided into two links of $|Sd(K)|$, every face into six faces and every cell into twenty four cells of $|Sd(K)|$.

The sense in which the knot-classes of graphs $[\gamma]$ are big is that every class contains a combinatorial graph, $h(\gamma_c) \in [\gamma]$. Given an arbitrary graph γ, the following series of steps generates a combinatorial graph γ_c and a piecewise analytic map $\phi : \Sigma \to \Sigma$ such that $\phi(h(\gamma_c)) = \gamma$.

1. Find n such that $|Sd^n(K)|$ separates the vertices of $h^{-1}(\gamma)$ to lie in different simplices. (The conventions are such that every point of the manifold belongs to the interior of one and only one simplex of a given triangulation).

2. Let $h_1 : |K| \to |K|$ be the piecewise linear map that fixes the vertices of $|Sd^n(K)|$ and sends the new vertices $v \in |Sd^{n+1}(K)|$ (the baricenters of the simplices of $|Sd^n(K)|$) to:

   (a) themselves $(h(v) = v)$, if there is no vertex of $h^{-1}(\gamma)$ in the simplex of $|Sd^n(K)|$ which has v as baricenter.

   (b) the vertex of the graph $(h(v) = w)$, in the case when the simplex of $|Sd^n(K)|$ which has v as baricenter contains a vertex of the graph $(w \in h^{-1}(\gamma))$ in its interior.

3. Find m such that $h_1(|Sd^{n+m}(K)|)$ separates the edges of $h^{-1}(\gamma)$ in the interiors of different simplices[4].

4. Let a *cell* be a (closed) image (by $h : |K| \to \Sigma$) of a simplex of $h_1(|Sd^{n+m}(K)|)$. Let $\phi = \phi_1 \circ h \circ h_1 \circ h^{-1} : \Sigma \to \Sigma$, where ϕ_1 is a piecewise analytic map that is equal to the identity when restricted to cells which do not intersect γ, and sends the cells which intersect γ to themselves, but has nontrivial analycity domains[5]. The analycity domains divide the cell into the *subcells* given by the image (by $h : |K| \to \Sigma$) of the simplices of $h_1(|Sd^{n+m+1}(K)|)$. ϕ_1 must be such that the intersection of γ and the cell lies in the the image (by ϕ_1) of the boundaries of the subcells; since only one (analytic) edge of γ intersects the interior of the original cell, a map ϕ_1 with the requested property always exists.

From the construction of $\phi : \Sigma \to \Sigma$ it is immediate that $\phi(h(\gamma_c)) = \gamma$.

The sense in which there are very few knot-classes of graphs is that the set of combinatorial graphs $\{\gamma_c\}$ is countable. One can easily convince oneself that this is the case because every γ_c belongs to $|Sd^n(K)|$ for some n, and there are countably many of these triangulations, each of which has finitely many links[6]. This property implies that the set of labels of the s-knot states is countable; that is, the Hilbert space of 'diffeomorphism' invariant states $\mathcal{H}_{\rm diff}$ is separable.

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[4]In the case of a graph γ with two or more edges meeting at a vertex this step needs to be refined. One needs to find an integer m and a piecewise analytic map $\psi : \Sigma \to \Sigma$ (with analycity domains given by $h \circ h_1(|Sd^{n+m}(K)|)$ see next footnote) such that $\psi \circ h \circ h_1(|Sd^{n+m}(K)|)$ separates the edges of γ. Using this refinement, the rest of the construction has a clear extension.

[5]A piecewise analytic map is a continuous map whose restriction to the interior of any of its analycity domains is analytic.

[6]One can triangulate a compact manifold with finitely many simplices and a paracompact manifold with countably many simplices. I sketch in the argument for the compact case, but it is immediate to extend it to the paracompact case, which includes all the Cauchy surfaces of asymptotically flat spacetimes.

I used the combinatorial graphs to prove the separability of the Hilbert space, but there is a deeper consequence of the existence of such a subspace of $\mathcal{H}_{\text{kin}}$. It has a manifestly combinatorial origin and is capable of generating all the states in the space of solutions to the diffeomorphism constraint. As far as observables are concerned, the combinatorial states are sufficient; meaning that the manifestly combinatorial framework yields a unitarily equivalent representation of the algebra of observables (see appendix B for a rigorous proof).

Equivalence with a manifestly combinatorial model is not so surprising if one remembers that observables in generally covariant theories are supposed to measure only relative 'positions' of the dynamical fields. One may object that in pure gravity there are not enough explicitly known observables as to serve as a basis of any argument. But, physically meaningful observables will arise if other fields are coupled to pure gravity (or to the Husain-Kuchař model). In these systems one can study observables that measure the gravitational field; for example, any covariant operator of pure gravity, say an area operator, whose labeling surface becomes dynamical after coupling other fields becomes an observable. They are generally covariant systems with plenty of observables measuring the gravitational field. Proving equivalence with a manifestly combinatorial model explicitly exhibits the relational nature of loop quantization.

In contrast with the treatment of diffeomorphism invariance presented here, the original study of the quantization of the Husain-Kuchař model considered the diffeomorphism group as the quantum gauge group. By using the same kinematical Hilbert space, but averaging over the diffeomorphism group instead of $\bar{D}$ to generate the solutions of the diffeomorphism constraint (5.7), they constructed the space of "physical" states

$\mathcal{H}_{\text{Diff}^*}$. This difference implies, in particular, that $\mathcal{H}_{\text{Diff}^*}$ is not separable [49] and that the nature of the theory is not combinatorial.

It was argued that classical functions which are diffeomorphism invariant/covariant induce, after loop regularization, $\bar{D}$ invariant/covariant operators on $\mathcal{H}_{\text{kin}}$. Because the operators are invariant under a larger group, the algebra of observables can be represented in $\mathcal{H}_{\text{Diff}^*}$; however, the representation of such operators yields a continuum of superselected sectors [57]. This superselection is not surprising after one knows that the same operators are naturally represented in the separable Hilbert space $\mathcal{H}_{\text{diff}}$.

5.3 Summary and discussion

In this chapter I studied the loop quantization of diffeomorphism invariant theories of connections. Such theories include general relativity described in terms of Ashtekar-Barbero variables and extension to Yang-Mills fields (with or without fermions [58, 59]) coupled to gravity. For the sake of concreteness the results were stated for the Husain-Kuchař model [42], which shares the phase space with general relativity, but it does not have a Hamiltonian constraint.

Loop quantization regularizes operators using the expression of a phase space function in terms of "loop variables" (functions of the holonomies of the connection along the edges of graphs and functions of surface smeared triads) and the quantization of the loop variables. The loop variables are a family of covariant functions with geometric labels whose quantization is a family of operators with the same geometric labels and an extended covariance. Since the quantum theory is built over the quantization of the loop variables, the extended covariance becomes a feature of the whole quantum theory.

Guidance from the classical theory tells us that the operators induced by $h_l(A)$ and $h_{\pi(l)}(A)$ for any smooth map π are gauge related. In the case of non-smooth maps, one can not say that the functions $h_l(A)$ and $h_l(\phi^*(A)) := h_{\phi(l)}(A)$ are gauge related since the non-smooth map ϕ does not define a canonical transformation because connections of the form $A' = \phi^*(A)$ are not in the configuration space of the classical theory. However, the quantum states are functions of generalized connections and $A \in \bar{\mathcal{A}}$ if and only if $\phi^*(A) \in \bar{\mathcal{A}}$ for any map $\phi \in \bar{D}$, where $\bar{D}$ is a completion of the diffeomorphism group.

Just as in the case of the internal gauge group, where the quantum internal gauge group is $\bar{\mathcal{G}}$, the same equations that defined the classical gauge group in terms of smooth connections are used to define the the quantum gauge group in terms of generalized connections.

A quantum diffeomorphism belongs to $\bar{D}$, which in the analytic category is the group of piecewise analytic diffeomorphisms.

The resulting quantum theory yields a representation of the algebra of observables in a *separable* Hilbert space. Furthermore, the quantum theory turns out to be equivalent to a model for diffeomorphism invariant gauge theories which replaces the space manifold with a manifestly combinatorial object (see chapter 4). Loop quantization yields a quantum theory which is sensitive only to the combinatorial information on the space manifold. Thus, it fulfills the expectations of a framework tailored to study generally covariant theories.

Since the Hamiltonian constraint is a diffeomorphism covariant function, it is natural for its loop regularization to be $\bar{D}$ covariant (and there are versions of the Hamiltonian constraint which are $\bar{D}$ covariant). Hence, the notion of space in loop quantum gravity is expected to remain combinatorial after the Hamiltonian constraint is imposed. It should be noticed that the original version of Thiemann's Hamiltonian constraint uses the Ashtekar-Lewandowski volume operator which is not $\bar{D}$ covariant. However, the modification of Thiemann's Hamiltonian constraint using the Rovelli-Smolin volume operator is $\bar{D}$ covariant, and it has been shown that it enjoys similar properties; in particular, the algebra of the constraints is not altered by using the $\bar{D}$ covariant version of the volume operator [52]. That the properties of the $\bar{D}$ covariant Hamiltonian constraint are the same as Thieman's is not necessarily a desirable property [51, 52]. In spite of this feature, a combinatorial view of loop quantization does suggest new treatments of dynamics. A discussion on this point and on new proposals for the dynamics of loop quantum gravity is the subject of chapter 6).

Apart from the analytic category, which I have used throughout this chapter, there is the smooth (C^∞) category [34]. The difference is that the allowed graphs have smooth edges; because of this, it is necessary to include "wild graphs," which are graphs whose edges intersect infinitely many times between vertices. Some aspects of this framework require a more careful analysis, but the quantization strategy is essentially the same. However, in view of the results presented in this thesis, part of the motivation to develop a refined version of the smooth category is lost. The quantum gauge group constructed by loop quantization is an appropriate completion of the diffeomorphism group, not the diffeomorphism group itself. Smoothness is considered as a semiclassical/macroscopic property of space by most approaches to quantum gravity. How to reconcile this notion with the quantization of the classical theory is a puzzling problem. This is part of the motivation behind a proposal by Louko and Sorkin of considering more general groups than the diffeomorphism group as the gauge group of general relativity [60].

If smoothness is not considered as fundamental, one has to find the characteristics of the arena of the fundamental theory. By completing the diffeomorphism group, loop quantization gives a precise replacement of classical smooth space: only the combinatorial information of the manifold is relevant in the quantum theory.

Chapter 6

On the dynamics

At this point the lattice approach (chapters 2 and 4) and the loop approach (chapters 4 and 5) share their fundamental assumptions. The difference between both approaches makes different ideas arise more naturally in one than in the other; for example, they suggest different regularizations of the Hamiltonian constraint. In spite of their differences, both success and problems are shared because they are directly related to the fundamental assumptions. The regularization of geometric operators and the solution of the diffeomorphism constraint are successes of both approaches. Similarly, essential problems, like the structural problem in the dynamics (described below), are shared by both approaches. In this chapter I discuss some problems in the dynamics and find their origins in the fundamental assumptions; in addition, I comment on a new proposal for dynamics, and why it may solve some of the problems in the dynamics of loop quantization.

As I argued in the previous chapter, according to the operator norm the diffeomorphism group is a discrete group; hence, it has no generators. In order to find generators of diffeomorphisms, one must modify the framework somehow. Finding a suitable space of distributions over the kinematical Hilbert space is a possibility [52]. In such a framework, one could proceed to regularize the Hamiltonian constraint. If the representation of the constraint algebra is anomaly-free, one may represent observables in the space of physical states defined by the constraints. On the other hand, if one is not willing to modify the kinematical arena, imposing a Hamiltonian constraint creates structural problems in loop quantization: The asymmetry in the treatment of the diffeomorphism and the Hamiltonian constraints makes it difficult to prove that the algebra of the constraints is anomaly-free. Having a regularization of the Hamiltonian constraint, while not having generators of diffeomorphisms is consistent with the assumptions discussed in chapters 1 and 3; assuming that space and time are fundamentally different makes an asymmetric treatment of the Hamiltonian and diffeomorphism constraints possible. In some sense, treating space and time as separate entities leads to the structural problems in the dynamics of loop quantum gravity. Ignoring the structural problems for the moment, one can explore the regularizations of the Hamiltonian constraint that arise from the combinatorial and from the continuum perspectives.

The combinatorial picture of space suggests a simple lattice-like regularization of the Hamiltonian constraint. As in regular lattice gauge theories, one can prove that the algebra of the constraints resembles the continuum algebra, but it has corrections that vanish in the continuum limit of regular lattice gauge theories. However, in loop quantization the correction terms do not vanish. The continuum limit (where the lattice spacing, measured in a background metric, is reduced to zero) was replaced by the projective limit, and the correction terms do not vanish in the projective limit. Apart from the structural problems alluded above, one is left with the difficult regularization problem of finding a refinement of the simple lattice regularization (see discussion in chapter 2).

As mentioned in the previous chapter, a version of Thiemann's Hamiltonian constraint can be defined the PL and the *piecewise* analytic categories of loop quantum gravity. This extension uses the Rovelli-Smolin volume operator instead of the Ashtekar-Lewandowski volume operator; however, the algebra of the constraints so defined remains unchanged [52]. By reproducing the algebra of Thieman's Hamiltonian constraint, we also adopt its problems. When the commutator of two Hamiltonian constraints acts on a diffeomorphism invariant state, one gets zero. In a sense, the problem is that the Hamiltonian constraint is too local [51]; it has also been shown that the constraint can be extended to a framework where the diffeomorphism group is faithfully represented, and that the commutator of the resulting Hamiltonian constraints also vanish [52].

I believe that the covariant analog of the strategy proposed in chapter 2 –the strategy behind the spin foam models– is a more promising avenue in understanding the dynamics of loop quantum gravity. It preserves most of loop quantization's kinematics while avoiding an asymmetric treatment of Hamiltonian and diffeomorphism constraints. The strategy, chiefly advocated by Reisenberger, gives a combinatorial basis for quantum spacetime that is manifestly 4d covariant. In the next paragraphs I will briefly describe the spin foam models highlighting the significance of the work presented in this thesis.

Given a triangulated four-manifold of the type $\Sigma \times [0,1]$, a spin foam model produces transition amplitudes from the lattice gauge theory state space defined by the triangulation of Σ_0 to the state space of Σ_1. These transition amplitudes are computed from a partition function derived from a discretization of the Plebanski action [5, 8]; this action constrains the B field of BF theory to come from a tetrad. One may also begin with the Hilbert-Palatini action and impose the geometricity conditions (simplicity conditions in the Barrett-Crane terminology) by hand [9]. That is, once again we are using BF theory as a stepping stone in our path towards quantum gravity. In this respect, the canonical approach gives some information about the nature of the geometricity conditions (2.41), which lie at the heart of the quantization program. For example, one of the consequences of the constraint imposed by the second term in the Plebanski action is $A = \omega^+$, where A is the configuration variable, and ω is the spacetime connection determined uniquely by the tetrad induced by the B fields solving the constraint. In the continuum, and in the lattice (see chapter 2), the geometricity conditions imply that the connection is torsion free. If the quantum constraint is replaced by a Gaussian potential, as proposed in [8], there is a danger of having microscopic torsion. In the presence of torsion the energy momentum tensor has torsion contributions usually associated to the spin of matter fields; only by adding these torsion terms do Einstein's equations ensure an automatic conservation of energy-momentum. Adding microscopic torsion that is not correlated to the matter fields could lead to "microscopic leaks of energy-momentum." A softened form of the geometricity conditions may be necessary and may lead to the correct classical limit; the previous observation only suggests that a careful analysis of its physical consequences is needed [61].

Another observation is that the geometricity conditions imposed in [9] yield second-class constraints that can be solved by using self-dual variables. A covariant analog of this fact should provide a proof of the equivalence of Reisenberger's approach (without softening the constraints) and the approach of Barrett and Crane [9].

In the spin foam models, one can write the transition amplitudes as a sum over branched surfaces whose weight depends only on topological features of the branched surface [8]; having a fixed triangulation only restricts the number of branched surfaces and the knot-classes of branched surfaces that are included in the sum. This is a step toward discarding the extra background structure added by the triangulation; the goal of the spin foam models is to provide finite transition amplitudes between the state spaces $\mathcal{H}_{\text{kin}_{\text{PL}}}(\Sigma_0)$ and $\mathcal{H}_{\text{kin}_{\text{PL}}}(\Sigma_1)$ (or their counterparts in the combinatorial category) defined in chapter 4. Finding finite transitions amplitudes from a sum over surfaces may be achieved by a normalization procedure and a method analogous to the group average method used to solve the diffeomorphism constraint [26] (see chapter 4). From the study done in chapters 4, 5, one may expect that the piecewise linear or piecewise analytic background structure does not affect the resulting amplitudes. At the heart of loop quantization is the quantization of flux; the spirit of the spin foam models is to find a covariant formalism which has quantized flux world-sheets as primary ingredient. The triangulated manifold is meant only as a combinatorial tool that keeps track of the flux world-sheets that form spacetime.

Apart from the conceptual satisfaction of having a 4d covariant framework, one also expects that a formulation of loop quantization in terms of path integrals will provide the necessary tools to study the semiclassical/macroscopic limit of the theory. Once this limiting behavior is understood, important problems, like providing a quantum background in which quantum field theories are naturally free of ultraviolet divergences, will be within reach (see for example [62]).

Some of the assumptions, on which our new framework relies, come from those of the combinatorial approach to canonical loop quantization; however, the covariant formalism modifies some of them. In the spin foam models developed up to today, spacetime is fundamental and has a combinatorial origin. A signature of this new assumption is that the spin foam models have manifest 4d covariance, and simultaneously they can restrict to four valent intertwiners; in the view of traditional loop quantum gravity, spatial diffeomorphism invariance makes a restriction on the valence of the intertwiners impossible. Treating spacetime as fundamental may be the assumption that avoids the structural problems, while retaining loop quantization's kinematics.

Most of the work has been done for Euclidean spacetimes. One may consider them as valuable toy models, but considering the Lorentzian nature of spacetime as a feature that may be incorporated later may not be wise. Exploring manifestly Lorentzian strategies ought to drive much of the work in the field in the years to come. An interesting suggestion is that a Wick rotation may be the only necessary modification of the models to treat the Lorentzian regime [63]. Another possibility, advocated by Markopoulou and Smolin, is to have the amplitudes assigned to the branched surfaces in the state sum depend on a microscopic causal structure encoded in the triangulation [64].

This thesis used three articles on quantum gravity and quantum gauge theory to study a path towards a complete theory. I showed how some of the initial problems were solved, and I discussed possible solutions for some of the remaining problems of this approach to quantum gravity and quantum gauge theory.

Appendix A

The symmetry group of the model

The symmetry group of the model (excluding $SO(3,1)$ symmetries) is the group of cell translations. It has dimension $4N_3$ and is a subgroup of the symmetry group of GLBF (the group of vertex translations) that has dimension $4N_0$. Thus, the fact that the symmetry group of the model is smaller than that of GLBF follows from the fact (to be shown bellow) that $N_0 > N_3$ in locally Euclidean simplicial lattices.

In three dimensions, the Euler number is zero, i.e., $N_0 - N_1 + N_2 - N_3 = \chi = 0$ where N_i is the number of i-dimensional simplices (points, links, faces, and cells). Then the difference between N_0 and N_3 is the same as that between N_1 and N_2. In a simplicial lattice there are three links in each face, and each link is shared by three or more faces (excluding lattices with zero volume cells). Thus, $N_1 \geq N_2$, the equality holding only in a case where every link is shared by three faces. A simplicial lattice with this connectivity can not be deformed in to a flat lattice as I explain now: It is easy to prove that in Euclidean three dimensional space one cannot draw a tetrahedron and its four neighbors in such a way that these five tetrahedra form a convex polyhedron. Start embedding one tetrahedron in R^3, then because every link is shared by three faces the four neighbors of the tetrahedron must have their faces identified with each other. Thus, the five tetrahedron embedded in R^3 form a convex polyhedron that has no boundary faces. The contradiction indicates that it is impossible to draw a three-dimensional simplicial lattice with more than five cells where three faces share each link or equivalently only simplicial lattices with $N_0 > N_3$ fit locally in three-dimensional Euclidean space.

Appendix B

Completeness of combinatorial graphs

First we will prove lemma 4.5, and then, indicate how the proof can be extended to link our models and the refined version of the analytic category that was mentioned in section 4.4.

Given an oriented PL graph $\vec{\gamma}_{PL} \subset |K|$ we will construct an oriented combinatorial graph $\vec{\gamma}$ and a piecewise linear homeomorphism (PL map) $h : |K| \to |K|$ such that $h(|\vec{\gamma}|) = \vec{\gamma}_{PL}$. The construction has four steps.

1. Let $\vec{\gamma}'_{PL}$ be a refinement of $\vec{\gamma}_{PL}$ such that for every $\Delta(x_n) \in |K|$ $e \in E_{\gamma'_{PL}}$ implies that $e \cap \Delta(x_n)$ is empty or linear according to the affine coordinates given by $\Delta(x_n)$.

2. Find n such that $M_n(|Sd^n(K)|)$ separates the vertices of γ'_{PL} to lie in different geometric simplices $M_n(\Delta(x_n))$, where $\Delta(x_n) \in |Sd^n(K)|$. Namely, we chose n as big as necessary to accomplish a fine enough refinement of $|K|$, where $v_1, v_2 \in M_n(\Delta(x_n))$ for two different vertices of the PL graph $v_1, v_2 \in V_{\gamma'_{PL}}$ does not happen.

3. Let $h_1 : |K| \to |K|$ be the PL map that fixes the vertices of $M_n(|Sd^n(K)|)$ and sends the new vertices $M_{n+1}(v(\Delta(x_n)))$ of $M_{n+1}|(|Sd^{n+1}(K)|)$ to
   (a) $v \in V_{\gamma'_{PL}}$ if v lies in the interior of $M_n(\Delta(x_n))$; symbolically, $v \in (M_n(\Delta(x_n)))^\circ$.
   (b) the baricenter of $M_n(\Delta(x_n))$ if there is no $v \in V_{\gamma'_{PL}}$ such that $v \in (M_n(\Delta(x_n)))^\circ$.

4. Find m such that $h_1(M_{n+m}(|Sd^{n+m}(K)|))$ separates the edges of γ_{PL}. Stated formally, find $m \geq 1$ such that $\gamma_{PL} \cap h_1(M_{n+m}(\Delta(x_{n+m})))^\circ$ has one connected component or it is empty.

5. Let $h = h_2 \circ h_1 : |K| \to |K|$, where h_2 is the PL map that fixes the vertices of $h_1(M_{n+m}(|Sd^{n+m}(K)|))$ and sends the new vertices $h_1(M_{n+m+1}(v(\Delta(x_{n+m}))))$ of $h_1(M_{n+m+1}|(|Sd^{n+m+1}(K)|))$ to
   (a) the baricenter of $\gamma_{PL} \cap h_1(M_{n+m}(\Delta(x_{n+m})))$ if $\gamma_{PL} \cap (h_1(M_{n+m}(\Delta(x_{n+m}))))^\circ \neq \emptyset$.
   (b) the baricenter of $h_1(M_{n+m}(\Delta(x_{n+m})))$ if $\gamma_{PL} \cap (h_1(M_{n+m}(\Delta(x_{n+m}))))^\circ = \emptyset$.

From the construction of $h \circ M_{n+m} : |Sd^{n+m}(K)| \to |K|$ it is immediate that $(h \circ M_{n+m})^{-1}(\gamma_{PL}) = |\gamma_{n+m}|$ if $\gamma_{n+m} \subset Sd^{n+m}(K)$ is defined by

- The zero-dimesional simplex $p \in Sd^{n+m}(K)$ belongs to γ_{n+m} if $(h \circ M_{n+m})^{-1}(\gamma_{PL}) \cap |p| \neq \emptyset$.
- The one-dimesional simplex $e \in Sd^{n+m}(K)$ belongs to γ_{n+m} if $(h \circ M_{n+m})^{-1}(\gamma_{PL}) \cap |e|^{\circ} \neq \emptyset$.

Then the obvious orientation of γ_{n+m} defines the oriented combinatorial graph $\vec{\gamma}$ and the pair h, $\vec{\gamma}$ satisfies

$$h(|\vec{\gamma}|) = \vec{\gamma}_{PL} \quad \diamondsuit \tag{B.1}$$

To link the combinatorial and the analytic categories we need to fix a map $N_0 : |K| \to \Sigma_{P\omega}$ that assigns a piecewise analytic curve in $\Sigma_{P\omega}$ to every PL curve of $|K|$. Then the map $N : Cyl_C \to Cyl_\omega$ defined by

$$N(f_\gamma) := f_{N_0 \circ M_n(|\gamma_n|)} = f_{N_0(|\gamma|)} \tag{B.2}$$

links the kinematical Hilbert spaces, and the map $N^* : Cyl^*_\omega \to Cyl^*_C$ links the spaces of physical states of the analytic and combinatorial categories. As it was argued in section 4.3 N^* is an isometry between $\mathcal{H}_{\mathrm{diff}_\omega}$ and $\mathcal{H}_{\mathrm{diff}_C}$, which means that the two Hilbert spaces are isomorphic if every knot-class of piecewise analytic graphs $[\gamma_\omega]$ has at least one representative that comes from a combinatorial graph $N_0(|\gamma|) \in [\gamma_\omega]$.

An extension of the lemma proved in this appendix solves the issue. Given a piecewise analytic graph $\gamma_\omega \subset \Sigma_{P\omega}$ we can construct a combinatorial graph γ and a piecewise analytic map ϕ such that $\phi \circ N_0(|\gamma|) = \gamma_\omega$. First find a refinement γ'_ω of γ_ω such that its edges are analytic according to the domains of analycity of $\Sigma_{P\omega}$. Then, define a graph in $|K|$ by $\alpha = N_0^{-1}(\gamma'_\omega)$ and do steps (2), (3) and (4) using α instead of γ_{PL}. At this moment $N_0 \circ h_1 \circ M_{n+m}(|Sd^{n+m}(K)|)$ separates the edges of γ'_ω; we only need to find a replacement for step (5). Our strategy is to find a map of the form $\phi = \phi_2 \circ N_0 \circ h_1$ to solve the problem. This would be achieved if the piecewise analytic diffeomorphism ϕ_2 fixes the mesh given by $N_0 \circ h_1 \circ M_{n+m}(|Sd^{n+m}(K)|)$ and at the same time matches the mesh given by $N_0 \circ h_1 \circ M_{n+m+1}(|Sd^{n+m+1}(K)|)$ and the graph γ'_ω. The map ϕ_2 needs to send every cell $N_0 \circ h_1 \circ M_{n+m}(\Delta(x_{n+m}))$ to itself and matche the graph with analytic edges. An explicit construction would be cumbersome, but the existence of such a *piecewise* analytic map is clear. After this is completed, the construction of the combinatorial graph follows the instructions given above to link the combinatorial and PL categories.

References

[1] Rovelli C 1998 Plenary lecture on quantum gravity at the GR15 conference, Poona, India. *e-Print Archive*: gr-qc/9803024
Isham C 1995 Plenary lecture on quantum gravity at the GR14 conference, Firenze, Italy. *e-Print Archive*: gr-qc/9510063

[2] Waelbroeck H and Zapata J A 1994 *Class. Quant. Grav.* **11** 989

[3] Horowitz G 1989 *Commun. Math. Phys.* **125** 417

[4] Ashtekar A and R. Tate 1991 *Lectures on non-perturbative canonical gravity.* Advanced Series in Astrophysics and Cosmology-Vol. 6. Singapore: World Scientific

[5] Plebanski J 1977 J. Math, Phys. **18** 2511
Capovilla R, Dell J and Jacobson T 1991 *Class. Quant. Grav.* **8** 59
Capovilla R and Jacobson T 1992 *Class. Quant. Grav.* **9** 1839
Capovilla R, Dell J and Jacobson T 1989 *Phys. Rev. Lett.* **63** 2325
Capovilla R, Dell J, Jacobson T and Mason L 1991 *Class. Quant. Grav.* **8** (1991) 41
Peldán P 1991 *Class. Quant. Grav.* **8** (1991) 1765

[6] Reisenberger M P 1995 *Nucl. Phys.* **B457** 643

[7] Ashtekar A, Rovelli C and Smolin L 1991 *Phys. Rev.* **D44** 1740
Rovelli C and Smolin L 1995 *Nucl. Phys.* **B442** 593; Erratum *Nucl. Phys.* **B456** 753
Ashtekar A and Lewandowski J 1997 *Class. Quant. Grav.* **14** A55

[8] Reisenberger M P 1997 *Classical and Quantum Gravity* **14** 1753; 1997 A lattice worldsheet sum for 4-d Euclidean general relativity *e-Print Archive*: gr-qc/9711052

[9] Barrett J W and Crane L 1998 To be published in *J. Math. Phys.*, *e-Print Archive*: gr-qc/9709028
Baez J 1997 Spin foam models, *e-Print Archive*: gr-qc/9709052

[10] Regge T 1961 *Nuovo Cimento* **19** 558

[11] Immirzi G 1994 *Class. Quan. Grav.* **11** 1971

[12] Katsymovsky V 1991 *Class. Quan. Grav.* **6**L249

[13] Waelbroeck H 1990 *Class. Quan. Grav.* **7**(1) 751

[14] Sorkin R 1975 *J. Math. Phys.* **16** 2432
Sorkin R 1978 *J. Math. Phys.* **19** 1800

[15] Brewin L 1989 *Phys. Rev.* **D39**(8) 2258

[16] Ashtekar A 1987 *Phys. Rev.* **D36**(6)1587, see also [4]

[17] Immirzi G 1996 *Class. Quan. Grav.* **13** 2385

[18] Thiemann T 1995 Contribution to the Workshop "Recent mathematical developments in classical and quantum gravity", Sintra, Portugal. *e-Print Archive*: gr-qc/9511049, CGPG-95/11-3, HUTMP-95-B-347; 1996 *Class. Quan. Grav.* **13** 1383

[19] Peldán P 1994 *Nuc. Phys.***B430** 460

[20] Renteln P and Smolin L 1989 *Class. Quan. Grav.* **6**275
Renteln P 1990 *Class. Quan. Grav.* **7**(3)493

[21] Waelbroeck H 1994 *Phys. Rev.* **D50** 4982

[22] Waelbroeck H 1990 *Phys. Rev. Lett.* **64** 2222

[23] Arnold V 1978 *Mathematical Methods of Classical Mechanics*
Graduate Texts in Mathematics; 60 Springer-Verlag, New York

[24] Boström O, Miller M and Smolin L 1994 A new discretization of classical and quantum general relativity, *e-Print Archive*: gr-qc/9304005, Göteborg ITP 94-5, SU-GP-93-4-1, CGPG-94/3-3

[25] Ashtekar A 1996 *Phys. Rev.* **D53** 2865

[26] Ashtekar A and Isham C J 1992 *Class. Quant. Grav.* **9** 1069
Ashtekar A and Lewandowski J 1995 *J. Math. Phys.* **36** 2170
Ashtekar A, Lewandowski J, Marolf D, Mourao J and Thiemann T 1995 *J. Math. Phys.* **36** 6456; 1994 in Cambridge Proceedings, Geometry of constrained dynamical systems, *e-Print Archive*: hep-th/9408108
Baez J 1996 *Adv. Math.* **117**, 253; 1996 in *The Interface of Knots and Physics*, ed. L. Kauffman, American Mathematical Society

[27] Loll R 1995 *Nuc. Phys.* **B444** 619

[28] Corichi A and Zapata J A 1997 *Nucl. Phys.* **B 493** 475

[29] Ooguri H 1992 Modern Physics Letters **A 7** 2799
Crane L, Kauffman L and Yetter D 1997 *J. Knot Theory Ram.* **6** 177

[30] Ashtekar A, Corichi A and Zapata J A 1997 Quantum theory of geometry III: non-commutativity of Riemannian structures (Presented at the Warsaw meeting on "Lower-dimensional topology and quantum gravity" 1996, *e-Print Archive*: gr-qc/9806041

[31] Mandelstam S 1962 *Ann. Phys.* (NY) **19** 1 Wilson K G 1974, *Phys.Rev.* **D10** 2445

[32] Rovelli C and Smolin L 1990, *Nuc. Phys.* **B331**(1) 80
Gambini R and Trias A 1986 *Nucl. Phys.* **B 278** 436
Gambini R and Pullin J 1996, *Loops, Knots, Gauge Theories and Quantum Gravity.* Cambridge University Press

[33] Ashtekar A, Lewandowski J, Marolf D, Mourao J and Thiemann T 1997 *J. Math. Phys.* **38** 5453

[34] Baez J and Sawin S 1995 Functional Integration on Spaces of Connections, *e-Print Archive*: q-alg/9507023

[35] Ashtekar A 1996 To appear in the Proceedings of GR-14, *e-Print Archive*: hep-th/9601054

[36] Ashtekar A and Lewandowski A 1993 in *Quantum Gravity and Knots*, ed. J. Baez, Oxford University Press

[37] Ashtekar A and Lewandowski J 1994 in *Knots and quantum gravity*, ed. J. Baez, Oxford University Press
Baez J 1993 in *Proc. of the Conference on Quantum Topology*, ed. D. Yetter, World Scientific

[38] Ashtekar and Lewandowski J 1995 *J. Geom. Phys.* **17** 191

[39] De Pietri R and Rovelli C 1996 *Phys. Rev.* **D54** 2664

[40] Fritsch R and Piccinini R A 1990 *Cellular Structures in Topology* Cambridge studies in advanced mathematics. Cambridge University Press

[41] Ashtekar A and Tate R 1994 *J. Math. Phys.* **35**, 6434
Ashtekar A 1995 in *Gravitation and Quantization*, ed. Julia B Elsevier Amsterdam
Marolf D 1995 Refined Algebraic Quantization: Systems with a single constraint, *e-Print Archive*: gr-qc/9508015

[42] Husain V and Kuchař K V 1990 *Phys. Rev.* **D42**, 4070

[43] Rado T 1925 *Acta Litt. Scient.* bf 2 101
Moise 1952 *Ann. Math.* **56** 96

[44] Streater R F and Wightman A S 1964 *PCT spin statistics and all that* Addson-Wesley

[45] Rovelli C and Smolin L 1994 *Phys. Rev. Lett.* **72** 446
Rovelli C 1995 *J. Math. Phys.* **36** 5629
Borissov R 1997 *Phys. Rev.***D55** 6099
Gambini R and Pullin J 1996 *Phys. Rev.***D54** 5935
Loll R. 1996 *Phys. Rev.***D54** 5381

[46] Thiemann T 1996 *Phys. Lett.* **B 380** 257

[47] Reisenberger M P and Rovelli C 1997 *Phys. Rev.* **D56** 3490

[48] Ashtekar A and Rovelli C 1992 *Class. Quant. Grav.* **9** 1121
Iwasaki J and Rovelli C 1993 *Int. J. Mod. Phys.* **D1** 533
Iwasaki J and C. Rovelli 1994 *Class. Quant. Grav.* **11** 1653
Rovelli C 1996 *Phys. Rev. Lett.* **77** 3288
Krasnov K 1997 *Phys. Rev.* **D55** 3505

[49] Grot N and Rovelli C 1996 *J.Math.Phys.* **37** 3014

[50] Barbero F 1995 *Phys. Rev.* **D51** 5507

[51] Smolin L 1996 The classical limit and the form of the hamiltonian constraint in nonperturbative quantum gravity, *e-Print Archive*: gr-qc/9609034

[52] Lewandowski J and Marolf D 1997 Loop constraints: A habitat and their algebra, *e-Print Archive*: gr-qc/9710016
Gambini R, Lewandowski J, Marolf D and Pullin J 1997 On the consistency of the constraint algebra in spin network quantum gravity, *e-Print Archive*: gr-qc/9710018

[53] Smolin L 1992 in *Quantum gravity and cosmology* eds. Pérez Mercader J. et al, World Scientific, Singapore

[54] Giles R 1981 *Phys. Rev.* **D24** 2160
Anandan J 1981 *Proc. Conf. Differential Geometric Methods in Physics* (Trieste 1981) eds. Denardo G et al, World Scientific, Singapore
Barrett J W 1985 *Int. Jour. Theor. Phys.* **30** 1171
Lewandowski J 1993 *Class.Quant.Grav.* **10** 879

[55] Penrose R 1971 in *Quantum theory and beyond* ed. Bassin T, Cambridge University Press; in *Combinatorial Mathematics and its Application* ed. Welsh D J A, Academic Press
Rovelli C and Smolin L 1995 *Phys.Rev.* **D52** 5743
Baez J 1996 in *The Interface of Knots and Physics*, ed. Kauffman L, American Mathematical Society

[56] De Pietri R 1997 *Class.Quant.Grav.* **14** 53
Frittelli S, Lehner L and Rovelli C 1996 *Class.Quant.Grav.* **13** 2921

[57] Thiemann T 1997 QSD III : Quantum Constraint Algebra and Physical Scalar Product in Quantum General Relativity, *e-Print Archive*: gr-qc/9705017

[58] Ashtekar A and Rovelli C 1992 *Class. Quant. Grav.* **9** 1121
Morales-Tecotl H A and Rovelli C 1994 *Phys. Rev. Lett.* **72**, 3642
Krasnov K 1996 *Phys. Rev.***D53** 1874

[59] Baez J, Krasnov K 1998 *J. Math. Phys.* **39** 1251

[60] Louko J and Sorkin R D 1997 *Class.Quant.Grav.* **14** 179

[61] Zapata J A *Work in progress.*

[62] Aroca J M, Fort H and Gambini R 1996 em Phys. Rev. **D54** 7751
Reisenberger M P 1994 Worldsheet formulations of gauge theories and gravity, *e-Print Archive*: gr-qc/9412035

[63] Crane L and Yetter D 1997 On the semiclassical limit of the balanced state sum, *e-Print Archive*: gr-qc 9712087

[64] Markopoulou F and Smolin L 1997 *Nucl. Phys.* **B508** 409
Markopoulou F 1997 Dual formulation of spin network evolution, *e-Print Archive*: gr-qc/9704013