The Pennsylvania State University
The Graduate School
Department of Physics

SPIN FOAM MODELS

A Thesis in
Physics
by
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Submitted in Partial Fulfillment
of the Requirements
for the Degree of

Doctor of Philosophy

August 1999

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Abstract

The term 'spin foam models' was introduced by Baez to refer to a new approach to the quantization of general relativity, which appeared as an offsping of loop quantum gravity. Although this new approach was motivated, logically and historically, by loop quantum gravity, by now it has become clear that the two approaches are rather independent. While loop quantum gravity attempts to give a canonical quantization of general relativity, the spin foam model approach aims at giving a precise meaning to the gravity path integral. Eventually, the two approaches will probably be shown to be essentially equivalent, but no rigorous result to this effect exists as of now. In this thesis I develop the spin foam quantization of gravity ab initio, referring to results from loop quantum gravity only for comparison. I start from a review of 2+1 gravity and discuss different routes to quantize it. While some of these, for example, using Chern-Simons theory, only exist in 2+1, others can be generalized to higher dimensions. Spin foam models give such a generalization. Developing it, we will encounter, in particular, a deep relation between group representation theory and geometry. I also discuss in some detail related topics, such as topological field theory, notably BF theory, higher-dimensional Euclidean geometry and quantum groups.

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Acknowledgments

I would like to thank the Center for Gravitational Physics and Geometry at Penn State, all its faculty, Post Docs and graduate students for creating one of the most stimulating environments I ever had a chance to work in. In particular, I would like to thank Abhay Ashtekar, the Director of the Center, for his labor that allows the Center to exist and run smoothly. I am grateful to our muse Karen Brewster, whose help in administrative matters is always deeply appreciated.

I am grateful to Abhay Ashtekar, who during these years was for me not only the Director of the Center and my advisor, but also one of the brightest people I ever met, and an excellent teacher. I thank him for all the lessons he gave me, both in science and in life.

I am grateful to members of the Center and visitors for many motivating discussions. In particular, I am grateful to John Baker, Max Banados, Chris Beetle, Roumen Borissov, Roberto De Pietri, Sameer Gupta, Eli Hawkins, Seth Major, Fotini Markopoulou, Roger Penrose, Jorge Pullin, Carlo Rovelli, Michael Reisenberger, Lee Smolin, Thomas Thiemann. I am especially grateful to my collaborators, Abhay Ashtekar, John Baez, Alejandro Corichi, Laurent Freidel, Raimond Puzio, for their understanding and patience.

The moral support of my parents and friends is especially acknowledged, for without them I would never be able to complete this work.

Finally, I have to thank the Department of Physics of Penn State for their support, and for nominating me for Braddock Graduate Fellowship, 1997 and David C. Duncan Graduate Fellowship, 1998. I acknowledge the support from the NSF grant PHY95-14240 and from the Eberly research funds of Penn State. I am also grateful to the Institute for Theoretical Physics, University of California, Santa Barbara, where this thesis was written. During my visit to ITP I was supported from NSF grant No. PHY94-07194.

Preface

The subject of quantum gravity has a long history. In fact, it is almost as old as general relativity itself. As early as 1916, Einstein discussed[1] a possibility that it may be assential to treat the gravitational field quantum mechanically [28]. Since then we have experienced an enormous progress in our understanding of both quantum theory and general relativity. However, the problem of quantum gravity is still open. There are many approaches to the subject, each enjoying a varying degree of success. The most popular nowadays is probably string theory. In this thesis I develop another approach to the problem. The approach pursued here takes very seriously the lesson given to us by Einstein: gravity is geometry. Thus, the problem of quantum gravity is treated here as the problem of quantum description of geometry.

More specifically, I develop an approach, for which the name "spin foam models" was proposed by Baez [9]. This approach is intimately related to loop quantum gravity [58], and was, in fact, inspired by it. Loop quantum gravity attempts to give the canonical quantization of general relativity; spin foam models can be thought of as the corresponding path integral description [51]. Although it is natural to expect this relation on general grounds, a precise argument to this effect does not exist as of now. In this thesis I take an alternative, independent route to spin foam models, and will refer to results of loop quantum gravity only for comparison. Namely, to arrive to spin foam models I will start from the classical action of general relativity and try to give meaning to the corresponding path integral. As a guide, I will use the intuition from topological field theories and 2+1 gravity.

At the outset, I wish to point out that some of the results I am using here are not mine. I will make clear which results are mine and which are not through references. Most of my results presented here were obtained in collaboration with Laurent Freidel. I am happy to use this as another occasion to thank him for his patience. I would also like to say that in several places I am using other author results giving them an interpretation different from that in the original work. This may not agree with the author's viewpoint, and I am the only one to be blamed for any misconceptions.

The organization of this thesis is as follows. In the next chapter I discuss in some details the quantization of 2+1 gravity. The subject of 2+1 gravity is vast, see e.g. the recent book [20]. I am presenting here only what is relevant to the viewpoint I am developing. The main point which I want to make in this chapter is that there exists a non-perturbative definition of the path integral for the theory given by Ponzano-Regge or Turaev-Viro models. This non-perturbative definition allows a generalization to higher dimensions, which will be given by spin foam models. Chapter 2 develops a relation between classical (and quantum) Euclidean geometry and group representation theory. I

[1] He wrote: "Nevertheless, due to the inner atomic movement of electrons, atoms would have to radiate not only electromagnetic, but also gravitational energy, if only in tiny amounts. As this is hardly true in nature, it appears that quantum theory would have to modify not only Maxwellian electrodynamics, but also the new theory of gravitation."

start here with the Ponzano-Regge formula for the asymptotics of (6j)-symbol, and then discuss quantization of a geometric simplex in 2+1 dimensions. This gives a geometric interpretation to various constructions that will be developed later. Motivated by all these results, I introduce in chapter 3 the idea of a spin foam model. In the approach I am presenting here, spin foam models arise as certain deformations of the topological field theory known as BF theory. Thus, in chapter 4 I describe BF theory, both classically and quantum mechanically. Chapter 5 is one of the central ones; here I calculate the generating functional for BF theory, from which all spin foam models will be derived later. Chapter 6 gives a classical description of a large class of physically interesting theories as deformations of the BF theory, where deformation means a modification of the BF theory action by addition to it of an interaction term. In chapter 7 I remind the reader the known state sum models for the theories described. Chapter 8 obtains these, as well as few other models using the technique of generating functional. Finally, using all the results obtained, Chapter 9 generalizes the relation between geometry and representation theory, which was discussed in Chapter 2, to higher dimensions. Chapter 10 is devoted to a discussion of results achieved.

Chapter 1

Introduction: 2+1 gravity

The logic of the thesis will be to first discuss the quantum description of geometry on the well-understood example of 2+1 gravity, and then generalize this description to higher dimensions. We start in section 1.1 with a review of different methods to quantize 2+1 gravity. The emphasis will be placed on path integral type quantization, which is described in section 1.2. Here I introduce Ponzano-Regge and Turaev-Viro models. In section 1.3 I discuss how these models can be derived starting from the classical action functional, and compare perturbative and non-perturbative derivations. The non-perturbative derivation presented here will be the starting point for introduction of spin foam models in chapter 3. Here we will also encounter a relation between classical (quantum) geometry and group representation theory. This relation will be the main theme of the next chapter.

1.1 Quantizations

There are many ways to quantize 2+1 gravity, see, e.g., the recent book [20] on the subject. Here we will discuss the different quantization methods only schematically, in the extent we need for later constructions. Most of the results presented here are about ten years old, developed in a spur of activity that followed Witten's pioneering work [63] on the subject.

Classical theory

Let us start with a brief review of the classical theory. There are two different, although equivalent, classical descriptions available. First, in the spirit of Einstein, one can describe 2+1 gravity as a theory of metrics. The action, which is a functional of the Lorentzian signature metric g_{ab}, is the standard Einstein-Hilbert action:

$$\frac{1}{2}\int_{\mathcal{M}} d^3x\,\sqrt{-g}\,(R-2\Lambda), \tag{1.1}$$

where g is the determinant of the metric, R is the trace of the Ricci tensor and Λ is the cosmological constant. We use the units in which $8\pi G = 1$ throughout. We shall also assume everywhere that the spacetime manifold $\mathcal{M}$ is smooth and is either compact or has the topology $\Sigma \times \mathbb{R}$, where Σ is compact. Which of the two choices is used will depend on the context.

The above, geometrodynamics description of 2+1 gravity is, however, not the one best suited for quantization. Let us now consider another, classically equivalent description of 2+1 gravity. This description treats it as a theory of connections rather than metrics. Namely, the action for gravity in the so-called first order formalism is a functional of the frame field, or triad e^I_μ, and the spin connection $w^{IJ}_{\mu\nu}$:

$$S[e,w] = \int_{\mathcal{M}} \left(e^I \wedge F^{JK} - \frac{\Lambda}{6} e^I \wedge e^J \wedge e^K \right) \epsilon_{IJK}. \tag{1.2}$$

Here F is the curvature of the spin connection w, μ, ν are the spacetime indices, and I, J, K are the internal ones. Varying this action with respect to w one obtains the equation saying that the frame is covariantly constant with respect to the connection w. This equation can be solved for w, which determines w as the spin connection compatible with the frame. Substituting the solution into the action, one obtains the original Einstein-Hilbert action. This means that the first order formulation is equivalent to the geometrodynamics one, at least classically, in the sense that all solutions of Einstein's gravity are also solutions of this theory. Note, however, that the reverse is not true: there are solutions of the first order gravity that are not solutions of general relativity. For example, the 'zero metric' solution $e, w = 0$ solves all equations of motion in the absence of the cosmological constant, but is quite pathological from the point of view of geometrodynamics. More generally, the first order gravity provides an extension of general relativity to degenerate metrics. As we shall see, degenerate metrics play an especially important role in the quantum theory. It is the fact that the first order action is defined for degenerate metrics that makes it more suitable for the quantization.

The theory becomes especially simple in the case of zero cosmological constant. Indeed, in this case, varying with respect to e one gets:

$$F(w) = 0, \tag{1.3}$$

which means that the space of classical solutions is the space of flat connections, together with the frames compatible with them. This means that the reduced phase space of the theory, which can be thought of as the space of solutions modulo gauge transformations, will usually be finite dimensional! This comes from the fact that the moduli space of flat structures on $\mathcal{M} = \Sigma \times \mathbb{R}$ can be parametrized by homomorphisms of $\pi_1(\Sigma)$ into ISO(3) or ISO(2,1) depending on the spacetime signature, see [63] for the explanation. It is then easy to show that the dimension of the moduli space is $2g-2$, where g is the genus of Σ, times the dimension of G, which is 6 in our case. This means that the dimension of the phase space will be $12g-12$. Thus, 2+1 gravity is a system with finite number of degrees of freedom! One can show that this also holds for a non-zero cosmological constant, in which case one simply has to replace the group of linear transformations by an appropriate conformal group, see [63]. The fact that the phase space of 2+1 gravity is finite dimensional is crucial for the quantization.

As always, there are two main routes to the quantization: canonical and path integrals. Let me first discuss the canonical quantization.

Canonical quantization

As we have seen, when the spatial manifold Σ is compact, 2+1 gravity has only a finite number of degrees of freedom. This makes it possible to treat it as a usual quantum mechanical system. One can apply to it, for example, the method of geometrical quantization. This is what was done in the original paper of Witten [63], and then developed extensively in the works that followed. The advantage of the canonical quantization method is that it is best suited in order to prove that quantum theory exists and is well-defined. However, if one wants to analyze the nature of quantum geometry, the canonical quantization framework *a la* Witten is hardly the best one. Indeed, this quantization deals only with global quantities, and the picture of quantum geometry is obscure. Moreover, the canonical quantization program is so successful in 2+1 because of the 'finitness' of the physical phase space. In higher dimensions one knows this not to be the case. Thus, because in higher dimensions one does not expect to be able to understand the reduced phase as well as in 2+1, the method that is successful in 2+1 cannot be generalized to higher dimensions. Because of this, we shall not develop the canonical quantization further, and turn to the path integral method. As we shall see, this method gives a more transparent picture of quantum geometry, and also serves as a starting point for the higher dimensional generalizations. Let us note, however, that one can canonically quantize 2+1 gravity without first reducing the phase space to that of a system with a number of degrees of freedom. Such canonical quantization can be generalized to higher dimensions. This is the subject of loop quantum gravity [58]. We shall not follow this route here and instead turn to path integrals.

Path integral quantization

Path integral quantization attempts to make sense of the following object:

$$\int \mathcal{D}w\mathcal{D}e\, e^{iS[w,e]}. \tag{1.4}$$

Here we have taken the action to be that in the first order formalism, for it is not known how to make sense of the path integral over the space of metrics.

In the above path integral, one can take the relevant gauge group to be either SO(2, 1) or SO(3), depending on the signature. An important delicate point is the presence of the imaginary unit in the exponential. For the case of Lorentzian signature this is as expected. However, in the case of metrics of Euclidean signature, the above path integral defines not what is usually called Euclidean gravity. Indeed, in the later case one considers the path integral of the exponentiated Euclidean action with no i in the exponential. This corresponds to statistical mechanics, rather than quantum mechanics, and is of importance, for example, for black hole thermodynamics. The path integral of the exponentiated Euclidean action with i in the exponential defines a *quantum theory* of metrics of Euclidean signature. The physical relevance of this theory is disputable. However, at the very least, this theory may be thought of as a playground for a more realistic quantum theory, for example, the one describing metrics of Lorentzian signature. The quantum theory of Euclidean metrics is much simpler due to the fact that the corresponding gauge group is compact. However, even in this case many of the difficult questions of quantum gravity can be addressed without one having to worry about subtleties related to non-compactness. Thus, this 'Euclidean quantum theory' is certainly worth studying. Most of our analysis will be devoted to this theory. There are, however, interesting results available for the Lorentzian case, see [15, 33]. We will comment on the case of Lorentzian signature in the last chapter.

The path integral (1.4) is not defined. The standard route to define the path integral is through perturbation theory. This route is not always available, as for example for the case of gravity in geometrodynamics variables, but, luckily, is available when the theory is rewritten in the first order formalism. Indeed, the first order formalism Hilbert-Palatini action is polynomial in the basic variables. There is the quadratic term $e \wedge dw$, which, according to the rules of perturbation theory, must be treated as the kinetic term, and two cubic terms: $e \wedge w \wedge w$ and $e \wedge e \wedge e$. Thus, up to the important subtleties related to the gauge fixing, the perturbation theory is straightforward. It turns out that not only this perturbation theory makes sense, it is finite! It is especially easy to see that in the case of zero cosmological constant. Then, we have only one interaction term: $e \wedge w \wedge w$. A simple argument due to Witten [64] shows that the theory must be one loop exact. Thus, in the case of a compact spacetime manifold $\mathcal{M}$, the path integral is given by the sum over the moduli space of classical solutions weighted with their one loop contributions. This weight is given by the ratio of certain determinants and is just the Ray-Singer analytic torsion for the corresponding flat connection. In particular, it is a topological invariant of the manifold $\mathcal{M}$. In the case when the cosmological constant is not zero, the situation is more complicated, for the theory is no longer one loop exact. Still, it turns out that the perturbation theory expansion is finite order by order. All these results are intimately related to Chern-Simons theory, knot theory and quantum groups, but we shall not develop this further now.

Thus, the path integral for gravity can be defined by the corresponding perturbation theory. A very important lesson one learns defining the theory this way is as follows. As we discussed above, when one attempts to construct the quantum theory considering perturbations around flat spacetime, the perturbation theory turns out to be non-renormalizable. However, as we just discussed, the perturbation theory around 'zero metric' is well-defined and actually finite. Thus, the flat metric is simply a wrong vacuum to expand around and this is an 'explanation' of the fact that 2+1 gravity seems to be non-renormalizable when described in geometrodynamics variables. This may be taken as a clue why perturbation theory does not make sense in higher dimensions: again, one may be simply expanding around a wrong vacuum! This is an important lesson that should be learned from the example of 2+1 gravity.

Encouraged by these beautiful results one may want to generalize them to higher dimensions. However, here one runs into a dead block. The first step of such a generalization is straightforward: rewrite the theory in the first order formalism. The action is again a functional of the frame and the spin connection:

$$\int_{\mathcal{M}} \mathrm{Tr}\left({}^{\dagger}(e \wedge e) \wedge F\right), \tag{1.5}$$

where $\dagger$ denotes the Hodge dual, and Tr is taken in the Lie algebra. One immediately sees the problem: this theory does not define any perturbation theory around $e = 0$, for there is no kinetic term. One can try to circumvent the problem and rewrite the theory in the form (see Section 6.5):

$$\int_{\mathcal{M}} \mathrm{Tr}\left(E \wedge F + \frac{1}{2}\Phi(E) \wedge E\right). \tag{1.6}$$

Here E is an independent field, which is a $(D-2)$-form taking values in the Lie algebra, and $\Phi(E)$ is a certain 2-form constructed from E and Lagrange multipliers Φ, and linear in both. Varying with respect to Lagrange multipliers one gets equations that guarantee that E field comes from the frame. This action solves the problem with the kinetic term: one can now have a perturbative expansion around $E = 0$, but introduces another problem: there is no kinetic term for the Lagrange multiplier fields.

Thus, there seems to be no perturbative way to define the path integral for gravity in spacetime dimensions $D > 3$. The perturbative definition works only in 2+1, but not in higher dimensions. This negative conclusion lead the idea of quantization of pure gravity to be abandoned by a majority of theoretical physisists, and the bulk of research activity on quantum gravity has shifted to approaches related to string theory, which require extra dimensions and a whole infinite tower of massive states to make sense of the theory.

However, as we shall see in the next section, there exists another, non-perturbative way to define the path integral for 2+1 gravity. It is given by the so-called Ponzano-Regge [48] and Turaev-Viro models [60]. This non-perturbative definition of the path integral can be generalized to higher dimensions and leads to spin foam models.

1.2 Ponzano-Regge and Turaev-Viro models

Ponzano-Regge and Turaev-Viro models are simplicial state sum models in the sense that they start by chopping the spacetime into simplices (tetrahedra) and associate each triangulation of the spacetime manifold an amplitude. These models give a definition of the path integral of the exponentiated action, with i in the exponential, for the case of Euclidean signature. Thus, they are theories of 'Euclidean quantum theory', in the meaning discussed above. The relevant gauge group in this case is SO(3) (or SU(2)).

More precisely, let us fix a triangulation Δ of $\mathcal{M}$. Let us label the edges of Δ by irreducible representations of SU(2), that is, by spins j. Thus, to each edge e we assign a label j_e. One can then construct the following sum over labellings

$$\mathrm{PR}(\Delta) = \sum_{j_e} \prod_e \dim_{j_e} \prod_t (6j)_t. \tag{1.7}$$

Here $\dim(j) = 2j + 1$ is the dimension of the representation j, the second product is taken over tetrahedra t of Δ, and $(6j)$ is the (normalized) classical (6j)-symbol (see the Appendix D for a definition) constructed from the six spins labelling the edges of t. Summing over spins one gets the triangulation amplitude $\mathrm{PR}(\Delta)$.

There is an important subtlety, however. Note that, because each spin runs over an infinite range of values, the sum over spins in (1.7) typically diverges. To make sense of it one must introduce a regularization. A possible regularization is given by the Turaev-Viro model, which we discuss below. After the introduction of this 'regularization', the Ponzano-Regge amplitude $\mathrm{PR}(\Delta)$ can be shown to be triangulation independent. Thus, (1.7) gives an invariant of $\mathcal{M}$: $\mathrm{PR}(\Delta) = \mathrm{PR}(\mathcal{M})$, which can be thought of as the quantum amplitude of $\mathcal{M}$. As we show below, this amplitude gives the value of the path integral for $\mathcal{M}$ in the case of zero cosmological constant.

Turaev-Viro model is very similar, but it is constructed using the quantum group $\mathrm{SU}(2)_q$ instead of the usual SU(2). Thus, let us again fix a triangulation Δ of $\mathcal{M}$. Let us label the edges e by irreducible representations of the quantum group $(\mathrm{SU}(2))_q$, where q is a root of unity

$$q = e^{\frac{2\pi i}{k}} \equiv e^{i\hbar}. \tag{1.8}$$

The parameter $\hbar$ in the above formula turns out to be the parameter of the deformation quantization. In this sense it plays the role of the Planck constant for this theory, and this explains why we used the usual notation for Planck constant to refer to it. This is the standard convention in the mathematics literature. The irreducible representations of $(\mathrm{SU}(2))_q$ are labelled by half-integers (spins) j satisfying $j \leq (k-2)/2$. Thus, we associate a spin j_e to each edge e. The vacuum-vacuum transition amplitude of the theory is then given by the following expression (see, for example, [54]):

$$\mathrm{TV}(q, \Delta) = \eta^{2V} \sum_{j_e} \prod_e \dim_q(j_e) \prod_t (6j)_q, \tag{1.9}$$

where η and the so-called quantum dimension $\dim_q(j)$ are defined in the Appendix A by (A.4) and (A.5) correspondingly, and V is the number of vertices in Δ. The last product in (1.9) is taken over tetrahedra t of Δ, and $(6j)_q$ is the (normalized) quantum $(6j)$-symbol constructed from the 6 spins labelling the edges of t (see Appendices D, E). It turns out that (1.9) is independent of the triangulation Δ and gives a topological invariant of $\mathcal{M}$: $\mathrm{TV}(q, \Delta) = \mathrm{TV}(q, \mathcal{M})$. This invariant is the amplitude of the manifold $\mathcal{M}$ for a non-zero cosmological constant, and Λ is related to the deformation parameter.

There are several ways to show that the sums defined have something to do with the gravity path integral. Some of them use perturbative definition of the path integral, the others are non-perturbative. We discuss and compare them in the next section.

1.3 Perturbative vs. Non-perturbative

A possible route to arrive to the models described is through Chern-Simons theory. One uses Witten's trick and rewrites the action of 2+1 gravity as the difference of two Chern-Simons actions. One then uses the Chern-Simons perturbation theory to calculate the path integral. The result can be shown to be equal to the Turaev-Viro amplitude [54], if one relates the deformation parameter q with Λ in such a way that $\hbar = \sqrt{\Lambda}$. Ponzano-Regge model then arises as the $\Lambda = 0$ limit of Turaev-Viro model. This will be described in details in Section 7.2.

The derivation, whose idea was sketched above, relates the result of path integration with the topological invariant given by the Turaev-Viro model. However, we would like to have a more direct derivation, which relates to the path integral each term of the state sum model, before the sum over spins is taken. One can do this using perturbation theory and non-perturbatively. For simplicity we consider only the zero cosmological constant case. Let us first sketch the perturbation theory derivation.

Perturbative derivation

Let us take the perturbative definition of the path integral. To arrive to Ponzano-Regge model starting from the classical action, let us split the integration over $\mathcal{D}w\mathcal{D}e$ into two parts. Namely, let us triangulate the spacetime manifold $\mathcal{M}$, and choose a labelling of edges e by irreducible representations j_e, as we did when we introduced the Ponzano-Regge model. Let us then integrate over $\mathcal{D}w\mathcal{D}e$ subject to the constraint that the holonomies around edges are in the conjugacy classes determined by labels j_e. In the second step, let us sum over all labellings j_e. This splitting of the integration into two steps does not change anything: at the end we have integrated over all $\mathcal{D}w\mathcal{D}e$. We would like to see now that, after performing the first step, one arrives at the Ponzano-Regge amplitude for a labelled triangulation.

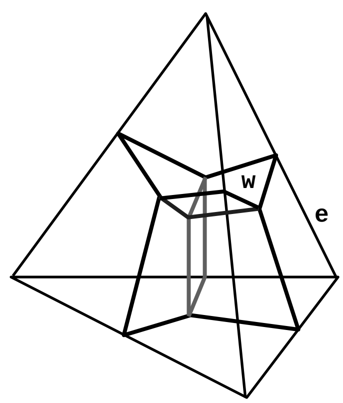


Fig. 1.1. Tetrahedron with parts of the dual cell that lie inside it.

To make this procedure more precise, let me specify what the constraint 'holonomy around an edge e is in the conjugacy class determined by j_e' means. Let us consider one of the tetrahedra of the triangulation, see Fig. 1.3. Given a triangulation, one can also construct a dual simplicial complex. Then faces of the dual complex are in one-to-one correspondence with the edges of the original triangulation. A part of the dual face that lies inside the tetrahedron is denoted by w in the figure. The boundary of the dual face has the topology of S^1 and goes around the edge e. In the first step of the integration over $\mathcal{D}w\mathcal{D}e$ we require the holonomy along the boundary of the dual face to be in the conjugacy class of the group element $e^{iJ^3 j_e}$, where J^3 is the generator of some U(1) subgroup of SU(2).

To see why the integration over $\mathcal{D}w\mathcal{D}e$ subject to this restriction must give the Ponzano-Regge amplitude

$$\prod_e \dim_{j_e} \prod_t (6j)_t, \tag{1.10}$$

we have to use two facts. First, it is known that in the path integral of BF theory, which is what our 2+1 gravity in the absence of the cosmological constant is, only the flat connections dominate the path integral (see [17], the chapter on BF theory, for a discussion of this fact). Second, as it was first noticed by Boulatov [18], the integral of products of holonomies over a number of copies of the group gives the Ponzano-Regge amplitude:

$$\prod_{d.e.} \int dg \prod_{d.f.} \dim_j \chi_j(\prod g) = \prod_{d.f.} \dim_j \prod_t (6j)_t. \tag{1.11}$$

Here the first product is taken over the dual edges, that is, the lines connecting the centra of tetrahedra. Each of such dual edges is associated a group element g, and the integration is taken over each g. The second product is over dual faces. The quantity $\chi_j(\prod g)$ is the character of representation j evaluated on the product of group elements taken around the dual face.

Using these facts one can show that the result of the first step of our procedure is just the Ponzano-Regge amplitude, up to the subtleties related to the gauge fixing. Actually what one gets after fixing the gauge properly is the regularized Ponzano-Regge amplitude, which is the limit of Turaev-Viro amplitude for $\hbar = \sqrt{\Lambda} \to 0$.

Thus, Ponzano-Regge model can be derived directly from the path integral of the theory, when the path integral is defined perturbatively. Similarly one could envisage the derivation of Turaev-Viro model. This, however, will be much more difficult because the path integral is not dominated by flat connections anymore, and is not one loop exact. It will be easier to perform the path integration, with the conjugacy classes of holonomies around edges fixed, by going to Chern-Simons theory.

To summarize, we see that Ponzano-Regge and Turaev-Viro models are correct definitions of the path integral, where correct means that it agrees with the that provided by the perturbation theory. We now consider another derivation of these models.

Non-perturbative derivation

The first direct, without a reference to the perturbation theory, sign suggesting that Ponzano-Regge model may have something to do with gravity is provided by the old result of Ponzano and Regge [48] on the asymptotics of the classical $(6j)$-symbol. It turns out that, for large spins, the $(6j)$-symbol behaves as the cosine of the Regge action for one tetrahedron, see section 2.1 for the precise statement. This fact, actually, motivated the Ponzano-Regge model in the first place. It means that the classical $(6j)$-symbol 'knows' about geometry of the corresponding simplex, and, thus, could have been derived using only the geometry considerations, without perturbation theory. This point of view will be further developed in the next chapter. In this section we will present another, although intimately related, derivation.

Regge calculus [49] was developed as a way to describe gravity 'without coordinates', as the title of the original paper suggests. The main idea is that one can approximate a curved spacetime by a large collection of simplices glued together. The interior of every simplex is flat, and the spacetime curvature resides where the simplices are glued to each other. In three spacetime dimensions this goes as follows. One starts by triangulating the spacetime into simplices, and assigning arbitrary positive numbers –length– to every edge. The Regge calculus version of Einstein-Hilbert action then reads:

$$\sum_{e \in t} l_e \theta_e. \tag{1.12}$$

Here, as before, we use units in which $8\pi G = 1$, and l_e, θ_e are the edge length and the dihedral angle deficit at the edge correspondingly. Note that the dihedral angle deficits θ_e are the functions of length l_e. Varying with respect to the length, one gets equations that can be shown to be the correct 'discretization' of vacuum Einstein equations.

Let us try to copy this procedure in the quantum theory. Namely, let us triangulate the spacetime into a collection of tetrahedra, and assume that all the interiors are flat, the curvature is concentrated only on edges. This means that the classical Lagrangian $\mathrm{Tr}(e \wedge F)$ integrated over a single tetrahedron gets contributions only from the edges:

$$\int_t \mathrm{Tr}(e \wedge F) \to \sum_e \int_e \mathrm{Tr}(e \wedge F). \tag{1.13}$$

Thus, in the case the spacetime is glued from flat tetrahedra, the action of the theory becomes a sum over the edges. It depends only on the values of the field e along the edges, and only on the behavior of the connection in the vicinity of the edges.

To make this more precise, we can further approximate the curved spacetime we are working with. Actually, the following construction can be taken as a part of the definition of the 'quantum Regge calculus'. Let us encode all information about the curvature on an edge e into the holonomy of the spin connection around this edge. This holonomy is the exponential of some Lie algebra element, which we will denote by Z_e:

$$h_{\gamma_e}(w) = e^{Z_e}. \tag{1.14}$$

Let us encode all information about the e field on the edge e into the integral of e (which is a one-form) along the edge:

$$X_e = \int_e e. \tag{1.15}$$

The analog of the Regge action then becomes:

$$\sum_e \mathrm{Tr}(X_e Z_e), \tag{1.16}$$

where Tr must be understood appropriately (as the one constructed with the help of the ϵ-symbol on the Lie algebra). There are certain subtleties related to the fact that Z_e is only defined up to conjugacy, but we will postpone their discussion until Chapter 5.

Thus, what we get is the action, which depends only on variables X_e and a collection of group elements. One can take this action as a starting point for the path integral quantization. To define the measure, one uses a number of copies of the Lebesque measure dX on the Lie algebra as the analog of the integral over e, and a number of copies of Haar measure dg as the analog of the integral over w. This defines the theory completely, up to the subtleties related to gauge transformations, which will be discussed later. The result of this path integral is exactly Ponzano-Regge model. We shall present all the details of this calculation in Chapter 5.

Thus, what we see is that, to arrive to Ponzano-Regge model, one does not have to refer to the perturbation theory. One can instead define a Regge type action, which takes into account the fact that the curvature is concentrated along the edges, and whose path integral gives the Ponzano-Regge model. In the following chapters we shall see that this strategy works in any dimension, and gives higher-dimensional analogs of Ponzano-Regge model. The procedure we shall develop will also give us an analog of the above derivation for a non-zero cosmological constant, that is, we will be able to see how one can derive Turaev-Viro model non-perturbatively. However, before we discuss in details this procedure, let us give more evidence in support of our non-perturbative definition of the path integral. Let us present another derivation of the Ponzano-Regge model.

Chapter 2

Geometry and representation theory: 2+1 dimensions

In this chapter we concentrate on the relation between the group representation theory and geometry, which is suggested by the Ponzano-Regge result on the asymptotics of the classical $(6j)$-symbol. We will show how representation theory is related to the quantization of a geometric tetrahedron in $\mathbb{R}^3$. We shall see how the $(6j)$-symbol arises as the quantum amplitude for a tetrahedron, and how Ponzano-Regge model can be constructed as giving the amplitude for the quantized spacetime. The results of this chapter will serve as an additional guide when we introduce the spin foam models in the next chapter. Most of the results presented here can be generalized to higher dimensions, which is done in Chapter 9. The following section is devoted to the $(6j)$-symbol asymptotics. Section 2.2 discusses the quantization of a geometric tetrahedron.

2.1 Asymptotics of the classical $(6j)$-symbol

A classical $6j$-symbol is a real number which can be associated to a labelling of the six edges of a tetrahedron by irreducible representations of SU(2). Its definition is as follows.

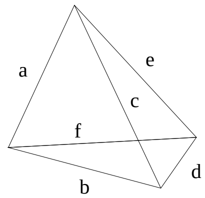
Let V_a, where a is an integer, denote the $(a+1)$-dimensional irreducible representation. The SU(2)-invariant part of the triple tensor product $V_a \otimes V_b \otimes V_c$ is non-zero if and only if

$$a \leq b+c \qquad b \leq c+a \qquad c \leq a+b \qquad a+b+c \text{ is even,} \tag{2.1}$$

in which case we may pick, almost canonically, a basis vector ϵ^{abc}.

Suppose we have a tetrahedron, labelled so that the three labels around each face satisfy these conditions: we will call this an *admissible* labelling. Then we may associate to each face an ϵ-tensor, and contract these four tensors together to obtain a scalar, the $6j$-symbol, denoted by a picture or a bracket symbol as in figure 2.1.

This tetrahedral picture is traditionally used simply to express the *symmetry* of the $6j$-symbol, which is naturally invariant under the full tetrahedral group S_4. However, it has a deeper *geometric* significance. To an admissibly-labelled tetrahedron we may associate a metric tetrahedron t whose side lengths are the six numbers $a, b, \ldots, f$. Its individual faces may be realized in Euclidean 2-space, by the admissibility condition 2.1. As a whole, t is either *Euclidean*, *Minkowskian* or *flat* (in other words has either a non-degenerate isometric embedding in Euclidean or Minkowskian 3-space, or has an isometric embedding in Euclidean 2-space), according to the sign of a certain polynomial in its edge-lengths, see below. If t is Euclidean, let $\theta_a, \theta_b, \ldots, \theta_f$ be its corresponding



$$\equiv \left\{ \begin{matrix} a & b & c \\ d & e & f \end{matrix} \right\}. \qquad (2.2)$$

Fig. 2.1. Pictorial representation

exterior dihedral angles and V be its volume. The following theorem has been proven by Roberts [55]:

THEOREM 2.1 (ROBERTS). *Suppose a tetrahedron is admissibly labelled by the numbers a, b, c, d, e, f. Let k be a natural number. As $k \to \infty$, there is an asymptotic formula*

$$\left\{ \begin{matrix} ka & kb & kc \\ kd & ke & kf \end{matrix} \right\} \sim \sqrt{\frac{2}{3\pi V k^3}} \cos\left\{ \sum (ka+1)\frac{\theta_a}{2} + \frac{\pi}{4} \right\} \qquad (2.3)$$

if t is Euclidean, and is exponentially decaying if t is Minkowskian.

The sum in the argument of cosine is taken over all edges of the tetrahedron. The quantity V in the denominator in front of the cosine is the volume of the corresponding geometric tetrahedron. A (slightly different) version of this formula was conjectured in [48] by the physicists Ponzano and Regge, building on heuristic work of Wigner; they produced much evidence to support it but did not prove it.

This formula means that there is a relation between representation theory of SU(2) and the geometry of $\mathbb{R}^3$. A possible way to unravel this relation is through geometric quantization, and this is the route followed by Roberts [55] in his proof of the theorem. In Chapter 9, where we obtain analogous results in higher dimensions, we shall follow another route, and study the relation between geometry and representation theory from the point of view of harmonic analysis on the group.

The fact that the $(6j)$-symbol, an object that appeared solely in the realm of representation theory, without any reference to geometry, has something to do with Euclidean geometry is fascinating. It motivated Ponzano and Regge to propose their model of quantum gravity, and it suggests that one can use group representation theory to describe quantum geometry.

The result on the asymptotics is not the only fact that suggests the relation between geometry and representation theory. In fact, there are analogous results that were known to mathematicians for a long time. It is known, for example, that the asymptotics of the classical $(3j)$-symbol is related to a triangle in Euclidean space, see, e.g., [61], Chapter 8. The relation between representation theory of SU(2) and geometry of Euclidean 3-space was also discovered by Penrose [45] in his work on spin networks. His results were of much importance for the development of loop quantum gravity.

From a mathematical point of view, this relation between geometry and representations is not that surprising. Indeed, it is well-known that representation theory of a particular group is intimately related with the harmonic analysis on this group. One can realize irreducible representations in the space of functions on the group that are eigenfunctions of the Laplace operator. These eigenfunctions 'know' a lot about the geometry of the group manifold, and the information gets encoded in the properties of irreducible representations.

The subject of representation theory and harmonic analysis on group manifolds is extremely rich mathematically. In fact, its development was very influential, and, in many ways, shaped the contemporary mathematics. However, in fundamental physics these beautiful constructions play hardly any role. It is one of the ambitious aims of this program to bring these results into physics, by using representation theory to describe quantum geometry. As the first step towards this goal, we consider the quantization of a geometric tetrahedron in 2+1 dimensions.

2.2 Quantization of a geometric tetrahedron

The results of this section build upon the work of Barbieri [11], Baez [9], Barrett and Crane [12]. The idea to study quantum geometry by considering quantization of a geometric simplex belongs to Barbieri. It was generalized to four dimensions by Barrett, Crane and Baez. What is presented here is the one-dimension-down analog of their procedure.

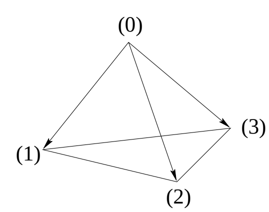


Fig. 2.2. Geometric tetrahedron

A tetrahedron in Euclidean 3-space is completely characterized (up to rotations and translations) by 6 numbers – edge length. The edge lengths must satisfy triangle inequalities, and a certain polynomial constructed from them must be positive. More precisely, let us introduce the following labelling of the edges. Let us label one of the vertices by (0), and other three vertices by $(1), (2), (3)$, see Figure 2.2. The edges are now labelled by two indices, for which we will use small Latin letters from the beginning of the alphabet. Thus, the triangle inequalities read:

$$l_{ab} < l_{ac} + l_{bc}, \tag{2.4}$$

where l_{ab} is the length of the edge going from vertex (a) to (b), $l_{ab} = l_{ba}, l_{aa} = 0$. In order for six numbers l_{ab} to correspond to the length of the edges of a tetrahedron that is isometricaly embeddable into Euclidean space, the following determinant must be positive:

$$V^2 = \frac{1}{2^3(3!)^2} \begin{vmatrix} 0 & l_{30}^2 & l_{20}^2 & l_{23}^2 & 1 \\ l_{30}^2 & 0 & l_{10}^2 & l_{13}^2 & 1 \\ l_{20}^2 & l_{10}^2 & 0 & l_{12}^2 & 1 \\ l_{23}^2 & l_{13}^2 & l_{12}^2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{vmatrix} \tag{2.5}$$

Here V is just the volume of the tetrahedron. If this determinant is negative, then the tetrahedron can be isometricaly embedded into 3-space of Lorentzian signature.

Another way to characterize a tetrahedron is by specifying (up to rotation) a triple of vectors e_{01}, e_{02}, e_{03}. Or, instead, one can characterize a tetrahedron by a collection of 6 vectors $e_{ab} = -e_{ba}$ satisfying an obvious constraint:

$$e_{ab} + e_{bc} + e_{ca} = 0. \tag{2.6}$$

One can easily see that we did not increase the number of degrees of freedom: each vector is specified by 3 numbers, and we have 6 of them, so we have 18 DOF. We then have to impose 4 constraints, one for each face, but only 3 of them are independent, which leaves us with 3×3 equations. Thus, we have 9 DOF left after we impose the constraints. After modding out by rotations, we get 6 DOF, as needed.

Each of the vectors e_{ab} is an element of the vector space $\mathbb{R}^3$. Let us now quantize these vectors. The idea is to identify $\mathbb{R}^3$, where the vectors live, with the Lie algebra of SU(2) (more precisely, with the dual to the Lie algebra, which is naturally a Poisson manifold). That is, we make our $\mathbb{R}^3$ a Poisson manifold, so that the components of vectors do not commute with respect to the Poisson structure. This Poisson manifold can then be quantized, which should be intuitively thought of as being the quantization of vectors e_{ab}.

After quantization, the vectors e_{ab} become operators in some Hilbert space. This Hilbert space can be obtained using representation theory of SU(2). Indeed, representation theory studies representations of group elements, and Lie algebra elements, by operators in a Hilbert space. This is exactly what we need, because our e_{ab} are identified with elements of the (dual to the) Lie algebra. Thus, for each vector e_{ab}, and, therefore, edge $(a)(b)$, one gets the Hilbert space:

$$\mathcal{H} = \oplus_j V^j, \tag{2.7}$$

where V^j is the Hilbert space of the irreducible representation of the dimension $2j + 1$, and the direct sum is taken over all irreducible representations. The original vectors e_{ab} become operators, each in its own copy of $\mathcal{H}$, so that different vectors commute. Note that each V^j in the sum can be thought of as the Hilbert space corresponding to the edge having the length $l^2 = j(j+1)$, which is the value of the quadratic Casimir in this representation. Thus, what we have starts to remind a quantization of geometry.

Thus, we have associated each edge a copy of the Hilbert space $\mathcal{H}$. Let us now try to find quantum analogs of constraints (2.6). Each of this constraints corresponds to a particular face of our tetrahedron. Thus, let us construct a Hilbert space corresponding to a face, an then impose the constraint (2.6) as an operator equation in this Hilbert space. Since face is a collection of 3 vectors, it is natural to associate with is the Hilbert space $\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}$. The constraint (2.6) becomes:

$$e \otimes 1 \otimes 1 + 1 \otimes e \otimes 1 + 1 \otimes 1 \otimes e = 0, \tag{2.8}$$

where e is the operator representing the vector e, and 1 is the identity operator. Equation (2.8) has an obvious meaning: its solutions are invariant tensors from the tensor product $\mathcal{H}^{\otimes 3}$. Recalling that $\mathcal{H}$ is the direct sum of irreducible representations, the space of invariant tensors can be nicely described as the space of all trivalent intertwiners. A trivalent intertwiner is a map $I : V^i \otimes V^j \otimes V^k \to C$ invariant under the diagonal action of the group on the tensor product. The space of such intertwiners is either zero or one-dimensional. It is one dimensional only if the collection of spins i, j, k satisfy the triangle inequalities, and $i + j + k$ is an integer.

Thus, what we find is that one can associate each face a Hilbert space, which is just the direct sum of one-dimensional Hilbert spaces generated by trivalent intertwiners. The trivalent intertwiners are just the usual $(3j)$-symbols, or Clebsch-Gordan coefficients. We find that they are naturally associated with the faces of our tetrahedron, or with triangles. This is in agreement with the known fact that the asymptotics of the $(3j)$-symbol for large spins "knows" about the area of the triangle constructed on the three spins as length: the asymptotics behaves as the inverse of the square root of the area, see [61], Chapter 8.

Having found the Hilbert space corresponding to a face, we now have to glue the faces to form a tetrahedron. This is achieved by "gluing" of the four Hilbert spaces corresponding to the faces. A natural way to do this is to look for maps from the tensor product of the four Hilbert spaces corresponding to the four faces to numbers. The result must be invariant under the diagonal action of the group, thus we have to take invariant maps. In other words, we have to construct a number from four $(3j)$-symbols corresponding to the faces, and this number must be invariant under the action of the group. There is only one such number: the classical $(6j)$-symbol.

Thus, the $(6j)$-symbol naturally arises when one quantizes a geometric tetrahedron. One can have two interpretations of this quantity. First, one can say that the $(6j)$-symbols are vectors in the Hilbert space, which is the result of quantization of the tetrahedron. This Hilbert space is the direct sum of one-dimensional Hilbert spaces generated by $(6j)$-symbols. The other interpretation is that each $(6j)$-symbol gives the quantum amplitude for the tetrahedron on the condition that the 6 length of the edges are specified.

This is this second interpretation that suggests the state sum model. Indeed, since each $(6j)$-symbol gives the amplitude for a tetrahedron, one can obtain the amplitude for the triangulation by multiplying the tetrahedron amplitudes. The interpretation of this number is the amplitude of the triangulation given that the length of the edges of all tetrahedra are specified.

This is almost the Ponzano-Regge amplitude, as it was described in the previous chapter, or the most important part of it: the product of $(6j)$-symbols. However, what we cannot get by quantizing single tetrahedra is the other part of the Ponzano-Regge amplitude: the product of dimensions of representations. This is natural, because our procedure of quantization of a single tetrahedron concentrated on a single simplex of the triangulation, not on the triangulation as a whole. Thus, we might have well missed some parts of the triangulation amplitude that have to do with how the simplices are glued together. In 2+1 this part can be found by bringing in the other input: the requirement of triangulation independence. In one wants the amplitude to be triangulation independent, one has to include in the amplitude the product of dimensions.

Thus, the quantization of a geometric tetrahedron allows us to understand the essential elements of the Ponzano-Regge model: $(6j)$-symbols. We see that they appear as the amplitudes for the simplices of the triangulation. What the results of this section cannot tell us is how to glue this amplitudes together to obtain the amplitude of the triangulation. This requires an additional input, which in 2+1 dimension is provided by the requirement of triangulation independence.

The quantization of a geometric simplex can be generalized to higher dimensions. In fact, this quantization was first proposed and studied for the case of a four-simplex [9, 12]. The results on quantization of a geometric four-simplex were very influential, for they motivated the development of the idea of spin foam models. However, we can also see the limitations of such results. Indeed, by concentrating on the quantization of a single simplex it is hard, if not impossible, to find the amplitude corresponding to the whole triangulation. This requires an additional input, which in 2+1 dimensions was given by the triangulation independence. In higher dimensions this property is not expected to be true, for it has to do with the topological property of gravity in 2+1 dimensions. Thus, in higher dimensions some other input is necessary. As we shall see in the following chapters, this input is given by the classical action of the theory. The procedure we propose allows one, by starting from the action, and defining the corresponding path integral, not only to obtain the amplitude for a particular simplex of the triangulation, but to get the amplitude for the triangulation as a whole. However, the geometric results of this chapter, and their higher-dimensional analogs described below serve as an important piece of intuition behind the spin foam models. Thus, they are certainly worth developing. These results are also exciting because they show that, by quantizing the classical geometry, without any reference to the classical action, one arrives to the quantum amplitude whose semi-classical limit is related to the usual Einstein-Hilbert action for gravity. This gives hope that quantum gravity indeed can be obtained as the quantization of geometry.

Chapter 3

Introduction to spin foams

In the previous two chapters we have seen how, by using a quantum version of Regge calculus, or by quantizing a tetrahedron in 2+1 dimensions, one can give sense to the path integral

$$\int \mathcal{D}w\mathcal{D}e\, e^{i\int_{\mathcal{M}} \mathrm{Tr}(e\wedge F)}. \tag{3.1}$$

In this chapter, motivated by the above results, we will try to give a more precise definition of such path integral. The results of the following chapters build upon the work of Freidel and myself [29].

To give a definition to the path integral of the type above, we consider a more general theory. Let us fix the dimension of spacetime to be D, and consider the action of the type

$$\int_{\mathcal{M}} [\mathrm{Tr}\,(E\wedge F) + \Phi(E)]\,, \tag{3.2}$$

where E is the Lie algebra valued $(D-2)$-form field (for instance, B field of BF theory, which we discuss below), F is the curvature of the connection form A and Φ is certain (polynomial) function of the E field, which can also depend on some Lagrange multipliers, as in the case of gravity, or on an additional background structure, as, for example, a fixed metric on $\mathcal{M}$ in the case of Yang-Mills theory. Below we will give many examples of theories belonging to this class. Thus, the action is that of the BF theory with an additional term. We call the B field E because of its relation with the non-abelian 'electric' field of the canonical formulation.

Below we will propose a definition of the path integral for such a theory. This definition will generalize the ideas we described above to an arbitrary spacetime dimension, and to a much larger class of theories than 2+1 gravity considered so far. We shall first propose a definition and then study it on various examples to see whether it makes sense.

The main idea of our proposal is very similar to the one that we used in the non-perturbative derivation of Ponzano-Regge model in section 1.3. Namely, we triangulate the spacetime manifold into simplices, and then approximate the continuous fields E, A by fields living on the elements of the triangulation. Thus, our procedure is a version of 'quantum Regge calculus', where smooth dynamical variables are approximated by distributional fields living on the boundaries of simplices.

Let us briefly remind the reader what was done in section 1.3. In 2+1 dimensions, we approximated the curvature F by a distributional field concentrated on the edges. The action of 2+1 gravity then becomes a sum over edges. It can be written in a simple form by introducing variables X_e and Z_e: analogs of variables e and w of the original action. The path integral over X_e, Z_e then makes sense and can be calculated. In 2+1 dimensions this procedure reproduces Ponzano-Regge model.

It turns out to be more convenient, however, to approximate the E field, not the curvature, by a distributional field. The idea is to say that E field is concentrated along the dual faces of the triangulation, see Fig. 1.3 to recall what the dual face is. In 2+1 dimensions, or, more generally, when one deals with the pure BF theory, it does not matter whether one makes E or F distributional. The resulting action in both cases is given by the sum over edges, or, because of the one-to-one correspondence between the edges and dual faces, by the sum over dual faces of $\mathrm{Tr}(X_e Z_e)$. However, when one considers more complicated theories, whose action is of the type described above, it is essential to make the E field distributional. This, as we shall see in numerous examples, allows us to make sense of the term $\Phi(E)$ in the action, and allows one to calculate the path integral.

Another reason why it is more natural to make the E field distributional is that the dual faces along which it is concentrated are always 2-dimensional, independently of the dimension of spacetime. On the other hand, the curvature F is concentrated along $(D-2)$-fases of triangulation, and this changes from dimension to dimension.

There is yet another reason to make the E field distributional, which also serves as an additional justification for our procedure. This justification comes from the canonical quantum theory. The phase space of all theories we work with, as it can be seen from the above action functional, is that of Yang-Mills theory. The space of quantum states of the canonical theory is then given by $L^2(\mathcal{A}/\mathcal{G})$, where $\mathcal{A}/\mathcal{G}$ is the space of connections on the spatial manifold modulo gauge transformations, and L^2 is defined by an appropriate choice of measure on this space. In the case the spatial manifold is one-dimensional, there exists quite a natural choice of this measure: the functionals on $\mathcal{A}/\mathcal{G}$ are just class functions, and it is natural to use the Haar measure on the group to define the inner product of such functionals. In higher dimensional cases, an analogous construction of $L^2(\mathcal{A}/\mathcal{G})$ was developed [6, 3] within the approach of canonical (loop) quantum gravity. In this construction, integrable functions arise by considering the so-called cylindrical functionals, that is, functionals that depend on connection only through holonomies along some paths in space. In other words, one constructs the space of states of the theory by considering spaces of states of lattice gauge theories on graphs in space, and then taking the projective limit over graph refinements (see [6, 3] for details). A characteristic feature of the quantum theory constructed with such a space of states is the distributional nature of the electric field operator. Let us illustrate this on the case of (2+1) dimensional theory. A typical state from $L^2(\mathcal{A}/\mathcal{G})$ depends on the connection only through holonomies along some paths in space. Let us denote the edges of such a graph by e_i. Then a typical state supported on this graph is of the form

$$\Psi\left(h_{e_1}(A), \ldots, h_{e_n}(A)\right),$$

where $h_{e_i}(A)$ is the holonomy of A along the edge e_i, and Ψ is a function with sufficiently good behavior. The electric field E, which classically is a canonically conjugate quantity to A, quantum mechanically becomes the operator $\delta/\delta A(x)$ of variational derivative with respect to A. This operator, when acting on a typical state, gives zero for all points x

except the points lying on the edges e_i, where the result is distributional. Thus, in a typical quantum state, the electric field E is distributional with support on edges e_i of the corresponding graph. One normally views a state of the canonical theory as a state "at a

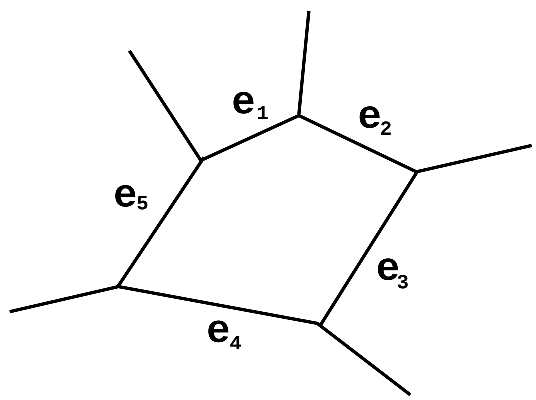


Fig. 3.1. A typical quantum state of (2+1) dimensional theory is labelled by a graph in space. The corresponding functional of connection depends on A only through holonomies along edges e_i. The electric field in such state is distributional and concentrated along the edges e_i.

given time". Thus, electric field "at each given time" is distributional and concentrated along edges of the graph. This suggests that one should think of the "history" of the electric field configuration as of a collection of *two-dimensional* surfaces in spacetime, along which the electric field is distributional. When sliced by a spatial hypersurface "at given time", such a collection of surfaces gives a graph in space, and electric field is distributional along the edges of this graph. These heuristic considerations suggest that one should consider configurations of E field that are distributional and concentrated along two dimensional surfaces in spacetime.

Thus, to define the path integral for a theory of the class we described above, we propose a version of "quantum Regge calculus", in which the smooth fields get replaced by the distributional ones. At the very least, this can be taken as a definition of the measure in the path integral. One must then test this definition on a number of examples to see whether it reproduces results one expects. We will test our approach on the exact results coming from Topological QFT's. As we shall see, for a large class of theories our approach gives results that are in a good agreement with the known state sum models.

To calculate the path integral, we will first calculate it over the distributional fields "living" on a fixed collection of surfaces in spacetime. In the case of topological theories, the result for a particular fixed collection of surfaces does not depend on the collection. One cannot expect this property to hold for such theories as gravity, which is not a topological theory. Thus, one might want to perform a sum over all possible collections of surfaces in spacetime, or take a limit as the triangulation becomes more refined. We comment on this problem only in the last chapter. Thus, the bulk of what we discuss is about how to make sense of the path integral for a fixed triangulation of the spacetime.

Our other main idea is to use a certain analog of perturbative expansion for the theories we consider. Usual interactive quantum field theories have been successfully understood using perturbative expansion in terms of Feynman graphs. In this case the interacting QFT is considered as a perturbation of a free field theory. From a technical point of view, what makes this approach successful is the exact solvability of the free field theory, i.e., our ability to compute all possible correlation functions. All correlation functions can be encoded in the generating functional, and Feynman graphs can be viewed as the evaluation of certain differential operators acting on the generating functional.

We will employ a somewhat similar strategy. In order to calculate transition amplitudes for our class of theories we will use the machinery of generating functionals. In our approach the role of the "free" field theory is played by the BF theory and the terms in the action polynomial in E are analogous to the "interaction" terms. The idea to use the BF theory as a "free" field theory is not new: it has been applied with some success in the context of Yang-mills theories (see [34]). However, the details of how the BF theory is used in our approach differ from those of [34]. In order to calculate a transition amplitude, i.e., the path integral of the exponentiated action

$$\int \mathcal{D}A\mathcal{D}E \; e^{i\int[\mathrm{Tr}(E\wedge F)+\Phi(E)]},$$

we formally rewrite this path integral as

$$\left(e^{i\int \Phi\left(\frac{\delta}{i\delta J}\right)} Z[J] \right)_{J=0},$$

where

$$Z[J] := \int \mathcal{D}A\mathcal{D}E \; e^{i\int[\mathrm{Tr}(E\wedge F)+\mathrm{Tr}(E\wedge J)]}.$$

We shall refer to $Z[J]$ as *generating functional.* Here J is Lie algebra valued two form field. Using terminology from field theory we will call the J field *current.* One of our main results is the exact computation, in the context of spin foam models, of the generating functional Z. It is obtained by integrating over fields A, E that "live" on a fixed triangulation of the spacetime manifold. Then the transition amplitude for any theory of the type we consider is given as a formal power series in variational derivatives $\delta/\delta J$ with respect to the current field. The series we get are quite reminiscent of the usual Feynman diagram expansions. Thus, the powerful technique of generating functionals allows us to study different theories –such as BF theory and gravity– from the same point of view.

Thus, our strategy will be to first calculate the generating functional in various dimensions, and then use the result to calculate the path integral for different theories. We will also be able to compare the results of our approach with exact results coming from topological field theories. The agreement we find serves as an additional justification for the procedure. Finally, we shall apply our quantization procedure to gravity. It will allow us to make sense of and study the gravity path integral in any dimension.

Chapter 4

BF theory

As it was clear from the previous chapter, BF theory will play a special role in our approach, in the sense that all theories we consider are obtained as perturbations of BF theory. This makes it desirable to first study the pure BF theory in more details. In this chapter we review what is BF theory and what is known about its quantization.

This theory, together with the name, was invented and first studied by Horowitz [35], the motivation being the results coming from 2+1 gravity. It was then extensively studied in the context of topological field theories.

The action of BF theory in any dimension is given by

$$\int_{\mathcal{M}} \mathrm{Tr}(E \wedge F), \tag{4.1}$$

that is, this is the action of the type we described in the previous chapter with zero "interaction" term Φ. The equations of motion that follow from this action state that

$$d_A E = 0, \qquad\qquad F = 0, \tag{4.2}$$

where $d_A E$ is the covariant derivative of E. Thus, the flat connections play the dominant role in BF theory.

The path integral for BF theory can be defined perturbatively, see, e.g., [17] for a discussion of this. The flat connections are known to dominate the path integral, and the theory is one loop exact. Another definition of the path integral is given by higher-dimensional analogs of Ponzano-Regge model. The case of three dimensions, which is what Ponzano-Regge model corresponds to, was described in section 1.2. A four-dimensional analog of this model was discovered by Ooguri [44], whose work was motivated by the work of Boulatov [18]. The description of this model is as follows. One starts by triangulating the spacetime manifold $\mathcal{M}$ into 4-simplices. For simplicity, let us fix the gauge group to be SU(2), and label the faces f of triangulation by spins j_f. Let us consider the following quantity:

$$\mathrm{O}(\Delta) = \sum_{j_f, j_t} \prod_f \dim_{j_f} \prod_t \dim_{j_t} \prod_h (15j)_h, \tag{4.3}$$

After a certain "regularization" (given by the Crane-Yetter model, which we describe below) that gives meaning to the infinite sums in (4.3), this can be shown to be triangulation independent: $\mathrm{O}(\Delta) = \mathrm{O}(\mathcal{M})$. In (4.3) one has introduced an additional label j_t for each tetrahedron of Δ. The spin j_t labels an intertwiner one has to assign to each tetrahedron (see the section on Crane-Yetter model for details). The last product is taken over the 4-simplices h of Δ. Then $(15j)$ is the (normalized) (15j)-symbol constructed from ten spins j_f labelling the faces of h and five spins j_t labelling the five tetrahedra composing h. See the Appendix D for a definition of the normalized (15j)-symbol.

As was realized by Boulatov and Ooguri [18, 44], this model, and the original Ponzano-Regge model, give a "discrete" realization of the path integral because they can be obtained from the requirement that the connection on $\mathcal{M}$ is flat. Indeed, formally the path integral of the BF theory is equal to

$$\int \mathcal{D}A\,\delta(F). \tag{4.4}$$

The state sums (1.7),(4.3) are in certain precise sense realizations of the integral over connections with the integrand being the delta-function at $F = 0$. Using this relation to the delta-function on the group, these models can be easily generalized to higher dimensions. The resulting model will be described in section 5.5, after we compute the generating functional.

Chapter 5

BF theory with a source. Generating functional

As we explained in Chapter 3, to calculate transition amplitudes for a theory of the type we consider here it is very convenient to first calculate the generating functional of BF theory. Then transition amplitudes can be obtained as formal power series in the derivatives with respect to the current. In this chapter we calculate the generating functional of the BF theory, i.e., the path integral

$$Z[J] = \int \mathcal{D}A\mathcal{D}E \; e^{i\int[\mathrm{Tr}(E\wedge F)+\mathrm{Tr}(E\wedge J)]} \tag{5.1}$$

in various dimensions. Here A is a connection on the principal G bundle over the spacetime manifold $\mathcal{M}$, where G is the gauge group of the theory, which we assume to be semi-simple, F is the curvature 2-form of the connection, E is a Lie algebra valued $(D-2)$-form field, where D is the dimension of $\mathcal{M}$, and J is a Lie algebra valued current 2-form. For simplicity, we consider here only the case when the spacetime manifold $\mathcal{M}$ is closed. Although for practical applications of spin foam models one is usually interested in the case when $\mathcal{M}$ has a boundary consisting of "initial" and "final" spatial hypersurfaces, we do not consider this here because all non-trivialities arise already in the case of closed $\mathcal{M}$. At any rate, the inclusion of boundaries is straightforward, and all our formulas can be easily generalized to this case. To give meaning to the path integration over spacetime fields A, E, we replace the E field by certain distributional field concentrated over two-dimensional surfaces in spacetime. To make contact with the known state sum models we put our fields on the 2-dimensional cellular complex dual to some triangulation of the spacetime manifold. In each particular calculation this triangulation is fixed and all results explicitly depend on it. We first investigate the generating functional in a general case, without specifying the dimension of spacetime manifold, and then find an explicit expression for it in each particular dimension.

5.1 General framework

Let us fix a triangulation Δ of a D-dimensional compact oriented spacetime manifold $\mathcal{M}$. This triangulation defines another decomposition of $\mathcal{M}$ into cells called dual complex. There is one-to-one correspondence between k-simplices of the triangulation and $(D-k)$-cells in the dual complex. We orient each cell of the dual complex in an arbitrary fashion, which also defines an orientation for all simplices of the triangulation. The $(D-2)$-form E can now be integrated over the $(D-2)$-simplices of the triangulation the result being a collection of the Lie algebra elements X — one Lie algebra element X for each 2-cell dual to a $(D-2)$-simplex of the triangulation. We would like to discretize our action by replacing the continuous E field by the collection of the Lie algebra elements X. It turns out, however, that this is not yet the most convenient set of variables for the theory. For reasons which we give below, we use another, more convenient set of variables introduced by Reisenberger [52]. These variables are motivated by the fact that, in the case of BF theory one *has* to use them if one wants to reproduce a triangulation-independent state sum model for the case the topology of dual faces is different from that of a disc. Another motivation for using the wedge variables is, as we explain below, that they appear naturally when one tries to implement the gauge symmetry in the theory.

Thus, instead of a single X for each dual face, we introduce a set of variables, which we call X_w. To do this, we divide each dual face into wedges (see Fig. 2). To construct wedges of the dual face one first has to find the center of the dual face. This is the point on the dual face where it intersects with the corresponding (D–2)-simplex of the triangulation. One then has to draw lines connecting this center with the centers of the neighboring dual faces. The part of the dual face that lies between two such lines is exactly the wedge. Thus, each dual face splits into wedges, and we assign a Lie algebra element X_w to each wedge w. Wedges of a given dual face are in one-to-one correspondence with the D-simplices of the triangulation neighboring this dual face. Thus, the physical meaning of each variable X_w can be said to be the integral of E over the (D–2) simplex of the triangulation "from the point of view" of a particular D-simplex containing this (D–2)-simplex. Note that the number of variables X_w that arises this way for a given triangulation is equal to the number of D-simplices times the number of (D–2)-simplices in each D-simplex.

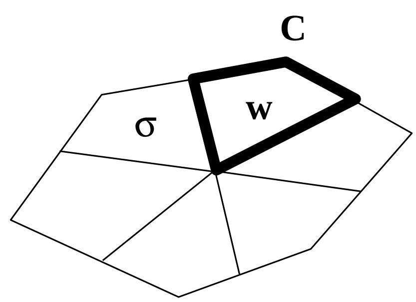


Fig. 5.1. A face σ of the dual triangulation. The portion of σ indicated by bold lines is what is called wedge. The point labelled by C is the center of one of the D-simplices neighbored by σ.

Having discussed the geometrical meaning of the wedge variables X_w we are ready to introduce the distributional E fields. Heuristically, our procedure of replacement of a smooth E field by a distribution concentrated along the wedges amounts to "squeezing" of the smooth E field that is "spread" over a (D–2)-simplex of the triangulation to a single point on this (D–2)-simplex, the point where the simplex intersects with the 2-cell of the dual complex. Thus, we define a distributional field E_w concentrated along a wedge w to be a 2-form satisfying the following relation:

$$\int_{\mathcal{M}} \mathrm{Tr}(E_w \wedge J) = \mathrm{Tr}(X_w \int_w J). \tag{5.2}$$

Here J is any $\mathrm{Ad}(P)$-valued 2-form, and w stands for a wedge. The integral on the right-hand-side is performed over the wedge w. This, in particular, implies that the Lie algebra element X_w is equal to the integral of E_w over the (D–2)-simplex of the triangulation that is dual to the 2-cell σ. We then define distributional field E to be

$$E = \sum_w E_w. \tag{5.3}$$

To calculate the generating functional Z as a function of a fixed triangulation Δ, we have to take the integral of the exponentiated action over the "discretized" dynamical fields A, E, that is, the fields "living" on the triangulation Δ. The "discretized" action, that is, the action evaluated on the distributional field E, becomes a function of the Lie algebra elements X, and a functional of connection A and current J. Using (5.2),(5.3), we find that this discretized action is given by

$$\sum_w \int_w \left[\mathrm{Tr}(F\, X_w) + \mathrm{Tr}(J\, X_w)\right], \tag{5.4}$$

where the sum is taken over the wedges of faces of the dual complex ('w' stands here for a wedge), and the integral is performed over each wedge, the integrand being the curvature of the connection A contracted with the Lie algebra element X_w "living" on that wedge plus the current J contracted with X_w. Each integral is performed using the orientation of the dual face to which the wedge belongs.

We could now substitute this discretized action into the path integral and integrate over X and A. However, we first have to discuss what measure has to be used to integrate over X, A. To give meaning to the integration over A, let us replace a continuous field A by a collection of group elements. To do this we use the following approximation

$$\int_w \mathrm{Tr}(F\, X_w) \approx \mathrm{Tr}(Z_w X_w), \tag{5.5}$$

where Z_w is the Lie algebra element corresponding to the holonomy of A around the wedge (see Fig. 3). The base point of the holonomy is not fixed at this stage (see below). In other words,

$$\exp Z_w = g_1 h_1 h_2 g_2, \tag{5.6}$$

where g_1, h_1, h_2, g_2 are the holonomies of A along the four edges that form the boundary of the wedge w. We assume a local trivialization of the bundle over w so that the holonomies are group elements. The order in which the product of group elements is

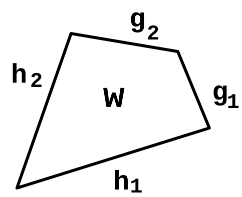


Fig. 5.2. A wedge w of the dual face and the group elements: holonomies along the edges of w.

taken is determined by the orientation of the dual face. Of course, $\exp Z_w$ is defined only up to its conjugacy class (any of the four edges can be taken to be the "first"). Thus, there is an ambiguity in the choice of the base point for the holonomy, which we have to fix in some way. To fix it we define a notion of "discretized" gauge transformation and fix the ambiguity requiring the discrete action to be gauge invariant.

But first, let us replace the continuous current field J by a collection of Lie algebra elements J_w:

$$J_w := \int_w J. \tag{5.7}$$

Here, to perform the integration, a trivialization of the bundle over the wedge w is chosen. The discretized action now becomes

$$\sum_w \mathrm{Tr}(Z_w\, X_w) + \mathrm{Tr}(J_w\, X_w). \tag{5.8}$$

We fix a definition of Z_w in such a way that this action is gauge invariant.

We define the discrete gauge transformation so that it "acts" in the center of each d-simplex. More precisely, a gauge transformation is parameterized by a collection of group elements: a group element g for each d-simplex. First of all, the gauge transformation is defined for the holonomy U of the connection along any loop that starts and finishes at the center C of the d-simplex. The transformation is as follows:

$$U \to gUg^{-1}. \tag{5.9}$$

The wedge variables X_w and the discrete current variables J_w transform as

$$X_w \to gX_wg^{-1}, \quad J_w \to gJ_wg^{-1}. \tag{5.10}$$

With this definition $\mathrm{Tr}(X_wZ_w)$ is gauge invariant only when $\exp Z$ is defined as the holonomy around the wedge w whose starting and final point is the center C of the d-simplex, as in (5.6). This fixes the ambiguity in Z_w. With Z_w defined this way the discretized action is gauge invariant.

The approximation (5.5) is good for Z_w close to zero element in the Lie algebra. Thus, this approximation is certainly justified for BF theory, where we expect only connections close to flat to matter in the quantum theory. It is harder to justify this approximation for theories, for which the classical equations of motion do not imply the connection to be flat, as is the case, for example, for BF theory with cosmological term or for gravity. However, even for such theories, one would expect the approximation (5.5) become better as the triangulation of the manifold becomes finer and the dual faces become smaller. In quantum gravity we expect finer triangulations to matter most, which can serve as a justification for the above approximation in the case of gravity. The approximation (5.5) will be additionally justified when we discuss the problem of the integration over X_w variables. To define the later we will refer to a certain standard field theory calculation for BF theory in two dimensions. As we shall see, the approximation (5.5) is quite natural from the point of view of 2D BF theory.

The approximation (5.5) finishes the discretization procedure for the classical action. We can now calculate the path integral for the generating functional Z by integrating the exponentiated discrete action (5.8) over the Lie algebra elements X_w and group elements g, h. This path integral is given by

$$Z(J,\Delta) = \prod \int_{\mathrm{SU}(2)} dg \prod \int_{\mathrm{SU}(2)} dh \int \prod_w dX_w \, e^{i \sum_w \mathrm{Tr}(X_w Z_w) + \mathrm{Tr}(X_w J_w)}. \tag{5.11}$$

Here the integrals are taken over all group elements g, h, entering the expression through Z_w, see (5.5). These integrals form the discrete analog of $\mathcal{D}A$, dg is the normalized Haar measure $\int dg = 1$ on SU(2). The integrals over X_w present here – one for each edge – form the analog of the integral over $\mathcal{D}E$. The measure dX here is some measure on the Lie algebra. For now, we will leave this measure unspecified.

Let us now investigate the structure of the path integral (5.11). To calculate Z we have to find the function of $\exp Z_w, \exp J_w$ that is given by

$$\int dX_w e^{i \,\mathrm{Tr}(X_w Z_w + X_w J_w)}. \tag{5.12}$$

In fact, it is not hard to see that this function is proportional to the δ-function of $\exp Z_w$ peaked at $\exp J_w$. The proportionality coefficient can be a gauge invariant function of J_w. As we explain below, this function must be set to be equal to $P(J_w)$, where P is the function that relates Lebesgue measure on the Lie algebra and Haar measure on the group (see the Appendix B). Thus, as the result of (5.12) we get:

$$P(J_w)\, \delta(\exp Z_w \exp J_w), \tag{5.13}$$

where δ is the standard δ-function on the group. In case the gauge group is SU(2), the function of Z_w, J_w given by (5.12) can be written as

$$P(J_w) \sum_j \dim_j \chi_j (\exp Z_w \exp J_w), \tag{5.14}$$

where we used the well-known decomposition of δ-function on the group SU(2) into the sum over characters $\chi_j(\exp Z_w \exp J_w)$ of irreducible representation. Here the sum is taken over all irreducible representations of SU(2) (labelled by spins j), and $\dim_j = (2j+1)$. For other groups one has a similar decomposition.

Thus, for the case of SU(2) we find that

$$Z(J,\Delta) = \prod \int_{\mathrm{SU}(2)} dg \prod \int_{\mathrm{SU}(2)} dh \prod_w P(J_w) \sum_{j_w} \dim_{j_w} \chi_{j_w}(\exp Z_w \exp J_w). \quad (5.15)$$

The integration over the group elements can now be easily performed using the well-known formulas for the integrals of the products of matrix elements. However, let us first give a systematic derivation of the result (5.13).

To give a systematic calculation of the integral (5.12) we relate it to a more complicated integral, for which a precise result is known from 2D BF theory calculation. Let us restrict our attention to a particular wedge w. We can restrict the bundle $\mathcal{P}$ to w and get a G bundle $\mathcal{P}_w$ over w. Let A be a connection on $\mathcal{P}_w$, and E be an $\mathrm{Ad}(\mathcal{P})$ valued 0-form. Consider the following path integral:

$$\int \mathcal{D}A\mathcal{D}E \exp\left(i \int_w [\mathrm{Tr}(F\,E) + \mathrm{Tr}(J\,E)]\right), \quad (5.16)$$

where the integration over A is performed subject to the condition that the connection on the boundary of w is fixed, and J is given by

$$J = \delta(p) J_w, \quad (5.17)$$

where p is an arbitrary fixed point on w, and J_w is the same as in (5.12),(5.13). The path integral (5.16) is just a partition function of BF theory on the disk with the distributional source given by (5.17). This partition function can be derived using results of [16]. The result is given by (5.13), where Z_w is the Lie algebra element that corresponds to the holonomy of A along the boundary of w. We will not present this calculation here. Instead, we refer to the calculation performed in [16] for the partition function of 2D BF theory on a punctured sphere. The result (5.13) can then be checked by taking the partition function on the disk (equal to $\delta(g)$, where g is the holonomy along the boundary of the disk), and integrating it with (5.13) over dg. This must give the partition function on a punctured sphere, and indeed reproduces the result given in [16]. The only cautionary remark we have to make is that the calculation performed in [16] finds a gauge invariant partition function, that is, the one in which one takes $J = \delta(p) h J_w h^{-1}$ and integrates over dh. The techniques developed in [16] can be used only to calculate gauge invariant quantities, and are not directly applicable to the integral (5.16). Thus, strictly speaking, using the results of [16] one can only argue that (5.16) is equal to

$$P(J_w)\delta(e^{Z_w} h e^{J_w} h^{-1}), \quad (5.18)$$

where h is some group element. To get rid of h in this expression, and, thus, to get (5.13), we will recall how "discretized" gauge transformations act on the Lie algebra elements Z_w, J_w, see (5.9),(5.10). The result of (5.12) must be invariant under this gauge transformations. It is not hard to see that this fixes h above to be unity, thus, giving (5.13).

Having discussed how one can calculate the path integral (5.16), let us now show that this path integral is, in fact, equivalent to (5.12). Indeed, the integration over E in (5.16) can be performed in two steps. First, one integrated over $E(x), x \neq p$, then one integrates over $E(p)$:

$$\int dX_w \int_{E(p)=X_w} \mathcal{D}E \int \mathcal{D}A \exp\left(i \int_w [\mathrm{Tr}(F\,E) + \mathrm{Tr}(J\,E)]\right). \tag{5.19}$$

Since the current is distributional and concentrated at point p, the last term in the exponential does not matter when one integrates over $\mathcal{D}E, \mathcal{D}A$. On the other hand, it is not hard to see that

$$\int_{E(p)=X_w} \mathcal{D}E \int \mathcal{D}A \exp\left(i \int_w \mathrm{Tr}(F\,E)\right), \tag{5.20}$$

where the integral over A is taken with A on the boundary of w fixed, is equal to

$$e^{i\mathrm{Tr}(Z_w\,X_w)}, \tag{5.21}$$

where $\exp Z_w$ is the holonomy of A along the boundary of w. Putting this back to (5.19) one gets exactly (5.12). This finishes our discussion of the derivation of (5.13).

Having discussed the derivation of the expression (5.15) for the generating functional, we can now perform the integrals over the group elements. Integration over the group elements h that correspond to edges dividing dual faces into wedges is the same in any dimension, and we perform it here. The rest of the group elements corresponds to edges that form the boundary of the dual faces. The integration over these group elements g is different in different dimensions, and we perform it in the following subsections. Each group element h is "shared" by two wedges; thus, we have to take the integral of the product of two matrix elements. Such an integral is given by (D.3) of the Appendix. Integrating over all these edges, and making a simple change of variables to eliminate trivial integrations, we find, for the case of SU(2):

$$Z(J,\Delta) = \prod_{\epsilon} \int_{\mathrm{SU}(2)} dg_\epsilon \times \tag{5.22}$$
$$\prod_{\sigma} \sum_{j_\sigma} (\mathrm{dim}_{j_\sigma})^{\kappa_\sigma} P(J_1)\cdots P(J_n) \chi_{j_\sigma}(\exp J_1\, g_{\epsilon_1}\, \exp J_2 \cdots \exp J_n\, g_{\epsilon_n}).$$

For other groups one gets a similar expression. Here the remaining integrals are over the group elements g_ϵ that correspond to the edges ϵ of the dual complex (edges that connect centers of 4-simplices). The second product is taken over the dual faces σ; j_σ is the spin labelling the dual face σ, κ_σ is the Euler characteristics of σ. It is equal to unity in case the dual face has the topology of a disc. In what follows we will always assume that this is the case. The order of the group elements in the argument of χ_{j_σ} is clear from the Figure 4.

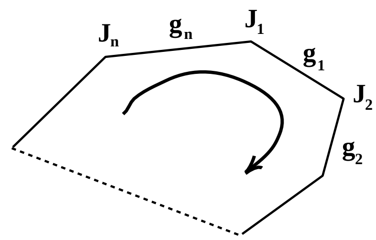


Fig. 5.3. The product of the group elements is the holonomy around the dual face σ with the insertions of the group elements $\exp J$ at each center of the corresponding d-simplex.

The expression (5.22) for the generating functional can be further simplified by integrating over the group elements g_ϵ. However, the result of this integration is different in different dimensions, and we will perform it separately for each particular case.

Before we calculate the generating functional for each particular spacetime dimension, let us pause to discuss some general properties of the expression for $Z(J, \Delta)$. The generating functional has several symmetries which we describe as follows. First, the generating functional is invariant under gauge transformations. Namely, let us consider a transformation $J_w \to g_{h(w)} J_w g^{-1}_{h(w)}$, where $h(w)$ denotes the d-simplex of Δ to which w belongs. Here $g_{h(w)}$ is the same for all w belonging to the simplex h. The generating functional (5.11) satisfies:

$$Z(gJg^{-1}, \Delta) = Z(J, \Delta). \tag{5.23}$$

This "discrete" gauge transformation is parametrized by one group element for each d-simplex of Δ. However the expression (5.22) for the generating functional has a bigger invariance. Namely, let us denote by $e_+(w), e_-(w)$ the two edges of the wedge w which meet at the center of the d-simplex $h(w)$. Let us associate with each edge $e_\pm(w)$ a group element $g_{e_\pm(w)}$ so that it is one and the same for different wedges w when $e_\pm(w)$ is one and the same. The transformation of all currents according to $J_w \to g_{e_+(w)} J_w g^{-1}_{e_-(w)}$ leaves the generating functional invariant. This transformation, which is parametrized by $d+1$ group element per simplex, contains the gauge transformation as its particular case. The later corresponds to $g_{e_+(w)} = g_{e_-(w)} = g_{h(w)}$ for all w that belong to h. The appearance and significance of the described extra symmetry is not yet clear to us.

The generating functional is covariant under the diffeomorphism group. Let f denote a diffeomorphism of the underlying manifold $\mathcal{M}$, and denote by $f(\Delta)$ the image of the embedded triangulation Δ under the diffeomorphism. If $J_w = \int_w J$, let us denote $f^*J \equiv \int_{f(w)} J = \int_w f^*J$. Then

$$Z(J, f(\Delta)) = Z(f^*J, \Delta). \tag{5.24}$$

The generating functional is generally not invariant under a refinement of the triangulation Δ unless the theory is a topological field theory (unless $J = 0$).

To define the generating functional we had to choose an arbitrary orientation of the wedges. However, the generating functional is independent of the orientation chosen. If one, for example, reverses the orientation of the wedge w this results: (i) in the change $J_w = \int_w J$ into $-J_w$; (ii) it changes the holonomy of the connection along the wedge into its inverse. The two effects cancel each other leaving the generating functional invariant.

5.2 2 dimensions

The case of two spacetime dimensions is somewhat special in the sense that dual faces (2-cells) cover the manifold (see Fig. 5). The general construction of the previous

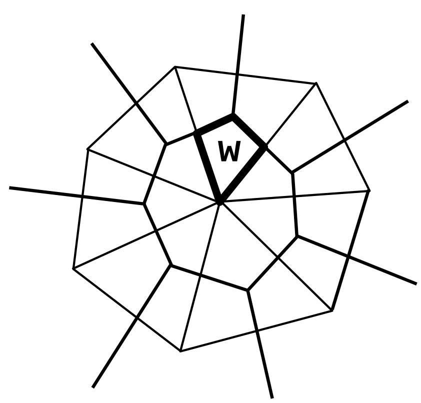


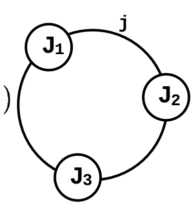
Fig. 5.4. Part of a triangulation of 2-dimensional spacetime manifold, with one face of the dual complex shown. Bold lines indicate a wedge of the dual face.

subsection prescribes to replace the continuous field E, which in the case of two dimensions is a zero form, by a distributional field concentrated along dual faces. However, since the spacetime is two-dimensional, the dual faces cover the manifold, and we just have to replace the E field by a field constant on each wedge, and equal to the Lie algebra element X_w.

Let us now see that it is quite a reasonable thing to do from the point of view of the canonical quantum theory. In the case $\mathcal{M}$ is compact, which is the case of interest for us here, a spatial slice Σ of $\mathcal{M}$ consists of a finite number of circles S^1. The states of the canonical theory are just class functions of the holonomies of $\underline{A}$, which is the pullback of A on Σ, around this circles. The corresponding field $\underline{E}$, which in the quantum theory becomes the operator of variational derivative with respect to $\underline{A}$, has thus a constant norm along each disjoint component of Σ. It thus makes sense to replace the E field by a collection of Lie algebra elements X_w constant on each wedge. Moreover, as we shall see, the integration over group elements g_ϵ in (5.22) renders all X_w to have the same norm in each disjoint component of $\mathcal{M}$, which is in agreement with what we get in the canonical approach.

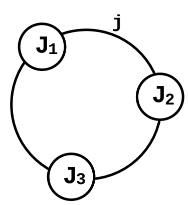
The group elements g_ϵ in (5.22) are "shared" by two dual faces. Thus, all integrals over g_ϵ in (5.22) have as the integrand the product of two matrix elements of g_ϵ. These can be taken using the formula (D.3) of the Appendix. To describe the result of this integration we will assume that $\mathcal{M}$ has a single connected component. Then the generating functional in two dimensions, in the case the gauge group is SU(2), is given by the following expression

$$Z_2(J,\Delta) = \sum_j (\dim_j)^{V-E} \prod_f P(J_1)P(J_2)P(J_3)$$



(5.25)

In the integration over the group elements g_ϵ in (5.22) survive only the terms in which all spins j_σ are equal. This spin is denoted by j in (5.25). The symbols V, E stand in (5.25) for number of vertices and number of edges in the triangulation correspondingly. The product here is taken over all centers of faces of Δ, or, equivalently, over faces f. The graphical notation stands for



$$= \chi_j(\exp J_1 \exp J_2 \exp J_3), \qquad (5.26)$$

where J_1, J_2, J_3 are the three Lie algebra elements (currents) "shared" by the face f. Recall that in two dimensions each face is just a union of three wedges, and J_1, J_2, J_3 are the currents J_w corresponding to these wedges (see Fig. 6). The expression (5.25) is the final result for the generating functional in two dimensions, for the case of $G = \mathrm{SU}(2)$. Generalization to other groups is straightforward.

Note that the graph in this graphical representation of the character can be obtained as a result of the following simple construction. Let us draw a circle S^1 centered

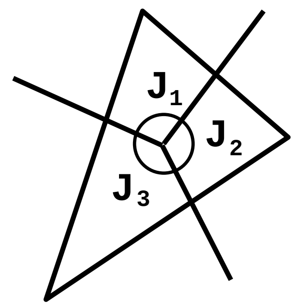


Fig. 5.5. In two dimensions each face is a union of three wedges, and J_1, J_2, J_3 are the currents J_w corresponding to these wedges.

at the center of the face (see Fig. 6). The intersection of wedges with the circle gives the circle itself. The intersection of this circle with the dual edges gives three points on the circle. Each wedge is labelled by a spin j, which is the same for all wedges, and we can assign a label j to the circle that intersects the three wedges. Then the face contribution to (5.25), that is, the contribution that we graphically represent by (5.2), is simply the *spin network* constructed with the labelled circle, with insertion of the three group elements $(\exp J_1 \exp J_2 \exp J_3)$ at the points where the circle is intersected by the corresponding wedges. This construction is trivial in two dimensions, but turns out to be generalizable to any dimension.

As the zeroth order check to our construction, let us see what the generating functional gives for $J = 0$. The transition amplitude given by (5.25), evaluated at $J = 0$, gives

$$\sum_j (\dim_j)^{\chi(\mathcal{M})}, \tag{5.27}$$

where $\chi(\mathcal{M})$ is the Euler characteristics of $\mathcal{M}$. One can recognize in this expression the volume of the space of flat connections on $\mathcal{M}$ modulo gauge transformations expressed in term of Riemann zeta function (see, for example, [16]). Thus, in two dimensions our calculation of the generating functional evaluated at $J = 0$ gives the expected partition function of BF theory.

5.3 3 dimensions

The three-dimensional case is more interesting. Here the dual faces no longer cover the spacetime manifold, and the E field acquires a true distributional character. The general construction prescribes to replace the E field, which is now a one-form, by a distribution concentrated along dual faces and constant along the wedges. It is illustrative to give a coordinate expression for such a distributional field. Let us consider an arbitrary wedge w (see Fig. 7). Let $u = 0$ be the equation of a plane containing w.

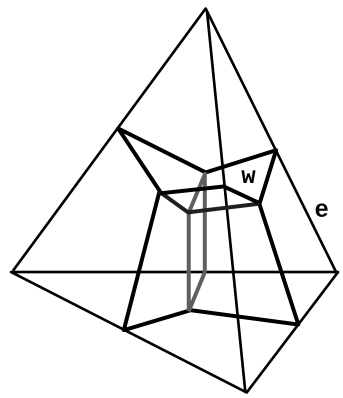


Fig. 5.6. Tetrahedron from a triangulation of a 3-dimensional spacetime manifold. The figure also shows the wedges that lie inside this tetrahedron.

Then the field E that has the correct distributional character is given by the expression

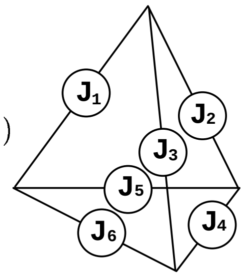
$$du\,\delta(u)\,X_w. \tag{5.28}$$

To calculate the generating functional let us note that each edge of the dual triangulation belongs to three dual faces. Thus, in (5.22), one has to take integrals over products of three matrix elements. The corresponding formula is given in the Appendix D. The result of this integration can be described as a generalization of the "circle" construction given at the end of the previous subsection. Let us draw a sphere S^2 centered at the center of each tetrahedron. The wedges belonging to a particular tetrahedron intersect the sphere and draw a graph on its surface. We will refer to this graph as Γ_t, where t refers to a tetrahedron of the triangulation that was used to construct Γ_t. It is not hard to see that Γ_t is a tetrahedron whose vertices come from the intersection of the sphere with the dual edges. The same graph Γ_t can also be obtained by looking at the boundary of tetrahedron t (this boundary has the topology of S^2), which is triangulated by the faces of t, and constructing the graph dual to the triangulation of this S^2. The resulting tetrahedron is the same as that in the previous construction with a sphere.

In (5.22) the sum is taken over spins j_σ labelling the dual faces. The result of the integration over the group elements g_ϵ will still be a sum over spins j_σ. Each wedge belongs to some dual face, and, thus, is labelled by spin. As we have just seen the edges of Γ_t are in one-to-one correspondence with the wedges. Let us label these edges with the same spins as those of the corresponding wedges. We have a current J_w associated with each of the 6 wedges belonging to tetrahedron t. Let us denote this currents by $J_1, \ldots, J_6$. We can now construct a function of the group elements $\exp J_1, \ldots, \exp J_6$ that is just a spin network function.

The generating functional in three dimensions, for the case of SU(2), is then given by the following expression

$$Z_3(J,\Delta) = \sum_{j_\sigma} \prod_\sigma \dim_{j_\sigma} \prod_t P(J_1) \cdots P(J_6) \tag{5.29}$$



where the sum is over all possible coloring of dual faces. This is the final expression we are going to use in the following section in order to derive spin foam models.

As the zeroth order check, it is instructive to find the value of the generating functional for $J = 0$. It is easy to see that this gives exactly Ponzano-Regge model, as expected.

5.4 4 dimensions

The case of four dimensions is analogous to the just analyzed case of three dimensions. The only difference is that it is harder to visualize a four-dimensional triangulated manifold, and that the final expression for the generating functional is more complicated. However, the final result for the generating functional follows the same pattern.

First, let us give a coordinate expression for the distributional field E in four dimensions. Let u, v be two functions such that $u = 0, v = 0$ is the equation for a 2-surface in $\mathcal{M}$ containing one of the faces σ of the dual complex, and $du \wedge dv$ is positive as defined by the orientation of σ. We then define E to be equal to

$$du \wedge dv \, \delta(u)\delta(v) \, X_w \tag{5.30}$$

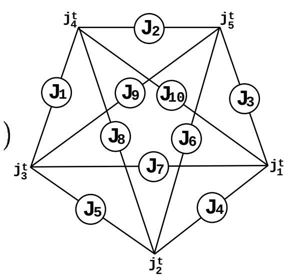
on each wedge w, and to be equal to zero everywhere else. We repeat this construction for all wedges, and add all these distributional forms together to get a two-form that is concentrated along the faces of the dual complex. Then the integral of such distributional E over a wedge of the triangulation is equal simply to X_w corresponding to that wedge.

Let us now describe the result of integration over the group elements g_ϵ in (5.22). In four dimensions each dual edge is shared by four dual faces. Thus, one has to take the integral of the product of four matrix elements. The required formula is given in Appendix D. The result of this integration can again be described using a certain spin network function of the group elements corresponding to the currents J_w. As in the case of two and three dimensions, let us introduce special graphs in the vicinity of the center of each 4-simplex h, which we will call Γ_h. We define graph Γ_h as the intersection of the sphere S^3 surrounding the center of 4-simplex h with the dual faces σ. The graph Γ_h lives in S^3, but it can be projected on a plane. This gives us the pentagon graph. Note that at this stage we do not care about types of crossings we get (that is, whether this is under- or over-crossing). The edges of this graph are in one-to-one correspondence with the wedges w belonging to the 4-simplex h. Thus, we can associate to each edge a current J_w, and label it with the spin j_σ labelling the dual face to which the corresponding wedge belongs. Additionally, let us label each of five vertices of the pentagon by a half-integer (spin). Vertices of the pentagon are in one-to-one correspondence with tetrahedra of the triangulation. Thus, we shall use the notation j_t for this spins.

Inside each four-simplex there are 10 wedges. Let us denote the corresponding currents by $J_1, \ldots, J_{10}$. One can then construct the function of the group elements $\exp J_1, \ldots, \exp J_{10}$.

Thus, for the case of $G = \mathrm{SU}(2)$, the generating functional Z_4 is given by the sum over spins of products over 4-simplices of the above functions of the currents J_w

$$Z_4(J, \Delta) = \sum_{j_\sigma, j_t} \prod_\sigma \dim_{j_\sigma} \prod_t \dim_{j_t} \prod_h P(J_1) \cdots P(J_{10}) \tag{5.31}$$



Here h labels 4-simplices of the triangulation and j_σ are spins labelling the dual faces. The resolution of each vertex in this pentagon graph is given in the Appendix D.

The value of the generating functional at $J = 0$ gives the Ooguri amplitude, as expected.

5.5 Higher dimensions

In higher dimensions, the result for the generating functional follows the same pattern. Since the general formula is very cumbersome, let us describe the result in words. The central element of the result for the generating functional is the spin network function evaluated on group elements corresponding to the currents. This spin network in any dimension can be obtained as the graph on the boundary of the simplex, and dual to the triangulation of the boundary. Equivalently, it can be obtained by considering the intersection of the sphere S^{D-1}, drawn inside the D-simplex, with the dual faces. The intersection of the dual faces with this sphere draws on the sphere the graph, which, together with the labels inherited from the dual faces, gives the spin network of interest. Every current comes with the function $P(J)$ multiplying the expression for the generating functional. There are other factors similar to the ones in the cases described above. It is easy to determine this factors in each particular dimension by performing the integral over the group of the product of a collection of characters. We will not develop this further here.

For $J = 0$ the generating functional described gives a model that can be checked to be triangulation independent. Thus, we obtain higher-dimensional generalizations of Ponzano-Regge and Ooguri models.

Chapter 6

BF theory with an interaction term. Examples

Having calculated the generating functional in various spacetime dimensions we are in the position to use our results to study concrete theories. However, let us first describe in more details the theories that can be treated within our approach, and give physically interesting examples. In this paper we will restrict our attention to a special class of theories. In most cases we for simplicity fix the gauge group to be $G = \mathrm{SU}(2)$. The only case where we consider a different gauge group is gravity in D dimensions, where the group is $\mathrm{SO}(D)$. Let $\mathcal{M}$ be an oriented smooth D-dimensional manifold, and let P be an SU(2)-bundle over $\mathcal{M}$. The basic fields of a theory of interest are a connection A on P, and an AdP-valued (D–2)-form, often called B, which we call E because of its relation with the SU(2) "electric" field of the canonical formulation of the theory. The action of the theory is of the form

$$S[A,E] = \int_{\mathcal{M}} \left[\mathrm{Tr}\,(E \wedge F) + \Phi(E) \right], \tag{6.1}$$

where F is the curvature of A, 'Tr' is the trace in the fundamental representation of SU(2), and Φ is a gauge invariant polynomial function that depends on E but not on its derivatives. Also, $\Phi(E)$ should be a D-form so that it can be integrated. It can also depend on other dynamical fields, for example Lagrange multipliers as in the case of gravity, or on additional non-dynamical fields (background structure), as in the case of Yang-Mills theories. Let us now give examples of theories belonging to this class.

6.1 2D Yang-Mills

Yang-Mills action can be written only if one introduces a background metric on $\mathcal{M}$. However, in two spacetime dimensions, Yang-Mills action turns out to depend only on a measure (area two-form) defined by the metric on $\mathcal{M}$. Thus, we will keep track only of this dependence of the action on a measure $d\mu$ on $\mathcal{M}$. Yang-Mills theory in 2D is described by the following BF-like action

$$\int_{\mathcal{M}} \mathrm{Tr}(EF) + \frac{e^2}{2} \int_{\mathcal{M}} d\mu \mathrm{Tr}(E^2). \tag{6.2}$$

Here E is a Lie-algebra valued zero-form, F is the curvature of the connection A. Solving equations of motion for E that follow from this action, and substituting the solution into the action, one can check that the action reduces to the standard Yang-Mills action

$$S_{\mathrm{YM}} = -\frac{1}{4e^2} \int_{\mathcal{M}} d^2x \sqrt{g} g^{ac} g^{bd} \mathrm{Tr}(F_{ab} F_{cd}), \tag{6.3}$$

where g_{ab} is a metric (of Euclidean signature) on $\mathcal{M}$ such that $d^2x\sqrt{g} = d\mu$, where $\sqrt{g}$ is the square root of the determinant of the metric.

Using the techniques developed in this paper, one can calculate the vacuum-vacuum transition amplitude of this theory, that is the path integral of $\exp(i \times \text{action})$. However, an interesting feature of this theory is that the same techniques can be used to calculate the *partition function*, that is, the path integral

$$\int \mathcal{D}A\mathcal{D}E \exp\left(-S_{\rm YM}\right). \tag{6.4}$$

The problem of calculation of the partition function can be reduced to the problem of calculation of the vacuum-vacuum transition amplitude of a somewhat different theory, that is of the path integral of $\exp(i \times \text{action})$, with the action given by

$$\int_{\mathcal{M}} \mathrm{Tr}(EF) - i\frac{e^2}{2}\int_{\mathcal{M}} d\mu \mathrm{Tr}(E^2). \tag{6.5}$$

Indeed, integrating over E in the path integral of $\exp(i \times \text{action})$, one gets (6.4). As one can see, the two actions (6.2),(6.5) differ by the factor of i in front of the second term. Using the two actions (6.2),(6.5) one can calculate both the vacuum-vacuum transition amplitude and partition function of the theory.

6.2 3D BF theory with the 'cosmological term'

In three dimensions the E field of BF theory is a one-form, and one can add a term cubic in E to the BF action to obtain

$$-\int_{\mathcal{M}} \mathrm{Tr}\left(E\wedge F + \frac{\Lambda}{12}E\wedge E\wedge E\right), \tag{6.6}$$

where $\mathcal{M}$ is assumed to be a three-dimensional orientable manifold. The action is a functional of an SU(2) connection A, whose curvature form is denoted by F, and a 1-form E, which takes values in the Lie algebra of SU(2). Thus, the action (6.6) is that of BF theory in 3d, with E field playing the role of B, and with an additional "cosmological term" added to the usual BF action. This theory is related to gravity in 3D as follows. Having the one-form E, one can construct from it a real metric of Euclidean signature

$$g_{ab} = -\frac{1}{2}\mathrm{Tr}(E_aE_b). \tag{6.7}$$

Here a, b stand for spacetime indices: $a, b, ... = 1, 2, 3$. We take the E field to be anti-hermitian, which explains the minus sign in (6.7). Thus, the E field in (6.6) has the interpretation of the triad field. One of the equations of motion that follows from (6.6) states that A is the spin connection compatible with the triad E. Taking the E field to be non-degenerate and "right-handed", i.e., giving a positive-definite volume form

$$\frac{1}{12}\tilde{\varepsilon}^{abc}\mathrm{Tr}(E_aE_bE_c), \tag{6.8}$$

and substituting into (6.6) the spin connection instead of A, one gets the Euclidean Einstein-Hilbert action

$$\frac{1}{2}\int_{\mathcal{M}} d^3x\sqrt{g}\,(R-2\Lambda). \tag{6.9}$$

We use units in which $8\pi G = 1$. The minus in front of (6.6) is needed to yield precisely the Einstein-Hilbert action after the elimination of A. Thus, Λ in (6.6) is the cosmological constant. An important difference of this theory and Einstein's theory is the fact BF theory, unlike gravity, is defined for degenerate metrics.

6.3 4D BF theory with the 'cosmological term'

In four spacetime dimensions the E field of BF theory is a two-form. Thus, one can construct the following action functional

$$\int_{\mathcal{M}} \mathrm{Tr}(E\wedge F) - \frac{\Lambda}{2}\mathrm{Tr}(E\wedge E), \tag{6.10}$$

where Λ is a real parameter, which we will refer to as "cosmological constant". If one, as in the following subsection, adds to this action an additional constraint that E is simple, that is, given by a product of two one-forms, then this theory is equivalent to Einstein's theory and Λ is proportional to the "physical" cosmological constant.

6.4 4D Euclidean self-dual gravity

The action for Euclidean general relativity in the self-dual first order formalism is given by [46, 37, 19]

$$\int_{\mathcal{M}} \mathrm{Tr}(E\wedge F) - \psi^{ij}\left(E^i\wedge E^j - \frac{1}{3}\delta^{ij}E^k\wedge E_k\right), \tag{6.11}$$

where, to write E^i, we have introduced a basis in the Lie algebra of SU(2). Here ψ^{ij} is a symmetric matrix of Lagrange multipliers. The variation of the action with respect to ψ yields equations

$$E^i\wedge E^j = \delta^{ij}\frac{1}{3}E^k\wedge E_k. \tag{6.12}$$

In the case E is non degenerate, i.e., the right hand side of (6.12) is non zero, these equations are satisfied if and only if E is the self-dual part of a decomposable 2-form, i.e., if and only if there exists a tetrad field e^I, $I = 0,1,2,3$ such that $E^i = \pm(e^0\wedge e^i + \frac{1}{2}\epsilon_{ijk}e^j\wedge e^k)$. In this case the action reduces to the self-dual Hilbert-Palatini action for 4-dimensional gravity and ψ_{ij} correspond to the components of the self-dual part of the Weyl curvature tensor. Thus, (6.11) is the BF action in four dimensions, with an additional "simplicity" constraint added to it.

6.5 Higher-dimensional gravity

It turns out that higher-dimensional gravity can also be written in the form of BF theory plus constraints. In other words, one can re-write the action in the following form:

$$\int_{\mathcal{M}} \mathrm{Tr}(E \wedge F) + \frac{1}{2}\mathrm{Tr}(E \wedge \Phi(E)). \tag{6.13}$$

The action is a functional of a connection A, and F is its curvature, of a (D–2)-form E, and of a collection of Lagrange multipliers Φ described below. Varying with respect to the Lagrange multiplier field, one gets equations guaranteeing that E comes from the metric. This generalizes what we have seen in the previous section in the case of self-dual gravity.

The existance of such a formulation is not obvious, for no generalization of the action of the previous section is possible to higher dimensions, where there is no analog of self-duality for the connection. Still, it turns out that one can write down the gravity action in the BF form in any dimension, but one has to work with the full internal group. The results of this section were discovered by Freidel and De Pietri [23] for the case of 4d, and by Freidel, Krasnov and Puzio for higher dimensions. The BF formulation exists both for Lorentzian and Euclidean signatures of the metric. Here we consider the case of Euclidean signature, which is what will be used in the quantum theory. Thus, the gauge group in this section is $\mathrm{SO}(D)$.

The action for gravity in the BF formulation is a functional of the E field, the connection form A, and Lagrange multipliers Φ. There are two equivalent formulations, which are both worth mentioning. In the first formulation, which is more customary in the context of BF theories, the E field is thought of as a Lie algebra valued $(D-2)$-form. In the second formulation one uses the metric-independent Levi-Civita density to construct from this $(D-2)$-form a densitized rank two antisymmetric covariant tensor, which we will call a bivector. We first present the action in this second formulation, for it looks exactly the same in any dimension $D \geq 4$. Thus, we start by writing E as a bivector $\tilde{E}^{\mu\nu}_{ij}$, where Greek characters are the spacetime indices, latin letters are the internal indices, and a single tilde over the symbol of E represents the fact that its density weight is one.

The action of the theory is then given by:

$$S[A, \tilde{E}, \underset{\sim}{\Phi}] = \int d^D x \ \tilde{E}^{\mu\nu}_{ij} F^{ij}_{\mu\nu} + \frac{1}{2} \underset{\sim}{\Phi}^{ijkl}_{\mu\nu\rho\sigma} \tilde{E}^{\mu\nu}_{ij} \tilde{E}^{\rho\sigma}_{kl} \, . \tag{6.14}$$

The action is a functional of an $\mathrm{SO}(D)$ gauge field A^{ij}_{μ}, bivector fields $\tilde{E}^{\mu\nu}_{ij}$, and Lagrange multiplier fields $\underset{\sim}{\Phi}^{ijkl}_{\mu\nu\rho\sigma}$. This action is generally covariant: the bivector fields scale as tensor densities of weight one, while the multipliers scale as densities of weight minus one, which is represented by a single tilde below the symbol 'Φ'.

The multiplier field $\underset{\sim}{\Phi}^{ijkl}_{\mu\nu\rho\sigma}$ must be such that it is completely anti-symmetric in one set of indices, and its anti-symmetrization on the other set of indices vanishes. There is a freedom, however, on which set of indices the anti-symmetrization is taken to vanish. It turns out to be more convenient for the quantum theory to choose the anti-symmetrization on the spacetime indices to vanish. This is the choice we make. Let us emphasize, however, that from the point of view of the classical theory the two possibilities are completely equivalent in the sense that they are both enough to guarantee the simplicity of the E field (for a generic field E).

The postulated properties of the Lagrange multiplier field Φ imply that it is of the form:

$$\underset{\sim}{\Phi}{}^{ijkl}_{\mu\nu\rho\sigma} = \epsilon^{[m]ijkl}\underset{\sim}{\Phi}{}_{[m]\mu\nu\rho\sigma},$$

where $\epsilon^{[m]ijkl}$ is the totally anti-symmetric form on the Lie algebra, $[m]$ is a completely anti-symmetric cumulative index of length $D-4$, and $\underset{\sim}{\Phi}{}_{[m]\mu\nu\rho\sigma}$ is a new Lagrange multiplier field, which we, by abuse of notation, also call Φ. This new Lagrange multiplier field also has density weight minus one. The field $\underset{\sim}{\Phi}{}_{[m]\mu\nu\rho\sigma}$ has a property that its anti-symmetrization on the spacetime indices vanishes:

$$\underset{\sim}{\Phi}{}_{[m][\mu\nu\rho\sigma]} = 0. \tag{6.15}$$

Using this new set of Lagrange multipliers the action can be written as:

$$S[A,B,\Phi] = \int d^D x\ \tilde{E}^{\mu\nu}_{ij} F^{ij}_{\mu\nu} + \frac{1}{2}\underset{\sim}{\Phi}{}_{[m]\mu\nu\rho\sigma}\epsilon^{[m]ijkl}\tilde{E}^{\mu\nu}_{ij}\tilde{E}^{\rho\sigma}_{kl}. \tag{6.16}$$

Let us now give another way the action (6.16) can be written, using the representation of the E field as a $(D-2)$-form. This is more standard in the context of BF theories. Using the definition of the bivector $\tilde{E}^{\mu\nu}$,

$$\tilde{E}^{\mu\nu}_{ij} = \frac{1}{2!(D-2)!}\tilde{\epsilon}^{\mu\nu\beta_1\ldots\beta_{D-2}}E_{\beta_1\ldots\beta_{D-2}\,ij}, \tag{6.17}$$

one can easily check that the action (6.16) can be rewritten as

$$S[A,E,\Phi] = \frac{1}{2!(D-2)!}\int d^D x\ E_{\beta_1\ldots\beta_{D-2}\,ij}F^{ij}_{\mu\nu}\tilde{\epsilon}^{\beta_1\ldots\beta_{D-2}\mu\nu} + \frac{1}{2}E_{\beta_1\ldots\beta_{D-2}\,ij}\Phi^{ij}_{\mu\nu}(E)\tilde{\epsilon}^{\beta_1\ldots\beta_{D-2}\mu\nu}, \tag{6.18}$$

where we have introduced a new two-form field $\Phi(E)$ with values in the Lie algebra. In the index notation it is given by:

$$\Phi^{ij}_{\mu\nu}(E) := \underset{\sim}{\Phi}{}^{ijkl}_{\mu\nu\rho\sigma}\tilde{E}^{\rho\sigma}_{kl}. \tag{6.19}$$

Therefore, in the abstract notations, one can write the action as in (6.13). Thus, there are two equivalent formulations of the theory. One can use the formulation in terms of forms, given by (6.13), or the formulation in terms of bivectors, given by (6.14).

Variation of the action (6.16) with respect to Φ gives the following equations:

$$\epsilon^{[m]ijkl}\tilde{E}^{\mu\nu}_{ij}\tilde{E}^{\rho\sigma}_{kl} = \tilde{\epsilon}^{[\alpha]\mu\nu\rho\sigma}\tilde{c}^{[m]}_{[\alpha]} \tag{6.20}$$

for some coefficients $\tilde{c}^{[m]}_{[\alpha]}$. Here $[m], [\alpha]$ are cumulative anti-symmetric indices of length $D-4$, Lie algebra and spacetime ones correspondingly. As one can see, when equations (6.20) are satisfied, the coefficients $\tilde{c}^{[m]}_{[\alpha]}$ are given by:

$$\tilde{c}^{[m]}_{[\alpha]} = \frac{1}{(D-4)!4!} \epsilon^{[m]ijkl} \tilde{E}^{\mu\nu}_{ij} \tilde{E}^{\rho\sigma}_{kl} \epsilon_{[\alpha]\mu\nu\rho\sigma}. \tag{6.21}$$

It is clear that when E comes from a frame field e, E identically satisfies (6.20). The following theorem states that the reverse is true.

THEOREM 6.1. *(Freidel, Puzio and Krasnov) In dimension $D > 4$ a generic (non-degenerate) E field satisfies the constraints (6.20) if and only if it comes from a frame field. In other words, a non-degenerate E satisfies the constraints (6.20) if and only if there exist e^{μ}_{i} such that:*

$$\tilde{E}^{\mu\nu}_{ij} = \pm |e| e^{[\mu}_{i} e^{\nu]}_{j}, \tag{6.22}$$

where $|e|$ is the absolute value of the determinant of the matrix e^{μ}_{i}.

The condition $D > 4$ is there because in four dimensions, under the same assumptions, there is another solution (see [23]) given by:

$$\tilde{E}^{\mu\nu}_{ij} = \pm |e| \epsilon_{ij}{}^{kl} e^{[\mu}_{k} e^{\nu]}_{l}. \tag{6.23}$$

Thus, our theorem implies that this other solution appears only in four dimensions. The proof of this statement is given in [31].

For later purposes, let us note that the constraints (6.20) can be subdivided into the following categories:

$$\text{simplicity:} \qquad \tilde{E}^{\mu\nu}_{[ij} \tilde{E}^{\mu\nu}_{kl]} = 0 \qquad \mu, \nu \text{ distinct} \tag{6.24}$$

$$\text{intersection:} \qquad \tilde{E}^{\mu\nu}_{[ij} \tilde{E}^{\nu\rho}_{kl]} = 0 \qquad \mu, \nu, \rho \text{ distinct} \tag{6.25}$$

$$\text{normalization:} \qquad \tilde{E}^{\mu\nu}_{[ij} \tilde{E}^{\rho\sigma}_{kl]} = \tilde{E}^{\mu\rho}_{[ij} \tilde{E}^{\sigma\nu}_{kl]} \qquad \mu, \nu, \rho, \sigma \text{ distinct} \tag{6.26}$$

The reason for this terminology has to do with the conditions imposed by the various constraints. Thus, constraints of the first type guarantee that $\tilde{E}^{\mu\nu}_{ij}$ is a simple two-form in the Lie algebra. Thus, they guarantee that E defines a plane in $\mathbb{R}^D$. The geometrical meaning of the intersection constraints is that they guarantee that the two planes defined by two simple E intersect along a line. The normalization constraint guarantees that the frame vectors defined by different E are normalized in the same way. For more details on this see [31].

Chapter 7

Known state sum models

In this chapter we present state sum models for the theories we just described. This section is mostly a review of known facts. Some of this models were already discussed before. Here we give more details, and also derive some facts that will be needed later, when we compare our results with the known models described here.

7.1 Migdal-Witten model

Migdal [42] has studied the Yang-Mills theory on a lattice, and, in particular, has proposed a lattice model for the Yang-Mills theory in two dimensions. This model was later studied by Witten [62] in the connection with topological field theories in two dimensions.

As we discussed above, the problem of calculation of the Yang-Mills partition function can be reduced to the problem of calculation of the path integral

$$Z^{\rm YM}(\rho e^2, \mathcal{M}) = \int \mathcal{D}A\mathcal{D}E \exp\left(i\int_{\mathcal{M}} {\rm Tr}(EF) + \frac{e^2}{2}\int_{\mathcal{M}} d\mu {\rm Tr}(E^2)\right), \qquad (7.1)$$

where ρ is the total area of $\mathcal{M}$, and other notations are explained in subsection (6.1). Migdal [42] has proposed the following lattice version of this path integral. For simplicity, we will formulate Migdal model on a triangulated manifold. Let us triangulate our two-dimensional manifold $\mathcal{M}$, and introduce the dual triangulation (see Fig. 5). Let us label the edges ϵ of the dual complex by group elements g_ϵ, and the dual faces σ by irreducible representations of SU(2), i.e., spins j_σ. The "discrete" version of (7.1) is then given by (see [62]):

$$\mathrm{YM}(\rho e^2, \mathcal{M}) = \prod_\epsilon \int dg_\epsilon \sum_{j_\sigma} \prod_\sigma \dim_{j_\sigma} \chi_{j_\sigma}(g_{\epsilon_1} \cdots g_{\epsilon_n}) \exp\left(-e^2 \rho_\sigma c(j_\sigma)/2\right), \qquad (7.2)$$

where the multiple integral is performed over all group elements g_ϵ and dg is the normalized Haar measure on SU(2), ρ_σ is the area of the dual face σ, as defined by the measure $d\mu$, $c(j)$ is the value of the quadratic Casimir operator in the representation j

$$c(j) = 2j(j+1), \qquad (7.3)$$

and $g_{\epsilon_1} \cdots g_{\epsilon_n}$ is the product of group elements around the dual face. After integration over the group elements g_ϵ the partition function takes the following simple form:

$$\sum_j (\dim_j)^{\kappa(\mathcal{M})} e^{-e^2 \rho c(j)}, \tag{7.4}$$

where $\kappa(\mathcal{M})$ is the Euler characteristics of $\mathcal{M}$. Thus, as we indicated in the argument of YM, the partition function depends only on the topological properties of $\mathcal{M}$. Also, the dependence on measure $d\mu$ enters only through the dependence on the total area ρ of $\mathcal{M}$.

7.2 Turaev-Viro model

The Turaev-Viro model gives a way to calculate the transition amplitude of the theory defined by (6.6), i.e., the path integral

$$Z_3^{\mathrm{BF}}(\Lambda, \mathcal{M}) = \int \mathcal{D}E\,\mathcal{D}A \exp\left(-i \int_{\mathcal{M}} \mathrm{Tr}\left(E \wedge F + \frac{\Lambda}{12} E \wedge E \wedge E\right)\right). \tag{7.5}$$

In this paper we consider only vacuum-vacuum amplitudes. Although the Turaev-Viro model can be used to calculate more general amplitudes between non-trivial initial and final states, we will not use this aspect of the model here. We consider the version of the model formulated on a triangulated manifold. Thus, let us fix a triangulation Δ of $\mathcal{M}$. Let us label the edges, for which we will employ the notation e, by irreducible representations of the quantum group $(\mathrm{SU}(2))_q$, where q is a root of unity

$$q = e^{\frac{2\pi i}{k}} \equiv e^{i\hbar}. \tag{7.6}$$

The parameter $\hbar$ above is the parameter of deformation quantization, and this explains why the usual notation of the Planck constant is used. The irreducible representations of $(\mathrm{SU}(2))_q$ are labelled by half-integers (spins) j satisfying $j \leq (k-2)/2$. Thus, we associate a spin j_e to each edge e. The vacuum-vacuum transition amplitude of the theory is then given by the following expression (see, for example, [54]):

$$\mathrm{TV}(q, \Delta) = \eta^{2V} \sum_{j_e} \prod_e \dim_q(j_e) \prod_t (6j)_q, \tag{7.7}$$

where η and the so-called quantum dimension $\dim_q(j)$ are defined in the Appendix A by (A.4) and (A.5) correspondingly, and V is the number of vertices in Δ. The last product in (7.7) is taken over tetrahedra t of Δ, and $(6j)_q$ is the (normalized) quantum $(6j)$-symbol constructed from the 6 spins labelling the edges of t (see Appendix C). It turns out that (7.7) is independent of the triangulation Δ and gives a topological invariant of $\mathcal{M}$: $\mathrm{TV}(q, \Delta) = \mathrm{TV}(q, \mathcal{M})$.

The construction that interprets the Turaev-Viro invariant (7.7) as the vacuum-vacuum transition amplitude of the theory defined by (6.6) is as follows. It has been proved (see *e.g.* [54]) that (7.7) is equal to the squared absolute value of the Chern-Simons amplitude

$$\mathrm{TV}(q, \mathcal{M}) = |\mathrm{CS}(k, \mathcal{M})|^2, \tag{7.8}$$

with the level of Chern-Simons theory being equal to k from (7.6). It is known, however, that the action (6.6) can be written as a difference of two copies of Chern-Simons action

$$S_{\mathrm{CS}}(A) = \frac{k}{4\pi} \int_{\mathcal{M}} \mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right). \tag{7.9}$$

Indeed, note that

$$S(A + \lambda E) - S(A - \lambda E) = \frac{k\lambda}{\pi} \int_{\mathcal{M}} \mathrm{Tr}\left(E \wedge F + \frac{\lambda^2}{3} E \wedge E \wedge E\right), \tag{7.10}$$

where λ is a real parameter. Thus, (7.10) is equal to (6.6) if

$$\lambda = -\sqrt{\Lambda}/2, \qquad k = \frac{2\pi}{\sqrt{\Lambda}}, \qquad \text{or} \qquad \hbar = \sqrt{\Lambda}. \tag{7.11}$$

This relates the deformation parameter q of the Turaev-Viro model with the cosmological constant Λ, and proves that the Turaev-Viro amplitude is proportional to the vacuum-vacuum transition amplitude of the theory defined by (6.6)

$$\mathrm{TV}(q, \mathcal{M}) \propto Z_3^{\mathrm{BF}}(\Lambda, \mathcal{M}). \tag{7.12}$$

To compare in Chapter 8 the spin foam model obtained via our techniques with the Turaev-Viro model, we will need the first-order term in the decomposition of (7.7) in the power series in Λ. This is proportional to the expectation value of the spacetime volume in BF theory at $\Lambda = 0$. Indeed, the expectation value of the volume is given simply by the derivative of the path integral with respect to $(-i\Lambda)$ evaluated at $\Lambda = 0$:

$$\langle \mathrm{Vol} \rangle = \frac{\int \mathcal{D}E\mathcal{D}A\, \mathrm{Vol}(\mathcal{M}) e^{iS}}{\int \mathcal{D}E\mathcal{D}A\, e^{iS}} = i \left(\frac{\partial \ln Z_3^{\mathrm{BF}}(\Lambda)}{\partial \Lambda}\right)_{\Lambda=0}, \tag{7.13}$$

$$\mathrm{Vol}(\mathcal{M}) = \int_{\mathcal{M}} \frac{1}{12} \tilde{\varepsilon}^{abc} \mathrm{Tr}(E_a E_b E_c).$$

Thus, the expectation value of the volume of $\mathcal{M}$ in BF theory is proportional to the first order term in Λ in the decomposition of (7.7).

Let us find this expectation value. An important subtlety arises here. It is not hard to show that (7.7) has the following asymptotic expansion in $\hbar$

$$\left(\frac{\hbar^3}{4\pi}\right)^V \mathrm{PR}(\Delta) \left(1 - \hbar^2 i \langle \mathrm{Vol} \rangle\right), \tag{7.14}$$

where PR is the amplitude of the Ponzano-Regge model (1.7) V is the number of vertices in Δ, and $i\langle \mathrm{Vol} \rangle$ is a real quantity independent of $\hbar$. Thus, apparently there is no term proportional to $\hbar^2 = \Lambda$ in this expansion. However, as it is explained, for example, in [30], the state sum invariant (7.7) does not exactly give the transition amplitude of BF theory. Instead, the Turaev-Viro amplitude is only proportional to the BF amplitude, and the proportionality coefficient depends on $\hbar$. This can be understood as follows. The integration over $(A+\lambda E), (A-\lambda E)$, which is carried out to obtain $|\mathrm{CS}(k,\mathcal{M})|^2$ in (7.8) and thus the Turaev-Viro amplitude, is different from the integration over A, E one has to perform to obtain (7.5). The difference in the integration measures is a power of $\hbar$. Thus, the amplitude (7.5) and the squared absolute value of the amplitude of the Chern-Simons theory are proportional to each other with the coefficient of proportionality being a power of $\hbar$. In the discretized version of the theory, given by the Turaev-Viro model, this power of $\hbar$ is replaced by $\hbar^{3V}$. Thus, the Turaev-Viro amplitude, in the limit of small cosmological constant, differs from the BF amplitude by a power of $\hbar^{3V}$.

This remark being made, we can write an expression for the expectation value of the volume in BF theory:

$$
\begin{aligned}
i\langle \mathrm{Vol}\rangle_\Delta &= -\frac{\partial}{\partial\Lambda}\left(\frac{\mathrm{TV}(\Lambda,\Delta)}{\mathrm{PR}(\Delta)(\hbar^3/4\pi)^V}\right)_{\Lambda=0} = \\
&\frac{1}{\mathrm{PR}(\Delta)}\sum_{j_e} i\mathrm{Vol}(\Delta,\mathbf{j})\left(\prod_e \dim(j_e)\prod_t (6j)\right),
\end{aligned}
\tag{7.15}
$$

where the function $\mathrm{Vol}(\Delta,\mathbf{j})$ of the triangulation Δ and the labels $\mathbf{j}=\{j_e\}$ is given by

$$
\begin{aligned}
i\mathrm{Vol}(\Delta,\mathbf{j}) = \sum_v\left(-\frac{\partial}{\partial\Lambda}\left(\frac{\eta^2}{(\hbar^3/4\pi)}\right)\right)_{\Lambda=0} + \sum_e\left(-\frac{\partial\ln(\dim_q(j_e))}{\partial\Lambda}\right)_{\Lambda=0} + \\
\sum_t\left(-\frac{\partial\ln((6j)_q)}{\partial\Lambda}\right)_{\Lambda=0} .
\end{aligned}
\tag{7.16}
$$

Here v stands for vertices of Δ, e stands for edges and t stands for tetrahedra. We intentionally wrote the expectation value of the volume in the form (7.15) to introduce the volume $\mathrm{Vol}(\Delta,\mathbf{j})$ of a labelled triangulation, which will be of interest to us in what follows. Indeed, (7.15) has the form

$$
\frac{\sum_{j_e} i\mathrm{Vol}(\Delta,\mathbf{j})\mathrm{Amplitude}(\Delta,\mathbf{j})}{\sum_{j_e}\mathrm{Amplitude}(\Delta,\mathbf{j})},
\tag{7.17}
$$

where

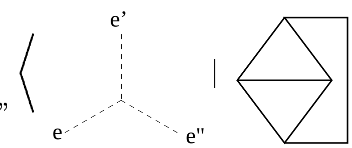
$$
\mathrm{Amplitude}(\Delta,\mathbf{j}) = \prod_e \dim(j_e)\prod_t (6j)
\tag{7.18}
$$

is the amplitude of Ponzano-Regge model. This shows that $\mathrm{Vol}(\Delta,\mathbf{j})$ indeed has the interpretation of the volume of a labelled triangulation. Note that the volume turns out to be purely imaginary. This has to do with the fact that in (7.5) one sums over configurations of E of both positive and negative volume. A more detailed explanation of this is given in [30].

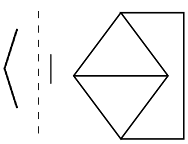
The volume (7.16) has three types of contributions: (i) from vertices; (ii) from edges; (iii) from tetrahedra. It is not hard to calculate the first two types of them. One finds that each vertex contributes exactly 1/12, and each edge contributes $j_e(j_e+1)/6$, where j_e is the spin that labels the edge e. It is much more complicated to find the tetrahedron contribution to the volume, that is, the derivative of $\ln((6j)_q)$ with respect to Λ. The result is described in [30]. Using the notations of the Appendix E, the final result for the expectation value of the spacetime volume in BF theory can be written as follows:

$$i\mathrm{Vol}(\Delta,\mathbf{j}) = \sum_v \frac{1}{12} + \sum_e \frac{j_e(j_e+1)}{6} + \quad (7.19)$$

$$\sum_t \frac{1}{16}\frac{1}{(6j)}\left(\frac{1}{24}\sum_{e\neq e'\neq e''}\left\langle\right.\right.$$



$$\left.\right\rangle - \frac{1}{4}\sum_e \left\langle\right.$$



$$\left.\left.\right\rangle\right)_0 . \quad (7.20)$$

Here $(6j)$ stands for the normalized classical $(6j)$-symbol.

It is interesting to note that not only tetrahedra t of Δ contribute to the volume, but also the edges e and vertices v. The contribution from the vertices is somewhat trivial – it is constant for each vertex. Nevertheless, when thinking about the triangulated manifold $\mathcal{M}$ one is forced to assign the spacetime volume to every vertex. The contribution from edges depends on the spins labelling the edges. Again, this implies that each edge of the triangulation Δ carries an intrinsic volume that depends on its spin. The contribution from tetrahedra is more complicated. It is given by a function that depends on the spins labelling the edges of each tetrahedron. It is interesting that this picture of the spacetime volume being split into contributions from vertices, edges and tetrahedra can be understood in terms of Heegard splitting of $\mathcal{M}$. Recall that Heegard splitting of a three-dimensional manifold $\mathcal{M}$ decomposes $\mathcal{M}$ into three dimensional manifolds with boundaries. Then the original manifold can be obtained by gluing these manifolds along the boundaries. For the case of a triangulated manifold $\mathcal{M}$, as we have now, the Heegard splitting proceeds as follows. First, one constructs balls centered at the vertices of Δ. Then one connects these balls with cylinders, whose axes of cylindrical symmetry coincide with the edges of Δ. Removing from $\mathcal{M}$ the obtained balls and cylinders, one obtains a three-dimensional manifold with a complicated boundary. One has to further cut this manifold along the faces of Δ. One obtains three types of "building blocks" that are needed to reconstruct the original manifold: (i) balls; (ii) cylinders; (iii) spheres with four discs removed. Each of this manifolds carries a part of the original volume of $\mathcal{M}$. Our result (7.19) provides one with exactly the same picture: the volume of $\mathcal{M}$ is concentrated in vertices (balls of the Heegard splitting), edges (cylinders), and tetrahedra (4-holed spheres).

7.3 Crane-Yetter model

Crane, Kauffman and Yetter [22] studied a state sum model that is very similar to the Ooguri model [44], however, instead of the gauge group SU(2) the quantum group

$(\mathrm{SU}(2))_q$ is used. The state sum invariant they proposed is given by (see, for example, [54]):

$$\mathrm{CY}(q,\Delta)=\eta^{-2V+2E}\sum_{j_f}\sum_{j_t}\prod_f \dim_q(j_f)\prod_t \dim_q(j_t)\prod_h (15j)_q. \tag{7.21}$$

Here $\dim_q(j)$ is the quantum dimension (A.5), η is defined by (A.4), $(15j)_q$ stands for a (normalized) quantum (15j)-symbol (see the Appendix D) associate with any 4-simplex. This quantum (15j)-symbol is the Reshetikhin-Turaev evaluation of the graph Γ_h described in subsection (5.4). The symbols V and E stand for the number of vertices and edges in Δ correspondingly. This state sum model is independent of a triangulation of the 4-manifold used to compute (7.21). Thus, it gives an invariant associated with the manifold $\mathcal{M}$: $\mathrm{CY}(q,\Delta)=\mathrm{CY}(q,\mathcal{M})$.

The quantity $\mathrm{CY}(q,\mathcal{M})$ is conjectured to give the transition amplitude for BF theory with cosmological constant, the cosmological constant Λ being related to q by

$$q=\exp i\Lambda. \tag{7.22}$$

Thus, unlike in the case of 3-dimensions, we have now $\hbar=\Lambda$. This relation can be established as follows. It has been shown (see *e.g.* [54]) that if $\mathcal{M}$ is a manifold with boundary then $\mathrm{CY}(q,\mathcal{M})$ is proportional to the Chern-Simons transition amplitude $\mathrm{CS}(k,\partial M)$ introduced in the previous subsection:

$$\mathrm{CY}(q,\mathcal{M})\propto \mathrm{CS}(k,\partial\mathcal{M}). \tag{7.23}$$

The deformation parameter q of the Crane-Yetter model is related to the level of Chern-Simons theory as in (7.6). Consider now the transition amplitude for BF theory with cosmological term:

$$Z_4^{\mathrm{BF}}(\Lambda,\mathcal{M})=\int \mathcal{D}A\mathcal{D}E\exp\left(i\int_{\mathcal{M}}\mathrm{Tr}(E\wedge F)-\frac{\Lambda}{2}\mathrm{Tr}(E\wedge E)\right). \tag{7.24}$$

Let us integrate over the field E in this path integral. One gets

$$Z_4^{\mathrm{BF}}(\Lambda,\mathcal{M})\propto\int \mathcal{D}A\exp\left(\frac{i}{2\Lambda}\int_{\mathcal{M}}\mathrm{Tr}(F\wedge F)\right). \tag{7.25}$$

Now, using the fact that

$$\mathrm{Tr}(F\wedge F)=d(A\wedge dA+\frac{2}{3}A\wedge A\wedge A) \tag{7.26}$$

we get

$$Z_4^{\mathrm{BF}}(\Lambda,\mathcal{M})\propto\int \mathcal{D}A\exp\left(\frac{i}{2\Lambda}S_{\mathrm{CS}}(A,\partial\mathcal{M})\right). \tag{7.27}$$

This, together with (7.23), then implies that

$$\mathrm{CY}(q,\mathcal{M})\propto Z_4^{\mathrm{BF}}(\Lambda,\mathcal{M}), \tag{7.28}$$

where q is related to Λ as in (7.22).

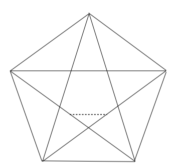
To compare the spin foam model we obtain below with the Crane-Yetter model we will need to know the first order term in the decomposition of $Z_4^{\mathrm{BF}}(\Lambda, \mathcal{M})$ in the power series in Λ. To find it, we have to take into account the fact that, similarly to the case of 3D BF theory, the transition amplitude is only proportional to the Crane-Yetter amplitude (7.21), with the proportionality coefficient being a function of $\hbar$. This proportionality coefficient is given by a power of $\hbar$. As one can check, the "zeroth" order term in $\hbar$ of (7.21) is equal to

$$\left(\frac{\hbar^3}{4\pi}\right)^{-V+E} O(\Delta), \tag{7.29}$$

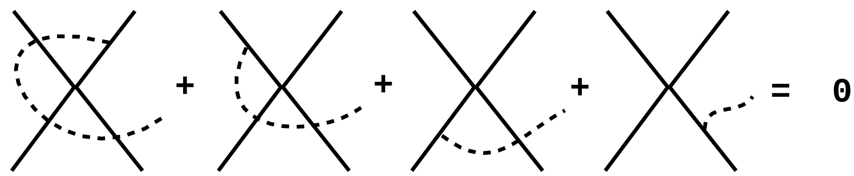
where $O(\Delta)$ is the Ooguri state sum invariant (4.3). Thus, the first order term in Λ of $Z_4^{\mathrm{BF}}(\Lambda, \mathcal{M})$ is given by

$$\left(\frac{\partial}{\partial \Lambda} Z_4^{\mathrm{BF}}(\Lambda, \mathcal{M})\right)_{\Lambda=0} = \left(\frac{\partial}{\partial \Lambda}\left(\frac{\mathrm{CY}(q, \mathcal{M})}{(\hbar^3/4\pi)^{-V+E}}\right)\right)_{\Lambda=0}. \tag{7.30}$$

However, unlike the case of 3D BF theory with cosmological term, in this case the only contribution to this expression comes from the quantum (15j)-symbol. All other "building blocks" that one encounters in (7.21) contain only the $\hbar^2 = \Lambda^2$ terms, and, thus, do not contribute to (7.30). The first derivative of the quantum (15j)-symbol with respect to $\hbar = \Lambda$ can be found by methods analogous to the ones used in the Appendix E to calculate the derivative of (6j)-symbol. However, in the case of (15j)-symbol, the calculation is much simpler due to the fact that only crossings contribute to the first order in $\hbar$. The (15j)-symbol used in (7.21) contains only one crossing (see, for example, [54]). Using the expression (E.2) for the R-matrix as a formal power series in $\hbar$ given in the Appendix E, one can easily check that the first $\hbar$-order term of the quantum (15j)-symbol is given by:

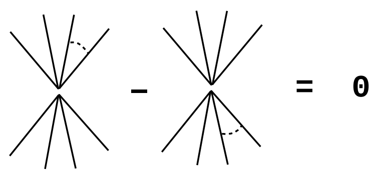
$\frac{i\hbar}{2}$  $+\cdots,$ (7.31)

where the dots stands for terms containing graspings between two edges sharing one vertex and correspond to different possible choices of the framing of the 15j-symbol. We will not keep track of these terms at this stage. Their relevance will be discussed below. This is, however, not quite the expression we want because it is not symmetrical. We will symmetrize it by putting the grasping at all pairs of lines of the graph in (7.31) that do not share a vertex. However, if one does that, than the quantity one obtains for each given 4-simplex h is not equal to (7.31). Only if one sums over all 4-simplices of the triangulation one obtains the quantity that is equal to the sum of (7.31) over h. This can be proved by using certain simple relations that hold for graspings. One such relation is the analogs of "closure" relations (B.4). It graphical representation is given by:



(7.32)

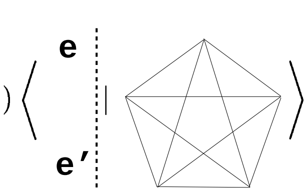
Such "closure" relation holds for every vertex of the graph in (7.31). Another relation that one has to use involves two different pentagon graphs. As one can convince oneself, the vertices of the pentagon graph in (7.31) are in one-to-one correspondence with the tetrahedra of Δ. Thus, every tetrahedron is shared exactly by two 4-simplices, and the following relations holds:



(7.33)

This allows one to cancel certain types of graspings from one 4-simplex with similar types of graspings from the neighboring 4-simplex. Certain cancellations occur for each tetrahedron of the triangulation. These relations allow one to obtain the following symmetric expression:

$$\left(\frac{\partial}{\partial\Lambda} Z_4^{\mathrm{BF}}(\Lambda, \mathcal{M})\right)_{\Lambda=0} = \frac{i}{2}\sum_h \frac{1}{30}\left(\sum_{e,e'} \mathrm{sign}(e,e')\left\langle \cdots \right\rangle\right)_0 + \cdots,$$



(7.34)

where the sum is taken over different pairs e, e' of edges that do not share a vertex. There are 30 terms in this sum, which explains the factor of $1/30$ in (7.34). The quantity $\mathrm{sign}(e, e')$ in (7.34) is plus or minus one depending on the orientation of the two edges e, e'. A consistent choice of orientations comes from the geometrical 4-simplex discussed in the Appendix B. With this choice, $\mathrm{sign}(e, e') = \mathrm{sign}(f, f')$, where we use the fact that every edge of the pentagon graph in (7.34) is in one-to-one correspondence with a face of the corresponding 4-simplex, and $\mathrm{sign}(f, f')$ is defined by the equation (B.5). The dots in (7.34) correspond to some symmetric expression that contains only terms with graspings between two edges sharing one vertex.

7.4 Reisenberger model

Reisenberger [50] has proposed a state sum model corresponding to the self-dual Plebanski action (6.11). The main idea of his construction is to modify the SU(2) Ooguri model in such a way that the constraints (6.12) are implemented. The model can be described as follows. As in the previous subsection, let $\mathcal{M}$ be a triangulated 4-manifold.

Let us associate a group element g_w to each wedge w of the dual triangulation (see Section 5). One can think of g_w as the holonomy of connection around the boundary of the wedge w with the basepoint being the center of the 4-simplex h to which the wedge belongs. Let D^i_w be a differential su(2) operator acting on functions of g_w as the sum of left and right invariant vector fields. Let us denote by Ω^{ij} the following operator acting in $(\mathrm{SU}(2))^{\otimes 10}$:

$$\Omega^{ij} = \sum_{w,w'} \epsilon(w,w') D^i_w D^j_{w'} - \frac{1}{3}\delta^{ij} \sum_{w,w'} \epsilon(w,w') D^k_w D_{w' k}, \tag{7.35}$$

where $\epsilon(w,w')$ is the sign of the four volume span by the wedges w, w' ($\epsilon(w,w') = 0$ if the two wedges don't span a four-volume). We choose a coloring of the wedges by spins j_w, denote by $R_j(g)$ the representation of g as an endomorphism of V_j, and define for each 4-simplex h the following function on $(\mathrm{SU}(2))^{\otimes 10}$:

$$\phi_{h,\vec{j}}(g_w) = tr_{\otimes_{w\in h} V_{j_w}} \left(\frac{1}{\sqrt{2\pi x}} e^{-\frac{1}{2x^2}\Omega^{ij}\Omega_{ij}} \otimes_{w\in h} V_{j_w} (2j_w+1) R_{j_w}(g_w) \right). \tag{7.36}$$

The state sum model proposed by Reisenberger is given by

$$\lim_{x\to 0} \sum_{j_w} \int \prod_w dg_w \prod_h \phi_{h,\vec{j}}(g_w) \tag{7.37}$$

The limit is a possible way of selecting only the states belonging to the kernel of Ω^{ij}.

7.5 Barrett-Crane model

Recently Barrett and Crane [12] proposed a model for four-dimensional quantum gravity based on the quantization of a geometric four-simplex. Similar ideas were pursued at the same time by Baez [9]. He independently found the solution of the so-called simplicity constraints (see below), but did not arrive at a spin foam model until the Barrett-Crane paper. The set of ideas behind the derivation of the Barrett-Crane model was already explained in Chapter 2, where a similar construction was presented for Ponzano-Regge model. Actually, the derivation of Ponzano-Regge model presented in section 2.2 is a one-dimension-down analog of the construction of Baez, Barrett and Crane.

The idea is to start from a geometric four-simplex, whose quantization will then give an expression for the quantum amplitude that must be associated with the simplex. Here we only give a sketch of the construction, for more details see [9, 12].

A geometric 4-simplex (see Fig. B, Appendix B), can be characterized by 4 vectors e^I_a. However, in 4d vectors are cannot be thought of as elements of the Lie algebra, and, thus, cannot be quantized in any natural way. Therefore, we instead choose to characterize the 4-simplex by the so-called bivectors $E^{IJ}_{ab} := e^{[I}_a e^{J]}_b$. Geometrically, the bivectors are associated with the faces of the simplex, and carry information about the orientation of the plane containing the face in $\mathbb{R}^4$, and about the area of the face. The problem is that not every bivector is of the form of the wedge product of two vectors, or, in other words, not every bivector is simple. One can, however, impose constraints which will allow only the simple bivectors. These are the so-called simplicity constraints. In four dimensions the simplicity constraints can be written in a nice form using the duality operation. Any bivector can be decomposed in its self-dual and anti-self-dual parts. Then the condition guaranteeing that the bivector is simple, i.e., is representable as the wedge product of two vectors, is simply that the norm of its self-dual part is equal to the norm of its anti-self-dual part. Understood in this way, the simplicity constraints are very easy to impose in the quantum theory, as we shall see below.

Thus, one can characterize a 4-simplex by 6 bivectors E^{IJ}_{ab}. It turns out to be more convenient, however, to introduce 4 more bivectors, corresponding to the other faces of the 4-simplex. One then gets 10 bivectors, corresponding to all 10 faces of the simplex. The bivectors are not linearly independent, but have to satisfy the so-called *closure relations* (B.4). As we have said, the bivectors have to satisfy in addition the simplicity constraints. It turns out, however, that there are even more constraints to impose if one wants a collection of 10 bivectors to determine a geometric 4-simplex. The other constraints are known as the *intersection constraints.* Each such constraint involves two bivectors, and, in the case bivectors are simple, guarantees that the two planes determined by them intersect along a line. In other words, this constraint guarantees that from a collection of vectors determined by simple bivectors, one is common to both bivectors. This constraint can be given the following nice form: one simply requires that the sum of bivectors is simple. One can then show that 10 bivectors satisfying the closure relations, the simplicity and intersection constraints determine a geometric 4-simplex.

To produce a quantum gravity model one then quantizes this collection of 10 bivectors by identifying them with Lie algebra elements of SO(4). It is very convenient to use the fact that SO(4) is isomorphic to $\mathrm{SU}(2)\times\mathrm{SU}(2)/Z_2$. Then the irreducible representations of SO(4) can be characterized by giving a pair of irreducible representations of SU(2), that is, by a pair of spins, with the condition that the sum of these spins is an integer. Thus, every face is associated the Hilbert space:

$$\sum_{(j_L,j_R)} V_L^{j_L}\otimes V_R^{j_R}, \tag{7.38}$$

where j_L, j_R are the spins determining the representations of the left and right SU(2), and V^j is the space of the irreducible representation of SU(2) of the dimension $2j+1$.

The above Hilbert space gives a quantization of a bivector, for the bivector becomes an operator in this space. One can now ask what is the quantum analog of the simplicity constraint, which each bivector has to satisfy. Recall that classically the simplicity constraint can be formulated as the requirement that the norm of the self-dual part of the bivector is equal to the norm of its anti-self-dual part. The norm of self- anti-self-dual parts has a very precise analog in the quantum theory: it is just the quadratic Casimir of the corresponding SU(2). Thus, the simplicity constraint has a precise quantum analog:

$$j_L = j_R. \tag{7.39}$$

Therefore, only such representations of SO(4) correspond to simple bivectors. Note that this is a true representation of SO(4) for the sum $j_L + j_R$ is always an integer in this case. Thus, it is easy to quantize simple bivectors, the corresponding Hilbert space is given by:

$$\mathcal{H} = \sum_j V_L^j \otimes V_R^j. \tag{7.40}$$

Having quantized the bivectors, or faces of the 4-simplex, one can ask what is the Hilbert space associated to a particular tetrahedron of this 4-simplex. Let us consider one of the 5 tetrahedra belonging to the 4-simplex. There are 4 faces making this tetrahedron, and the sum of bivectors corresponding to these faces must add up to zero, which expressed the closure constraint (B.4). In quantum theory this condition tells us that the Hilbert space associated to a tetrahedron is made of invariant elements of $\mathcal{H}^{\otimes 4}$. In other words, the Hilbert space of a tetrahedron is the space of all 4-valent intertwiners of simple representations of SO(4).

Note, however, that we still did not impose the intersection constraints. This constraint tell us that two faces that belong to a tetrahedron must intersect along a line. As we discussed before, this constraint can be expressed as the condition that the sum of the corresponding bivectors is a simple bivector. This condition can be expressed as an equation on 4-valent intertwiners allowed. It turns out that, given 4 simple representations, there is a unique solution. It was first discovered by Barrett and Crane [12] and then proved to be unique by Reisenberger [53]. Thus, finally, the Hilbert space corresponding to a tetrahedron is made of the Barrett-Crane intertwiners.

Having found the Hilbert space of a tetrahedron, we now have to glue 5 such Hilbert spaces to form the Hilbert space of a 4-simplex. There is a natural way to do this given by the pentagon spin network, constructed from the Barrett-Crane intertwiners. Thus, the Hilbert space of a 4-simplex is spanned by pentagon spin networks, with all representations labelling the edges being simple, and all 4-valent intertwiners being the Barrett-Crane intertwiners. Having specified 10 spins, this spin network gives one a number, whose interpretation is that of the quantum amplitude of a 4-simplex with areas of its 10 faces being specified.

Taking these simplex amplitudes, replacing with them the $(15j)$-symbols of the SO(4) Ooguri model, and summing only over simple representations of SO(4), one gets Barrett-Crane model of 4d quantum gravity.

Chapter 8

Applications of the generating functional technique

Having described some known state sum models, we are in the position to apply our method to various theories to obtain their spin foam models. Thus, in this chapter we use the generating functional computed in Chapter 5 to construct systematically state sum models of the theories described in Chapter 6. We then compare the results of this systematic approach with the known results described in Chapter 7.

8.1 2D Yang-Mills

Our general strategy is to calculate the 'interaction' term $\Phi(E)$ of the action (6.1) on the configuration of E that is distributional along the wedges of the dual complex

$$E = \sum_w E_w, \tag{8.1}$$

express the result as a polynomial function $\Phi(X)$ of the variables X_w, introduced in the previous section, and then look for the vacuum-vacuum transition amplitude of the theory as given by

$$\left(e^{i\Phi(-i\delta/\delta J)}\, Z[J,\Delta] \right)_{J=0}. \tag{8.2}$$

In the case of 2d Yang-Mills theory, the 'interaction' term $i\Phi$ is given by (see (7.1))

$$\frac{e^2}{2}\int_{\mathcal{M}} d\mu \operatorname{Tr}(E^2), \tag{8.3}$$

Calculating this on the configuration of E given by (8.1), with each E_w being constant along the wedge w, we get

$$\frac{e^2}{2}\sum_w \rho_w \operatorname{Tr}(X_w^2) = -e^2 \sum_w \rho_w X_w^i X_w^i, \tag{8.4}$$

where ρ_w is the area of the wedge w, as measured with respect to $d\mu$, and we have introduced the SO(3) indices (A.3). The partition function of the theory is then given by

$$\left(\exp\left(-e^2 \sum_w \rho_w \left(\frac{\delta}{\delta i J_w^i} \frac{\delta}{\delta i J_w^i} \right) \right) Z[J,\Delta] \right)_{J=0}. \tag{8.5}$$

To compare this with the partition function of the Migdal-Witten model, it is more convenient to take the generating functional $Z[J]$ in its general form (5.22). Then, using

$$e^{\alpha\left(\frac{\delta}{\delta iJ^i_w}\frac{\delta}{\delta iJ^i_w}\right)}\cdot P(J)R_{(j)}(e^J)|_{J=0} = e^{\alpha(j+1/2)^2} = e^{\alpha(c(j)/2+1/4)} \tag{8.6}$$

where $R_{(j)}(e^J)$ is the group element e^J taken in the j representation, and $c(j)$ is the quadratic casimir of this representation, we get for the partition function

$$\prod_\epsilon \int dg_\epsilon \sum_{j_\sigma} \prod_\sigma \dim_{j_\sigma} \chi_{j_\sigma}(g_{\epsilon_1}\cdots g_{\epsilon_n}) \exp\left(\sum_{w\in\sigma} -e^2\rho_w(c(j_\sigma)/2+1/4)\right). \tag{8.7}$$

This can be seen to be equal to

$$\mathrm{YM}(\rho e^2, \mathcal{M})e^{-e^2\rho/4} \tag{8.8}$$

by noting that

$$\sum_{w\in\sigma}\rho_w = \rho_\sigma, \qquad \text{and} \qquad \sum_\sigma \rho_\sigma = \rho.$$

Thus, our approach gives the correct expression (7.2) for the partition function of the Yang-Mills theory, apart from the factor $\exp(-e^2\rho/4)$. However, the later is what is called a "standard renormalization" of the partition function. That is, the result for the partition function of 2D Yang-Mills may depend on a regularization procedure that was used to calculate it, the two different schemes giving results that may differ, in particular, by factors of $\exp(-\alpha e^2\rho)$, where α is some coefficient. See, for instance, [62] for a more detailed discussion of the standard renormalizations. Thus, the partition function obtained by our method differs from that of the Migdal-Witten model just by a standard renormalization factor.

8.2 3D BF theory with cosmological term

As one can see from (6.6), the 'interaction' term $i\Phi(E)$ for this theory is given by

$$i\Phi(E) = -\frac{i\Lambda}{12}\int_{\mathcal{M}} \mathrm{Tr}(E\wedge E\wedge E). \tag{8.9}$$

As in the case of Yang-Mills theory in two dimensions, one has to find a polynomial function $\Phi(X)$ of variables X_w by evaluating (8.9) on the distributional field E given by (5.2). Unfortunately, in the case of three dimensions the result is not well-defined as in the case of 2D Yang-Mills theory. Indeed, the E field is concentrated along the wedges (see Fig. 7), and non-trivial contributions to the integral (8.9) come from the points where wedges intersect. Thus, the contributions to (8.9) come from the integrals

$$\int_{\mathcal{M}} \mathrm{Tr}(E_{w_1}\wedge E_{w_2}\wedge E_{w_3}), \tag{8.10}$$

where w_1, w_2, w_3 are three wedges that intersect. The wedges can intersect at points, as, for example, they do at the center of each tetrahedron, or they can have more general intersections, as, for example, an intersection of three wedges sharing a dual edge. Or, instead, one can have in (8.10) $w_1 = w_2$. Whatever the type of an intersection is, the result of (8.10) is ill-defined because of the distributional nature of E field. Here we show how the ambiguity in (8.10) can be resolved for the simplest, and most important type of intersections: intersections of three different wedges at the centers of tetrahedra. As we shall see, for intersections of this type there is a simple way to resolve the ambiguity in (8.10) using geometrical considerations. As we show, this type of intersection is the most important one, for it is responsible for, in certain sense, the most intricate contribution to the transition amplitude of the theory. Thus, in this section, we restrict our consideration only to this special type of intersection, calculate the corresponding interaction term $\Phi(X)$, find the corresponding spin foam model, and compare it with the Turaev-Viro model. In the first order in the decomposition of Turaev-Viro amplitude in power of Λ, we will be able to identify a term analogous to the term we obtain in our model and compare them. We will find a very delicate matching between the two, including the numerical coefficients. The importance of other types of intersections, not treated here, will be emphasized later.

Thus, the intersections we consider are the ones for which three different wedges intersect at the center of a tetrahedron. Given a tetrahedron t, there are 20 (without counting the permutations) different triples of wedges $w_1 \neq w_2 \neq w_3$. Four of these triples do not span a three-volume; thus, it is natural at first to take the integral (8.10) for such triples to be zero. These are exactly the triples of wedges that share a line – a part of the dual edge. Thus, there are only 16 different triples of wedges (without counting the permutations) that contribute to (8.10) for a given tetrahedron. Recall now that each E_w is given by (5.28). Thus, for any given triple of wedges $w_1 \neq w_2 \neq w_3$, the integral (8.10) is proportional to

$$i\Lambda \mathrm{Tr}(X_{w_1} X_{w_2} X_{w_3}), \tag{8.11}$$

but the proportionality coefficients are not fixed, because of the indeterminacy in the value of the integral

$$\int \delta(u_1)\delta(u_2)\delta(u_3) du_1 \wedge du_2 \wedge du_3. \tag{8.12}$$

Of course if the wedges were infinite planes without boundaries this integral is a well-defined quantity: it is just the intersection number and as such should be ± 1 depending on the orientations of the wedges. Because of the existence of boundary we expect this integral to be a number smaller than 1. We fix the proportionality coefficient by requiring that

$$\frac{1}{12}\int_t \mathrm{Tr}(E \wedge E \wedge E), \tag{8.13}$$

where the integral is taken over a tetrahedron t, is equal to the geometrical volume of t. For any triple of wedges $w_1 \neq w_2 \neq w_3$, the integral (8.12) is equal to $\alpha \cdot \mathrm{sign}(w_1, w_2, w_3)$, where α is a coefficient that is independent on a triple, and $\mathrm{sign}(w_1, w_2, w_3)$ is plus or minus one depending on the orientation of the form $du_1 \wedge du_2 \wedge du_3$. Thus, (8.13) is equal to

$$\frac{1}{12} \sum_{w_1 \neq w_2 \neq w_3 \in t} \alpha \cdot \mathrm{sign}(w_1, w_2, w_3) \mathrm{Tr}(X_{w_1} X_{w_2} X_{w_3}). \tag{8.14}$$

Here the sum is taken over triples of different wedges in t, taking into account all different permutations of w_1, w_2, w_3. Taking into account the fact that the number of permutations of w_1, w_2, w_3 is 6, we get for (8.13):

$$\frac{1}{2} \sum_{w_1 < w_2 < w_3} \alpha \mathrm{Tr}(X_{w_1} X_{w_2} X_{w_3}), \tag{8.15}$$

where the notation $w_1 < w_2 < w_3$ means that the sum is taken over 16 different triples $w_1 \neq w_2 \neq w_3$ such that $\mathrm{sign}(w_1, w_2, w_3) = 1$.

To relate (8.15) to the volume of tetrahedron t, we recall the geometrical interpretation of variables X_w. They were introduced earlier as the variables that carry information about the length of edges of the triangulation. At this stage it is more convenient to introduce SO(3) indices (A.3). Thus, each X_w is characterized by $X^i_w, i = 1, 2, 3$. Recall that wedges w are in one-to-one correspondence with edges e of the triangulation. Thus, let us view each X^i_w as the vector representing the corresponding edge (each edge can be viewed as a vector pointing from one vertex of Δ to another), and the norm squared $X^i_w X^i_w$ of X^i_w as the length squared of this vector. Indeed, it is not hard to check that the interpretation of $X^i_w X^i_w$ as the length of the corresponding edge is consistent with the other known facts. Let us consider the operator corresponding to $X^i_w X^i_w$. According to our general prescription, this quantity is represented by the operator $(\delta/\delta i J_w)^2$. We have already dealt with this operator in the previous section, see (8.6). Its eigenvalue is given just by the half of the Casimir (plus 1/4). Thus, in the sense of eigenvalues, we can write

$$X^i_w X^i_w = (j + 1/2)^2, \tag{8.16}$$

where j is the spin from (5.29) labelling the dual face that contains w. Thus, (8.16) tells us that, in the limit of large spins j, the norm of X_w grows as j. This is to be compared with the length spectrum of the canonical quantum theory

$$(\mathrm{length}) = \sqrt{j(j+1)}. \tag{8.17}$$

This also grows as j for large spins. The expression (8.17) can be easily derived in the context of canonical (loop) quantum gravity in three dimensions (note that we use units in which $8\pi G = 1$). Another motivation for interpreting the spin j as the length of an edge (for large j) is that it is exactly this interpretation that must be used to reproduce correctly the Regge calculus version of Einstein-Hilbert action in the Ponzano-Regge model of quantum gravity [48]. Thus, we learn that the interpretation of the norm of each vector X_w must be that of the length of the corresponding edge of Δ. Having this fixed we can relate (8.15) to the volume of tetrahedron t. We have

$$\mathrm{Tr}(X_{w_1}X_{w_2}X_{w_3}) = 2\epsilon_{ijk}X^i_{w_1}X^j_{w_2}X^k_{w_3}, \tag{8.18}$$

where we have introduced the SO(3) indices, see (A.3). Each X^i_w has the interpretation of the vector corresponding to one of the edges of t. Thus, (8.18) is equal to $12V$, where V is the volume of t. In (8.15) we have 16 such terms. Thus, it is equal to

$$\frac{1}{2}\alpha \cdot 16 \cdot 12V.$$

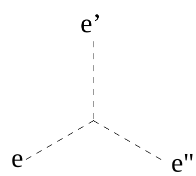
The requirement that (8.13) is equal to the volume of t fixes the parameter α to be $1/(16 \cdot 6)$. Thus, finally, we obtain the interaction term $i\Phi(X)$ to be

$$-\frac{i\Lambda}{6}\frac{1}{16}\sum_{w_1<w_2<w_3}\epsilon_{ijk}X^i_{w_1}X^j_{w_2}X^k_{w_3}, \tag{8.19}$$

where the sum is taken over 16 terms. To obtain the transition amplitude of the theory we have to replace each X^i_w by the operator $\delta/\delta i J^i_w$ and act by the exponential of (8.19) on the generating functional. Thus, the first order term in Λ in the decomposition of the transition amplitude is given by:

$$\left(\left(-\frac{i\Lambda}{24}\frac{1}{16}\sum_{w_1<w_2<w_3}4\epsilon_{ijk}\frac{\delta}{\delta i J^i_{w_1}}\frac{\delta}{\delta i J^j_{w_2}}\frac{\delta}{\delta i J^k_{w_3}}\right)Z_3(J,\Delta)\right)_{J=0}. \tag{8.20}$$

This is to be compared with (7.19). We will now show that the most complicated term in that expression – the term that involves trivalent graspings – exactly matches our result (8.20), including the numerical coefficient and the sign. To see this we just have to relate the trivalent grasping in (7.19) to the cubic operator in (8.20). As explained in the Appendix D, a single grasping in (7.19) acts by inserting $(\sigma^i/\sqrt{2})$, where σ^i are the Pauli matrices. The operator $(\delta/\delta i J^i)$, when applied only one time, acts by inserting just σ^i. Also, taking into account the definition of the trivalent grasping in (7.19), one can show that



$$= 4\epsilon_{ijk}\frac{\delta}{\delta i J^i_e}\frac{\delta}{\delta i J^j_{e'}}\frac{\delta}{\delta i J^k_{e''}} \tag{8.21}$$

So our approach *does* account for the most intricate part of the 3D transition amplitude in the first Λ-order, reproducing correctly even the numerical coefficients. Note, however, that we do not get all of the terms appearing in (7.19). The discrepancy we find between the model we obtain and the usual state sum model can be understood by comparing the two different expressions for the spacetime volume: the one obtained within our approach and the one obtained within the Turaev-Viro model, see (7.19). The Turaev-Viro model tells us that we must associate the spacetime volume not only to tetrahedra of the triangulation, but also to edges and vertices. Within our approach, only the tetrahedron part of the spacetime volume is accounted for. The reason for this was that, when evaluating the cosmological term of the action on the distributional E field, we took only the terms which came from intersections of different wedges at the centers of tetrahedra. As we discussed, even these terms are ill-defined because of the distributional nature of the E field. However, the corresponding ambiguities can be successfully resolved for these types of intersections by using geometrical considerations, and, after the ambiguities are eliminated, the result exactly matches the analogous terms of the usual state sum models, including the matching of the numerical factors. The geometrical considerations we used were exactly the ones that relate the terms we considered to the volume of a geometrical 3-simplex. Thus, it is not very surprising that only the "part" of the Turaev-Viro model that accounts for the volume of 3-simplices was reproduced correctly: as the geometrical considerations we used above tell us, we considered only the terms that are relevant for the volume of individual 3-simplices. However, there are other types of terms that we did not consider and that may be crucial to reproduce the corresponding state sum model correctly. Let us consider, for example, the terms of the type:

$$\int_t \mathrm{Tr}(E_{w_1} \wedge E_{w_1} \wedge E_{w_2}),$$

for some $w_1 \neq w_2$. The result of such an integral is ill-defined. Indeed, there is an indeterminacy of the type $0 \cdot \infty$, where 0 comes from $du_1 \wedge du_1$ (see (5.28)), and ∞ comes from the square of the δ-function $\delta^2(u_1)$. These terms are proportional to

$$\mathrm{Tr}(X_{w_1} X_{w_1} X_{w_2}).$$

Other terms that may arise are

$$\int_t \mathrm{Tr}(E_w \wedge E_w \wedge E_w),$$

for some w. They arise as the result of the indeterminacy $0^2 \cdot \infty^2$, where 0^2 comes from $du \wedge du \wedge du$ (see (5.28)), and ∞^2 comes from the cube of the δ-function $\delta^3(u)$. These terms are proportional to

$$\mathrm{Tr}(X_w X_w X_w).$$

There is no obvious reason to set these two types of terms to zero. In fact, as one can see, these are exactly the types of terms that are needed to account for the other terms appearing in the first Λ-order in the Turaev-Viro model. Although the structure of terms appearing this way is clear, at the present state of the development of our approach, no geometrical arguments is available to fix the ambiguity in the coefficients in front of such terms. However, the precise agreement of the terms for which such arguments do exist gives hope that, ones the above ambiguities are resolved, we will have an exact agreement between the models obtained using our procedure and the usual state sum models.

8.3 4D BF theory with cosmological term

In the case of 4D BF theory with cosmological term the interaction $i\Phi$ is given by:

$$i\Phi(E) = -\frac{i\Lambda}{2}\int_{\mathcal{M}} \mathrm{tr}(E\wedge E). \tag{8.22}$$

Our general prescription is to evaluate this interaction term on the distribution (5.3) and find a polynomial function $\Phi(X)$. Thus, one has to evaluate integrals

$$\int_{\mathcal{M}} \mathrm{Tr}(E_w \wedge E_{w'}) \tag{8.23}$$

with E_w given by (5.30). The integral (8.23) is non-zero only if wedges w, w' intersect. However, similarly to the case of 3D, the result when the wedges intersect is ill-defined because of the distributional nature of E_w. Thus, again some independent considerations have to be used to fix the ambiguity. We use a strategy similar to the one adopted in the case of 3D BF theory. We consider only the terms coming from wedges w, w' intersecting at the center of a 4-simplex. As we shall see, these are the most important terms in the sense that they are responsible for the main terms in the first order in Λ of the Crane-Yetter model. The relevance of other types of terms will be emphasized below. To fix the ambiguity in (8.23) when w, w' are two wedges that intersect at the center of a 4-simplex h we use geometrical considerations.

Let us consider the integral

$$-\frac{1}{2}\int_h \mathrm{Tr}(E\wedge E) \tag{8.24}$$

over the interior of a particular 4-simplex h. It is equal to the sum

$$-\frac{1}{2}\sum_{w,w'\in h}\int_h \mathrm{Tr}(E_w \wedge E'_w). \tag{8.25}$$

Each of the integrals here is proportional to

$$\mathrm{Tr}(X_w X_{w'}) = -2X^i_w X^i_{w'}, \tag{8.26}$$

with the proportionality coefficient given by:

$$-\frac{1}{2}\int_h \delta(u)\delta(v)\delta(u')\delta(v')du\wedge dv\wedge du'\wedge dv'. \tag{8.27}$$

The later is equal to $-(1/2)\alpha\cdot\mathrm{sign}(w,w')$, where α is a numerical parameter, whose value is not fixed due to the ambiguity referred to above, and $\mathrm{sign}(w,w')$ is the sign of the volume form in (8.27). Thus, (8.24) is equal to

$$\sum_{w,w'\in h}\alpha\cdot\mathrm{sign}(w,w')X^i_w X^i_{w'}, \tag{8.28}$$

where the sum is taken over wedges w, w' inside h that span a 4-volume. There are exactly 30 terms summed over in (8.28).

We will fix the parameter α relating the quantities X^i_w to the geometrical 4-simplex in $\mathbb{R}^4$. First, let us note that when E in (8.24) is equal to the self-dual part of the wedge product of two copies of the frame field:

$$E_{ab} = {}^{+}\Sigma_{ab}, \quad \Sigma^{IJ}_{ab} = \theta^I_{[a}\theta^J_{b]}, \tag{8.29}$$

where θ^I_a is a frame (tetrad) field, then (8.24) is equal to

$$\int_h \frac{1}{2}{}^{+}\Sigma^i_{ab}\frac{1}{2}{}^{+}\Sigma^i_{cd}\tilde{\epsilon}^{abcd} = \int_h \frac{1}{2}{}^{+}\Sigma^{IJ}_{ab}\frac{1}{2}{}^{+}\Sigma_{cd\,IJ}\tilde{\epsilon}^{abcd} = \tag{8.30}$$

$$\frac{1}{16}\int_h \epsilon_{IJKL}\Sigma^{IJ}_{ab}\Sigma^{KL}_{cd}\tilde{\epsilon}^{abcd} = \frac{4!}{16}V_h, \tag{8.31}$$

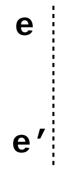
where V_h is the volume of h with respect to the metric defined by θ^I_a. Thus, we will fix α in such a way that (8.28) is equal to $(3/2)V_h$ when X^i_w can be related to the quantities characterizing a geometrical 4-simplex.

Recall that a geometrical 4-simplex in $\mathbb{R}^4$ is characterized (up to translations) by four vectors: the four vectors pointing from one of the vertices to the other four (for more details see the Appendix B). For each face of h one can also construct the so-called bivectors, which are given by the wedge products of any two of the edges belonging to that face. The bivectors live in the second exterior power of $\mathbb{R}^4$. One can take the self-dual part of a bivector to obtain an element of $\mathbb{R}^3$. As is shown in the Appendix B (see (B.10)), the norm (obtained using the usual flat metric in $\mathbb{R}^3$) of the self-dual part of each bivector is equal to the squared area of the corresponding face. There exists also a simple expression (B.11) for the volume of a 4-simplex involving only the self-dual parts of bivectors. Recall now that wedges $w \in h$ are in one-to-one correspondence with faces of h. Then, if X^i_w has the interpretation of the self-dual part of the bivector corresponding to a face of h.

A comparison of (B.11), (8.30) fixes the value of α to be $\alpha = 1/30 \cdot 4$. Thus, the first Λ-order term of the transition amplitude is given by:

$$\left(\Lambda\frac{1}{30}\sum_{w,w'\in h}\frac{1}{4}\mathrm{sign}(w,w')\frac{\delta}{\delta iJ^i_w}\frac{\delta}{\delta iJ^i_{w'}}Z(J)\right)_{J=0} \tag{8.32}$$

To compare this with (7.34) we need the relation between the grasping there and the operator in (8.32). The corresponding relation is given by



$$= \frac{1}{2} \frac{\delta}{\delta i J^i_w} \frac{\delta}{\delta i J^i_{w'}} \tag{8.33}$$

Using this correspondence we see that (8.32) agrees with the result (7.34). As in the 3D case, the matching includes the numerical coefficient and the sign of the expressions.

Let us now discuss the role of the terms denoted by dots in (7.34). Let us recall that those terms are determined by a framing of the (15j)-symbol used in the Crane-Yetter model. These terms are given by a sum of graspings of edges of the graph Γ_h that share a vertex. Recall that vertices of Γ_h are in one-to-one correspondence with the tetrahedra of h. Thus, the general structure of these terms is such that they can be grouped according to a 3-simplex (tetrahedron) to which they "belong". Therefore, these terms can be thought of as the contribution to the 4-volume of the BF theory coming from individual tetrahedra of Δ. Thus, these terms are in certain sense analogous to the terms in 3D that carry 3-volume corresponding to the edges of Δ. As in 3D our approach did not reproduce correctly the terms corresponding to edges, in 4D we did not reproduce correctly the contribution to the 4-volume coming from tetrahedra. Similarly to the case of 3D, these terms may possibly be reproduced if one takes into account the types of intersections in (8.23) other than the ones considered here.

8.4 4D self-dual gravity

In order to apply our strategy to the self-dual Plebanski model let us consider the "interaction" term of the action (6.11):

$$i\Phi(E) = -i \int_{\mathcal{M}} \psi_{ij} \left(E^i \wedge E^j - \frac{1}{3}\delta^{ij} E^k E_k \right). \tag{8.34}$$

Using the same procedure as in the case of the 4-dimensional BF theory with cosmological constant, this term, when evaluated on distributional E field, gives

$$-i \sum_h \psi_{ij}(v_h) \Omega^{ij}_h(X) \tag{8.35}$$

where v_h denotes the center of the 4-simplex and

$$\Omega^{ij}_h(X) = \sum_{w,w' \in h} \operatorname{sign}(w,w') \left(X^i_w X^j_{w'} - \frac{1}{3}\delta^{ij} X^k_w X_{w'\,k} \right) \tag{8.36}$$

is a quadratic function in the wedge variables X_w. In order to write (8.35) we have absorbed some unimportant numerical constants into the Lagrange multipliers ψ_{ij}. The generating functional approach tells us that the spin foam transition amplitude of the self dual Plebanski model is given by $\tilde{Z}(0)$ where

$$\tilde{Z}(J) = \int \prod_h d\psi^{ij}(v_h)\, e^{-i\sum_h \psi_{ij}(v_h)\Omega_h^{ij}(\frac{\delta}{i\delta J})} \cdot Z(J). \tag{8.37}$$

The integration over ψ^{ij} gives rise to delta functions. Thus, $\tilde{Z}(J)$ can be characterized as the solution of

$$\Omega_h^{ij}(\frac{\delta}{i\delta J})\tilde{Z}(J) = 0, \tag{8.38}$$

for all 4-simplices h, or, equivalently, as the solution of

$$\Omega_h^{i_1 j_1}(\frac{\delta}{i\delta J}) \cdots \Omega_h^{i_n j_n}(\frac{\delta}{i\delta J})\tilde{Z}(J)|_{J=0} = 0. \tag{8.39}$$

Using the fact that $Z(J)$ and, therefore, $\tilde{Z}(J)$ are gauge invariant and the property that an $su(2)$ symmetric traceless tensor Ω^{ij} is totally characterized up to gauge transformation by its square $\Omega^{ij}\Omega_{ij}$ we see that the preceding equation is equivalent to

$$(\Omega_h^{ij}\Omega_{ij\,h})^n(\frac{\delta}{i\delta J}) \cdot \tilde{Z}(J)|_{J=0} = 0. \tag{8.40}$$

The solution of the later can be written as

$$\tilde{Z}(0) = \lim_{x\to 0}\prod_h \frac{1}{\sqrt{2\pi x}} e^{-\frac{1}{2x^2}\Omega_h^{ij}\Omega_{ij\,h}(\frac{\delta}{i\delta J})} \cdot Z(J)|_{J=0} \tag{8.41}$$

In this form the result is very similar to the one given by the Reisenberger model. There are however several differences. The first one is the fact that the derivative operators appearing in our result are the commutative ones, while Reisenberger uses the non-commutative right and left invariant vector fields. The second difference is due to the presence of the factors of $P(J)$ in the generating functional.

8.5 Higher-dimensional gravity

Similarly to the case of the self-dual model considered in the previous section, in the case of higher-dimensional Palatini gravity the "interaction" term in the action must be treated not as a perturbation of BF theory, but as giving constraints that are to be imposed. Thus, we have to take the generating functional and impose all the constraints discussed in section 6.5. In general it is not clear what it means "to impose constraints on the generating functional". Indeed, it either satisfies the corresponding differential equations or does not. However, in our case, because we understand the geometrical meaning of the constraints, the situation is different. Because the generating functional has the form of the sum over representations of the product of simplex amplitudes, one can simply pick up those terms from this sum, which satisfy the constraints. The result has a very natural geometrical interpretation in terms of quantization of a geometric simplex, along the lines of section 7.5. The results of this section are based on Ref. [31].

First of all, one has to find the "discrete" analogs of the constraints of section 6.5. The result is as follows. There are several types of constraints that one has to impose. First, there are the so-called closure constraints. These arise because one is considering a set of Lie algebra two-forms E_{ij}, one for each $(D-2)$-simplex, that are obtained by integrating the E field, which is a $(D-2)$-form, over these $(D-2)$-simplices, and there are linear dependences between E_{ij} obtained this way. It is straightforward to solve the differential equations corresponding to these constraints for they simply require the simplex amplitude to be gauge invariant.

Second, there are simplicity constraints for each $(D-2)$-simplex, or for each two-form E_{ij}, which require this two-form to be simple. This constraints can also be solved in quantum theory. They imply that only a part of the space $L^2(G)$ is relevant. This relevant part can be written as a direct sum over special representations that can be called *simple representations*. We shall describe these representations below.

Third, there are analogs of intersection constraints. In the quantum theory these constraints appear as constraints on intertwiners. We describe solutions to these constraints below.

Finally, there is a problem of imposing analogs of normalization constraints. These turn out to be the most non-trivial ones. At this stage, we can say very little about these constraints. They, however, may turn out to be very important physically, as we comment on below.

Let us first describe how the simplicity constraints are solved. Let us denote the basis of the Lie algebra of $\mathrm{SO}(D)$ by X_{ij}, $i,j \in \{1,\ldots,D\}$. Our general framework tells us that the two-forms E_{ij} must be promoted to derivative operators acting on the generating functional. Since Lie algebra is generated by derivatives (vector fields) on the group, this means that E_{ij} is promoted in the quantum theory to an element X_{ij} of the Lie algebra of $\mathrm{SO}(D)$. Then the quantum analog of the simplicity constraints is given by:

$$X_{[ij}X_{kl]} = 0,\ \forall i,j,k,l \in \{1,\ldots,D\}, \tag{8.42}$$

where $[ijkl]$ means that we consider the total anti-symmetrization on these indices. It turns out that, among all irreducible representations of $\mathrm{SO}(D)$, there is a special subset of representations satisfying (8.42). These representations are related to harmonic analysis on the sphere S^{D-1}. Some of their properties are reviewed in Appendix F. The result of [31] shows that the spherical harmonics representations described in the appendix are the only irreducible simple representations.

Thus, it is not hard to solve the quantum analogs of the simplicity constraints: one simply has to restrict oneself to simple representations of $\mathrm{SO}(D)$. Let us now discuss the quantum version of the intersection constraints. As we mentioned above, these become equations on intertwiners. The simple representations can be realized in the space of homogeneous harmonic polynomials of a fixed degree. Let us first write down the quantum analog of the intersection constraints. Consider a given spin network. Let $e_1,\ldots,e_n$ be the incoming edges and $e'_1,\ldots,e'_p$ be the outgoing edges at the vertex v. Then $\mathcal{H}^{(D)}_{N_{e_i}}, \mathcal{H}^{(D)}_{N_{e'_j}}$ are simple representations associated with edges meeting at v. An intertwiner from the tensor product of incoming simple representations to the product of outgoing ones is given by a multi-linear map

$$I(P_1, \cdots, P_n, \bar{Q}_1, \cdots, \bar{Q}_p), \tag{8.43}$$

where P_i (Q_j respectively) are harmonic homogeneous polynomials of degree N_{e_i} ($N_{e'_j}$ respectively), and $\bar{Q}$ denotes the complex conjugate of Q. The intertwining property reads

$$I(P_1, \cdots, P_n, \bar{Q}_1, \cdots, \bar{Q}_p) = I(g \cdot P_1, \cdots, g \cdot P_n, g \cdot \bar{Q}_1, \cdots, g \cdot \bar{Q}_p), \tag{8.44}$$

and an example of the intersection constraint is given by:

$$I(X_{[ij} \cdot P_1, X_{kl]} \cdot P_2, \cdots, P_n, \bar{Q}_1, \cdots, \bar{Q}_p) = 0 \tag{8.45}$$

There exists a very simple and beautiful solution of these constraints. This solution was discovered for the case $D = 4$ by [12]. However, in that work, it was written in a rather cumbersome way as a sum over a product of intertwiners of $SU(2)$. Moreover, the proof that the intertwiner satisfies the intersection constraints used heavily the fact that the universal covering of $SO(4)$ can be written as the product $SU(2) \times SU(2)$. This uses the duality available in $D = 4$, which makes this dimension very special. Thus, it was not at all clear that this solution could be generalized to higher dimensions, where there is no notion of duality. The solution that was given in [31] shows that the central notion, allowing the construction to work, is not self-duality, but the fact that simple representations are realized in the space of polynomials on the sphere S^{D-1}.

Let P_i (Q_j respectively) be harmonic homogeneous polynomial of degree N_{e_i} ($N_{e'_j}$ respectively) and consider the following intertwiner between

$$\otimes_{i=1}^{n} \mathcal{H}_{N_{e_i}}^{(D)} \text{ and } \otimes_{j=1}^{p} \mathcal{H}_{N_{e'_j}}^{(D)}$$

given by

$$I_{n,p}(P_1, \cdots, P_n, \bar{Q}_1, \cdots, \bar{Q}_p) = \int_{S^{D-1}} d\Omega(x)\, P_1(x) \cdots P_n(x) \bar{Q}_1(x) \cdots \bar{Q}_p(x). \tag{8.46}$$

Here $d\Omega$ denotes the invariant measure on the unit sphere S^{D-1}. For definiteness we choose the normalization of this measure such that

$$\int_{S^{D-1}} d\Omega(x) = 1$$

The fact that $I_{n,p}$ is an invariant intertwiner can be easily seen using the invariance of the measure, integration by parts and the Leibniz rule. This is, in a sense, the simplest possible intertwiner one can imagine. Note that the above intertwiner in the case $D = 3$ is the usual intertwiner of SO(3) that is constructed from Clebsch-Gordan coefficients.

It is remarkable that such a simple entity gives a simple intertwiner, as it was shown in [31].

Thus, we found the quantum analogs and solved all the constraints but the normalization constraints. Taking the spin networks build of the simple representations, and using the intertwiners described below, and interpreting these spin networks as giving amplitudes for a quantum D-simplices of a fixed triangulation, one obtains spin foam models of D-dimensional quantum gravity. These models generalize the Barrett-Crane model of 4D gravity described above.

We must make a cautionary remark on the intersection constraints, however. In the way we treated these constraints, we considered each D-simplex independently of all other D-simplices of the triangulation. We then concentrated on (D–2)-faces of each D-simplex and required them to intersect properly by imposing the intersection constraints. Note, however, that a pair of intersecting (D–2)-faces belongs not just to a single D-simplex. In fact, it belongs exactly to two D-simplices that share a (D–1)-simplex containing these (D–2)-faces. The fact that the intersection constraints were imposed independently for each D-simplex means that, for each pair of intersecting (D–2)-faces, the intersection constraint was imposed *twice* in our model, instead of once as it should be. The same applies not only to our higher-dimensional generalizations, but also to the Barrett-Crane model. This might mean that, when a D-simplex is viewed as an element of a triangulations of spacetime, one must look for some other way to impose the intersection constraints, which would avoid imposing them twice. For more discussion on this see [25].

We conclude this section by remarking on a possible role played by the normalization constraints. As a preliminary analysis shows, there are no local solutions to these constraints, in the sense that no modification of a simplex amplitude solves them. Instead, one must look for solutions in the form of linear combinations of simplex amplitudes for different simplices of the triangulation. This, while at first seems problematic, may in fact be a blessing, for these non-local solutions may be related to the gravitational waves. The fact that normalization constraints only appear in four spacetime dimensions is consistent with the fact that the are no local degrees of freedom in dimensions smaller than four.

Chapter 9

Geometry and representation theory: higher dimensions

In the previous chapter we have checked our procedure against known state sum models and found a good agreement. We then applied it to higher-dimensional gravity and obtained a spin foam model for the later. This spin foam model is a generalization of Ponzano-Regge model for 2+1 gravity. We derived it using a definition of the path integral for the theory. In this section we show how the resulting model can be understood as realizing a relation between group representation theory and geometry. As we shall see, there is a natural generalization of the results we have obtained for 2+1 gravity in chapter 2 to higher dimensions. In this chapter we will encounter the relation between simple representations of $\mathrm{SO}(D)$ and Euclidean geometry of $\mathbb{R}^D$. As we shall see, the spin foam model derived in the previous chapter realizes this relation in an explicit fashion.

As we saw in the previous chapter, the quantum amplitude for a simplex is given by the value of a spin network, whose edges are labelled by simple representations, and whose intertwiners are the simple intertwiners in the sense described in section 8.5. As we saw in chapter 2, an analogous construction in 2+1 gives an amplitude that is related to the classical Regge action. In this chapter we show that the same holds in higher dimensions. To show this, it turns out to be very convenient to realize spin networks as Feynman graphs of a special sort. The general construction that realizes spin networks as Feynman graphs is presented in the following section. Section 9.2 gives details of this construction for the case of simple $\mathrm{SO}(D)$ spin networks. Section 9.3 proves the asymptotic formula for the simplex amplitude. It is in this section that we shall see the Regge action arising. Thus, this completes the circle: having started with the formula for the (6j)-symbol asymptotics, and being led by it to the idea of spin foam models, we introduced spin foam models of higher-dimensional gravity; we will now see how the asymptotics formula for the symplex amplitude in higher dimensions yeilds the Regge action.

9.1 Spin networks as Feynman graphs

Before we present the construction interpreting simple spin networks as Feynman graphs, we will need the standard notion of a *representation of class 1* (see, e.g., [61]). Spin networks that can be represented as Feynman graphs are the ones constructed using only these special representations of G.

DEFINITION 1. *Let ρ be an irreducible representation of G, and let H be a subgroup of G. If the representation space V^ρ contains vectors invariant under H, and if all operators $U^\rho(h), h \in H$ are unitary, then ρ is called a representation of class 1 with respect to H.*

The significance of these representations comes from the fact that they can be realized in the space of functions on the homogeneous space $H\backslash G$. As we describe below, spin networks that are constructed using only representations of class 1 with respect to H can be viewed as Feynman graphs on $H\backslash G$. Simple SO(D) spin networks of section 8.5 are just such spin networks. In this case $H = \mathrm{SO}(D-1)$ and $H\backslash G = S^{D-1}$.

The realization of a representation of class 1 in the space of functions on a homogeneous space $H\backslash G$ is a particular case of a general description of an irreducible representation ρ by shift operators in the space of functions on the group. Let us remind the reader this description. Consider matrix elements

$$U^\rho_{\mathbf{x},\mathbf{a}}(g) := (U^\rho(g)\mathbf{x}, \mathbf{a}),$$

where $\mathbf{x}, \mathbf{a}$ are vectors from the representation space V^ρ. Let us fix $\mathbf{a}$. Then the functions $U^\rho_{\mathbf{x},\mathbf{a}}(g), \mathbf{x} \in V^\rho$ span a subspace in the space $L^2(G)$ of square integrable functions on the group. One can then show that the right regular action of the group G on this subspace gives an irreducible representation equivalent to ρ. The scalar product in the representation space is then given by the integral over the group. In the case ρ is a representation of class 1 with respect to H, and a is a vector invariant under H, the functions $U^\rho_{\mathbf{x},\mathbf{a}}(g)$ are constant on the right cosets Hg and can be regarded as functions on the homogeneous space $X = H\backslash G$. The scalar product is then given by an integral over X.

We are now ready to describe spin networks constructed from representations of class 1 with respect to H as Feynman graphs on X. Let us denote by $P^{(\rho)n}(x), x \in X$ an orthonormal basis in the representation space ρ realized in the space of functions on X. The matrix elements of the group operators are then given by:

$$U^\rho(g)^n_m = \int_X dx\, \overline{P^{(\rho)}_m(x)} P^{(\rho)n}(xg),$$

where dx is the invariant normalized measure on X. This gives realization of the matrix elements as integrals over X. The other building block necessary to construct a spin network is an intertwiner. Intertwiners can be characterized by their integral kernels. For a k-valent vertex one defines the integral kernel $I_v(x_1, \ldots, x_k)$ so that:

$$I_{v\, m_1 \ldots m_i}^{\;\; n_{i+1} \ldots n_k} = \int_X dx_1 \cdots dx_k\, I_v(x_1, \ldots, x_k)$$
$$\overline{P^{(\rho_1)}_{m_1}(x_1)} \cdots \overline{P^{(\rho_i)}_{m_i}(x_i)} P^{(\rho_{i+1})n_{i+1}}(x_{i+1}) \cdots P^{(\rho_k)n_k}(x_k).$$

The integral kernels $I_v(x_1, \ldots, x_k)$ must satisfy the invariance property

$$I_v(x_1 g, \ldots, x_k g) = I_v(x_1, \ldots, x_k).$$

A special important set of intertwiners is given by:

$$\tilde{I}_v(x_1, \ldots, x_k) = \int_X dx\, \delta(x, x_1) \cdots \delta(x, x_k)$$

or

$$\tilde{I}_{v\, m_1 \dots m_i}^{\ n_{i+1} \dots n_k} = \int_X dx\, \overline{P_{m_1}^{(\rho_1)}(x)} \cdots \overline{P_{m_i}^{(\rho_i)}(x)} P^{(\rho_{i+1}) n_{i+1}}(x) \cdots P^{(\rho_k) n_k}(x).$$

These special intertwiners are the ones that appear in simple spin networks, as we described in section 8.5. Exactly for such intertwiners it is possible to represent the spin network evaluation as a Feynman graph. Let us now introduce what can be called Green's function:

$$G^{(\rho)}(x, y) := \sum_n \overline{P_n^{(\rho)}(x)} P^{(\rho) n}(y).$$

This Green's function satisfies the "propagator" property:

$$\int_X dz\, G^{(\rho)}(x, z) G^{(\rho)}(z, y) = G^{(\rho)}(x, y).$$

Let us also introduce a propagator "in the presence of a source":

$$G^{(\rho)}(x, y; g) := \int_X dz\, G^{(\rho)}(x, z) G^{(\rho)}(zg, y).$$

It is clear that $G^{(\rho)}(x, y; e) = G^{(\rho)}(x, y)$, where e is the identity element of the group. One can now check that, in the case all spin network intertwiners are of a special type $\tilde{I}_v$ described above, the spin network function $\phi_{(\Gamma,\rho,\tilde{I})}$ of the group elements $g_1, \dots, g_E$ is given by the Feynman graph with the following set of Feynman rules:

- With every edge e of the graph Γ associate a propagator $G^{(\rho_e)}(x, x'; g_e)$.
- Take a product of all these data and integrate over one copy of X for each vertex.

These rules can be summarized by the following formula:

$$\phi_{(\Gamma,\rho,\tilde{I})}(g_1, \dots, g_E) = \prod_v \int_X dx_v \prod_e G^{(\rho_e)}(x, x'; g_e). \tag{9.1}$$

Thus, in the case intertwiners are given by $\tilde{I}$ the evaluation of a spin network on a string of group elements is given by a Feynman graph: one associates the Green's function to every edge and integrates over the positions of vertices.

Before we illustrate this general construction on the example of simple SO(D) spin networks, let us note that this construction can be readily generalized to the case of an arbitrary spin network. Indeed, the restriction of representations labelling the spin network to be those of class 1 with respect to a fixed subgroup H was necessary only to guarantee that the resulting Feynman graph lives in the homogeneous space $X = H\backslash G$. It can be dropped at the expense of Feynman graphs becoming graphs in the group manifold. The restriction of intertwiners to be of a special type $\tilde{I}_v$ can be dropped with the result that the set of Feynman rules specified above changes: in this case one has to associate with every vertex the integral kernel $I_v(x_1, \dots, x_k)$ and then integrate over all the arguments. Thus, in the case of arbitrary intertwiners, the evaluation formula takes the form:

$$\phi_{(\Gamma,\rho,I)}(g_1,\dots,g_E) = \prod_v \int_X d\mathbf{x}_v\, I_v(\mathbf{x}_v) \prod_e G^{(\rho_e)}(x,x';g_e). \tag{9.2}$$

Here $\mathbf{x}_v$ stands for a string of arguments $x_1,\dots,x_k$ of a k-valent intertwiner, and x,x' in the argument of the Green's function $G^{(\rho_e)}(x,x';g_e)$ must be the same as those in two intertwiners: x must the appropriate argument in $I_v, e\in S(v)$ and x' must be the argument of $I(w), e\in T(w)$.

9.2 Simple SO(D) spin networks

In this section we illustrate the general construction presented above on the example of simple SO(D) spin networks. Their relevance to quantum gravity in D dimensions was explained in section 8.5.

Simple SO(D) spin networks are the ones constructed from special representations of SO(D). As is well-known, group SO(D) has a special class of representations, called spherical harmonics, that appear in the decomposition of the space of functions $L^2(S^{D-1})$ on S^{D-1} into irreducible components. Some properties of these representations are described in Appendix F. Using the terminology introduced in Sec. 9.1 these representations of SO(D) can be described as representations of class 1 with respect to SO($D-1$). They are characterized by a single parameter that we will denote by N in what follows; N is required to be an integer. These are the representations that were called simple in section 8.5. A simple SO(D) spin network was defined in 8.5 as a spin network which is constructed only from simple representations and whose intertwiners are the special intertwiners $\tilde{I}$ introduced in Sec. 9.1.

In the case intertwiners are given by $\tilde{I}$, the value of a simple spin network on a sequence of group elements can be evaluated using the general formula (9.1). In what follows we will be concerned only with a special case of spin network evaluated on all group elements being equal to the identity element. This "evaluation" of a spin network gives a number that depends only on the graph and on the labelling of its edges by integers N_e. Evaluation of a spin network is of special importance for quantum gravity because this is the way to obtain an amplitude for a spacetime simplex, see section 8.5. Thus, according to our Feynman graph formula (9.1), the evaluation of a simple spin network is given by

$$\phi_{(\Gamma,\rho)} = \prod_v \int_{S^{D-1}} dx_v \prod_e G_{N_e}(x,x'). \tag{9.3}$$

Here

$$G_N(x,y) = \sum_K \overline{\chi_K(x)}\chi^K(y), \tag{9.4}$$

where we have introduced an orthonormal basis χ^K, $K=(k_1,\cdots,k_{D-2})$ $N\geq k_1\cdots\geq k_{D-3}\geq|k_{D-2}|$ in the representation space (see Appendix F for a construction of such a basis). The invariance property $G_N(xg,yg)=G_N(x,y)$ implies that $G_N(x,y)$ depends only on the scalar product $(x \cdot y)$, and it is a standard result [61] that

$$G_N^{(D)}(x,y) = \frac{D+2N-2}{D-2} C_N^{(D-2)/2}(x \cdot y), \tag{9.5}$$

where C_N^p is the Gegenbauer polynomial, see Appendix G for the definition. The expression (9.3) for the evaluation of a simple spin network is a generalization of the result [13] for the evaluation in $D = 4$.

9.3 Asymptotics of the simplex amplitude

In this section we use the Feynman graph representation of the simple spin networks to study the asymptotics of a D-simplex amplitude for large N. The results of this section generalize those of [14] to the case of arbitrary dimension. Most of the labor necessary to get the asymptotics is done in Appendix H. Here we simple use the asymptotics (H.1) of the Gegenbauer polynomial obtained there.

As is explained in section 8.5, the amplitude for a D-simplex is given by the evaluation of the spin network that is dual to the boundary of the simplex. The $(D-2)$-simplices are labelled by simple representations of $\mathrm{SO}(D)$, i.e., by integers N. The edges of the spin network dual to the boundary of the simplex are in one-to-one correspondence with the $(D-2)$-simplices, and inherit the labels of $(D-2)$-simplices. As one can easily check, all vertices of the spin network in question are D-valent. All intertwiners are of the special type described in Sec. 9.1, and, thus, the formula (9.3) can be used for the evaluation. Using the asymptotics (H.1) and the formula (9.3) we present an asymptotic evaluation the amplitude: we will use the stationary phase approximation for the integral. Our discussion follows closely that of [14].

To get a feeling about the behavior of the amplitude, we will concentrate only on the oscillatory part of $C_N^p(\cos\theta)$. Thus, dropping all multiplicative constants, which are unimportant for us, we get

$$\phi_{(\Gamma,\rho)} \sim \sum_{\{\epsilon_{kl}\}} \left(\prod_{k<l} \epsilon_{kl} \right) \int_{S^{D-1}} dx_1 \cdots dx_{D+1}\, e^{i \sum_{k<l} \epsilon_{kl}((N_{kl}+p)\theta_{kl} + (1-p)\pi/2)},$$

where the integral is taken over $(D+1)$ points – vertices of the spin network – on the unit $(D-1)$-sphere, and k, l are indices labelling the vertices $k, l = 1, \ldots, D+1$. Thus, a pair kl labels a spin network edge, and $\theta_{kl} : \cos\theta_{kl} = x_k \cdot x_l$. The quantity ϵ_{kl} takes values ± 1 and the sum is taken over both possibilities for every edge. The rest of the analysis is exactly the same as in [14]. Taking into account the fact that the variation of the angles satisfy the following identity (see [14]):

$$\sum_{k<l} V_{kl} \delta\theta_{kl} = 0,$$

where V_{kl} are the volumes of $(D-2)$-simplices inside a geometric D-simplex, one finds that all ϵ_{kl} are either positive or negative, and that the stationary phase values of θ_{kl} are the ones corresponding to a geometric D-simplex determined by $N_{kl} + p$ interpreted as volumes of $(D-2)$-simplices. Then, in the case the number $D(D+1)/2$ of edges in the simplex is even, we get

$$\phi_{(\Gamma,\rho)} \sim \cos\left(\sum_{k<l}(N_{kl}+p)\theta_{kl} + \kappa\frac{\pi}{4}\right), \tag{9.6}$$

where θ_{kl} are the higher-dimensional analogs of the dihedral angles of the geometric D-simplex determined by $N_{kl}+p$ and

$$\kappa = \frac{(D+1)D}{2}(4-D)$$

is the integer determined by D. In the case $D(D+1)/2$ is odd one gets 'sin' instead of 'cos' in the asymptotics (9.6). Thus, the simplex amplitude has the asymptotics of the exponential of the Regge action, as expected.

Chapter 10

Conclusions

Thus, we have finished our journey. Having started with 2+1 gravity, and motivated by the relation between group representation theory and geometry that was suggested by Ponzano-Regge model, we have introduced the idea of spin foam models, and derived models for various theories. For some theories the model we found is just the model known to be the "correct" one for that theory. For other theories our method was able to reproduce the known "correct" models to certain degree, as, for example, in the case of BF theories with the cosmological constant we were able to see how the first order terms in Λ arise. For other theories, like higher-dimensional gravity, the models we found are new. The higher-dimensional models we found are in a precise sense generalizations of Ponzano-Regge model in 2+1 dimensions. Studying the asymptotics of the simplex amplitude for these models, we encountered the relation between representation theory and geometry, similar to the one that we had in 2+1 dimensions. On the way, we even learned something new about 2+1 gravity, for the Feynman graph representation of spin networks developed in the previous chapter gives one a new derivation of the $(6j)$-symbol asymptotics formula.

There are other recent developments on the subject of spin foam models that we did not discuss in this work. One of such developments is the recent analysis [15, 33] of the theories of metrics of Lorentizn signature. This is the physically relevant case, and all the developments presented in this thesis must be taken only as preliminaries for the Lorentian case. However, as the results [15, 33] show, the Lorentian case can be sucessfully analyzed using the same methods as the ones defeloped here. The structures one encounters are even more beautiful and exciting, due to the rich character of the Lorentizn geometry. The Lorentian case is the subject of an active study now.

These developments are exciting, but they by no means solve all the problems of quantum gravity. This is only the beginning of the attempt to understand the problem of quantum gravity as the problem of quantization of geometry. The models we found show that this idea can be realized. However, there were many problems on the way that we were not able to solve. Some of these problems are technical, while others are conceptual. Both will probably require a collective effort of many researchers in the field.

One of the main conceptual problems that was left unsolved, both in the case of Euclidean and Lorentzian signatures, is the problem of interpretation of the amplitude for a particular fixed triangulation. Recall that all spin foam models considered give an amplitude to a fixed triangulation of the spacetime manifold. In the case of 2+1 gravity, this amplitude was triangulation independent, so it was natural to interpret it as the amplitude of the spacetime itself. The higher-dimensional spin foam models for gravity do not have this property. The amplitude for a fixed triangulation does depend on the triangulation. There are two possible ways to interpret this amplitude. First, because each labelled triangulation has the interpretation of "quantum spacetime geometry", one may attempt to sum over all triangulations, which would have the interpretation of summing over all quantum geometries. This is conceptually appealing, but technically complicated, for it is very hard, if not impossible, to control such a sum. The other option is to take a (projective) limit of the triangulation becoming more refined. In case this limit exists, one can take the limiting value as the definition of the path integral. This is more in the spirit of the usual definition of the path integral, which is usually defined as a limit of a repeated integral of a cylinder function. In this procedure, the original integrand is being approximated by cylinder functions. This interpretation is very much in the spirit of what was done here, for the main idea of our approach was to approximate the exponential of the action by a cylinder function that depends only on a collection of "discrete" variables "living" on a particular triangulation. Which of this two interpretations is correct (if any) will become clear in the future.

Another problem which was not addressed by our higher-dimensional spin foam models is that of normalization constraints, introduced and discussed in section 6.5. Recall that in the models we considered all the constraints except this one were imposed. This problem may actually be a blessing. Indeed, as a preliminary analysis suggests, there is no local way to impose this constraints, and the only solutions are global, as the only solutions to the eigenvalue problem in the interacting one-dimensional Ising chain are global spin waives. This problem is currently the subject of an active investigation.

Despite all the problems left unsolved, the progress achieved by now by a collective effort is impressive. One sees how beautiful mathematical constructions of representation theory start to play role in physics, giving an adequate language for the quantum description of geometry. This starts to give a concrete realization of old ideas, notably by Penrose [45], to use group representation theory to describe geometry quantum mechanically. Whether this program succeeds or fails remains to be seen. However, one can only see what is at the end of the path by walking all the way. And I hope that the point of view developed in this thesis will serve as one further small step ahead along this path.

Appendix A

Conventions and notations

We use the following conventions for differential forms and integrals:

$$A = \frac{1}{p!} A_{a_1 \ldots a_p} dx^{a_1} \wedge \ldots \wedge dx^{a_p}, \tag{A.1}$$

$$\int A_{a_1 \ldots a_d} dx^{a_1} \wedge \ldots \wedge dx^{a_d} = \int d^d x A_{a_1 \ldots a_d} \tilde{\epsilon}^{a_1 \ldots a_d}, \tag{A.2}$$

where $\tilde{\epsilon}^{a_1 \ldots a_d}$ is the Levi-Civita density, taking the values plus minus one in any coordinate system. Our convention for the curvature form is: $F = dA + A \wedge A$.

The following symbols stand for the following elements of the triangulation

e — for an edge
f — for a face
t — for a tetrahedron
h — for a 4-simplex

We will also use the symbol ϵ to denote edges of the dual complex (dual 1-cells), and σ to denote dual faces (dual 2-cells).

All traces that we use in this paper are in the fundamental representation.

Our SO(3) index conventions:

$$X = i\tau^i X^i, \qquad J = \frac{i}{2}\tau^i J^i. \tag{A.3}$$

$$\eta = \frac{(q^{1/2} - q^{-1/2})}{i\sqrt{2k}}. \tag{A.4}$$

The quantity $\dim_q(j)$ is the so-called quantum dimension of j $\dim_q(j) = [2j+1]_q$, where $[n]_q$ is the quantum number

$$[n]_q = \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}} \tag{A.5}$$

having the property that $[k]_q = 0$.

Appendix B

Geometric 4-simplex

In this Appendix we list some facts about the geometry of a 4-simplex in $\mathbb{R}^4$. The geometrical considerations used in (8.3) are based on some of these facts.

A 4-simplex in $\mathbb{R}^4$ is characterized (up to translations) by four vectors. These, for instance, can be vectors pointing from one of the vertices, which we will denote by (0), to the other four vertices (1)-(4). See Fig. 10. Let us denote these vectors by e^I_a.

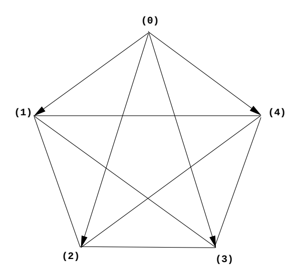


Fig. B.1. Geometric 4-simplex.

Here $I = 1 \ldots 4$ is an index for a vector in $\mathbb{R}^4$, and $a = 1 \ldots 4$ indicates a vertex at which the vector is directed. Thus, e^I_1 is a vector pointing from vertex (0) to the vertex (1).

Instead of vectors it is sometimes more convenient to use the so-called bivectors. A bivector E^{IJ} is an element of the second exterior power of $\mathbb{R}^4$. In particular, it can be obtained by wedging two vectors. Thus, bivectors that characterize a 4-simplex are obtained as $E^{IJ}_{ab} = e^{[I}_a e^{J]}_b$. Here the brackets denote the operation of antisymmetrization. It is not hard to see that bivectors E^{IJ}_{ab} are in one-to-one correspondence with faces of the 4-simplex h. For example, the bivector E^{IJ}_{12} corresponds to the face whose vertices are (0),(1),(2). The norm of each bivector is proportional to the squared area of the corresponding face

$$E^{IJ}_{ab} E_{ab\,IJ} = 2A^2_{ab}, \qquad \text{(B.1)}$$

where A_{ab} is the area of the face (0),(a),(b) and no summation over a, b is assumed. The volume of h can be obtained by wedging two bivectors that correspond to faces that do not share an edge. For example, with our choice of orientation of $\mathcal{M}$, the volume is given by

$$V_h = \frac{1}{4!}\epsilon_{IJKL}E^{IJ}_{12}E^{KL}_{34}. \qquad \text{(B.2)}$$

It is sometimes convenient to introduce bivectors corresponding to all 10 faces of h. So far we have introduced 6 bivectors E^{IJ}_{ab} corresponding to 6 faces of h. Bivectors that correspond to other 4 faces can be obtained as linear combinations of these 6 bivectors. When working with all 10 bivectors, it is convenient to label bivectors by 3 different indices instead of just two. We will employ for this purpose small Greek letters. Thus, bivectors are denoted by $E^{IJ}_{\alpha\beta\gamma}, \alpha, \beta, \gamma = 0\ldots 4$. These bivectors are defined by

$$E^{IJ}_{\alpha\beta\gamma} = e^{[I}_{\alpha\beta}e^{J]}_{\alpha\gamma}, \qquad \text{(B.3)}$$

where $e^I_{\alpha\beta}$ is a vector that points from the vertex α to vertex β. The norm of all bivectors (B.3) is equal to the twice of the squared area of the corresponding face, as in (B.1). As we have said above, only 6 of 10 bivectors (B.3) are independent. Thus, there are certain relations between them. One can write one such relation for each tetrahedron of h. One gets 5 relations only 4 of which are independent. With our definition (B.3) these relations are

$$\begin{aligned}
E_{012} + E_{023} - E_{013} - E_{123} &= 0, \\
E_{013} + E_{034} - E_{014} - E_{134} &= 0, \\
E_{024} + E_{234} - E_{023} - E_{034} &= 0, \\
E_{014} + E_{124} - E_{012} - E_{024} &= 0, \\
E_{123} + E_{134} - E_{124} - E_{234} &= 0,
\end{aligned} \qquad \text{(B.4)}$$

where we have suppressed the indices I, J for brevity. The volume of h can be expressed as a wedge product of any two of the bivectors (B.3) corresponding to faces that do not share an edge. This can be written as

$$\text{sign}(f, f')V_h = \frac{1}{4!}\epsilon_{IJKL}E(f)^{IJ}E(f')^{KL}, \qquad \text{(B.5)}$$

where we have introduced a notation $E(f)^{IJ}$ for a bivector that corresponds to face f, and $\text{sign}(f, f')$ is the sign of the right-hand-side in (B.5). The expression (B.5) gives the volume of h for any two pairs of faces f, f' that do not share an edge. One can use the expression (B.2) and the "closure" relations (B.4) to work out the correct sign of any of such formula for the volume.

Any bivector can naturally be split in its self-dual and anti-self-dual parts:

$$E_{\alpha\beta\gamma} = {}^{+}E_{\alpha\beta\gamma} + {}^{-}E_{\alpha\beta\gamma}. \qquad \text{(B.6)}$$

The self-dual and anti-self-dual parts are given correspondingly by

$$ {}^{+}E_{\alpha\beta\gamma} = \frac{1}{2}(E_{\alpha\beta\gamma} + {}^{*}E_{\alpha\beta\gamma}), \qquad \text{(B.7)} $$
$$ {}^{-}E_{\alpha\beta\gamma} = \frac{1}{2}(E_{\alpha\beta\gamma} - {}^{*}E_{\alpha\beta\gamma}), $$

where the Hodge star duality operation is defined as

$$ {}^{*}E^{IJ}_{\alpha\beta\gamma} = \frac{1}{2}\epsilon^{IJ}_{\ \ KL}E^{KL}_{\alpha\beta\gamma}. \qquad \text{(B.8)} $$

Since the space of self-dual (and anti-self-dual) bivectors in $\mathbb{R}^4$ is three-dimensional, we can introduce a new set of indices to label them. Thus, as the index for self-dual (anti-self-dual) bivector we will use lower case Latin letters from the middle of the alphabet: $i, j, k, \ldots = 1, 2, 3$. The norm of any self-dual (anti-self-dual) bivector calculated by contracting indices I, J will be the same as the norm calculated by contracting the single index i:

$$ {}^{+}E^{i}\,{}^{+}E_{i} = {}^{+}E^{IJ}\,{}^{+}E_{IJ}, \qquad \text{(B.9)} $$

where we have suppressed the indices $\alpha, \beta, \ldots$.

Not any bivector in $\mathbb{R}^4$ is simple, that is, not any bivector is a wedge product of two vectors. The necessary and sufficient requirement of simplicity is that the norm of the self-dual part is equal to the norm of the anti-self-dual part of the bivector. Thus, using (B.1), we can conclude that when a bivector is simple, its self-dual part norm is equal to the squared area of the corresponding face.

$$ {}^{+}E^{i}_{\alpha\beta\gamma}\ {}^{+}E_{\alpha\beta\gamma\, i} = A^{2}_{\alpha\beta\gamma}. \qquad \text{(B.10)} $$

There exists an expression for the volume of h that involves only the self-dual parts of bivectors (B.3):

$$ V_h = \frac{1}{3!}\frac{1}{30}\sum_{f,f'} \mathrm{sign}(f,f')\,{}^{+}E^{i}(f)\,{}^{+}E_{i}(f'), \qquad \text{(B.11)} $$

where ${}^{+}E^{i}(f)$ is the self-dual part of the bivector (B.3) corresponding to the face f, the sum is taken over all pairs f, f' of faces that do not share an edge, and $\mathrm{sign}(f, f')$ is the function introduced above by equation (B.5). The factor of $1/30$ in (B.11) appears to cancel the factor that comes from the sum over 30 terms f, f'.

Appendix C

Summary of facts on SU(2)

Here we give a short summary of some standard facts about the group SU(2). One can parameterize an element g of SU(2) by vectors Z from the Lie algebra. The corresponding relation is given by the exponentiation map:

$$g = e^Z = e^{i\psi n^i \sigma^i/2}, \tag{C.1}$$

where n^i is a unit vector $n^i n_i = 1$, ψ is a real positive parameter, and σ^i are the usual Pauli matrices:

$$(\sigma^i \cdot \sigma^j) = i\epsilon^{ijk}\sigma^k + \delta^{ij}. \tag{C.2}$$

Thus, ψn^i is an element of $\mathbb{R}^3 \sim \mathrm{su}(2)$, and (C.1) gives the exponentiation map. As one can easily check,

$$g = \cos(\psi/2) + in^i\sigma^i \sin(\psi/2). \tag{C.3}$$

Thus, to cover the whole SU(2) just ones, the parameter ψ should takes values in the range: $\psi \in [0, 4\pi]$.

The Haar measure on the group can be related to the usual Lebesgue measure in $\mathbb{R}^3$ by introducing a function $P(Z)$ on the Lie algebra:

$$P(Z) = \left(\frac{\sin(\psi/2)}{\psi/2}\right). \tag{C.4}$$

Then $P^2(Z)dZ/32\pi^2$ gives the normalized Haar measure on SU(2) in terms of the Lebesgue measure dZ.

The characters of irreducible representations are given by:

$$\chi_j(e^Z) = \frac{\sin(\psi(j + 1/2))}{\sin(\psi/2)}, \tag{C.5}$$

where j are half-integers (spins).

The Fourier transform on SU(2) maps any function on the group into a function on the space dual to the Lie algebra. Let f be a coordinate on the space $\mathrm{su}(2)^*$. Then the Fourier transform is given by:

$$\tilde{\phi}(f) := \int dZ\, P(Z)\, e^{-i\, f(Z)} \phi(\exp Z). \tag{C.6}$$

The inverse Fourier transform is given by:

$$\phi(\exp Z) = \sum_j \dim_j \frac{1}{P(Z)} \int_j d\Omega_f e^{i\, f(Z)} \tilde{\phi}(f), \tag{C.7}$$

where the integrals are taken over the co-adjoint orbits – spheres of radius $j + 1/2$, and the measure $d\Omega_f$ on each orbit is normalized so that

$$\int_j d\Omega_f = \dim_j = 2j + 1. \tag{C.8}$$

A particular case of (C.7) is the following simple formula for characters (C.5):

$$\chi_j(e^Z) = \frac{1}{P(Z)} \int_j d\Omega_f e^{i\, f(Z)}. \tag{C.9}$$

This is the famous Kirillov formula [38].

Appendix D

Haar measure, intertwiners and spin networks

Let us denote by V_i the spin i representation of SU(2) and by V_i^* the dual representation, which in the case of SU(2) is isomorphic to V_i. Let us denote by ϵ_i the corresponding isomorphism, and by $R_i(g)$ the representation of the group element g in V_i.

Throughout the paper we use the normalized Haar measure

$$\int dg = 1. \tag{D.1}$$

Let K be an intertwiner between the trivial representation and a representation V, that is $K \in Hom_G(\mathbb{R}, V)$. We will denote by $\bar{K}$ its dual $\bar{K} \in Hom_G(V, \mathbb{R}^*)$. Intertwiners are the basic building blocks of the so-called *spin networks*. Another usage of the intertwiners is to express the result of integration of a product of group elements. Let us denote by $K_\alpha^{i_1,\ldots,i_n}$ an orthonormal basis of $Hom_G(\mathbb{R}, V_{i_1} \otimes \cdots \otimes V_{i_n})$. The intertwiners $K_\alpha^{i_1,\ldots,i_n}$ have the property that

$$\bar{K_\alpha} K_\beta = \delta_{\alpha,\beta},$$

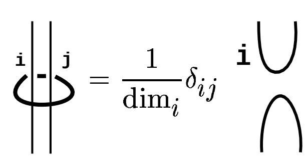
where the product between $\bar{K}$ and K is defined by the duality bracket $V \otimes V^* \to \mathbb{R}$. Then the integral of n group elements is given by

$$\int dg\, R_{i_1}(g) \otimes \cdots \otimes R_{i_n}(g) = \sum_\alpha K_\alpha^{i_1,\ldots,i_n} \bar{K}^\alpha_{i_1,\ldots,i_n}, \tag{D.2}$$

where $R_i(g)$ is considered as an element of $V_i \otimes V_i^*$. To integrate a product where both g and g^{-1} appear one has to use the duality relation $\bar{R}_i(g) = R_i(g^{-1}) = \epsilon_i R_i(g) \epsilon_i^*$. For instance

$$\int dg\, (\overline{R}_i)_{mn}\, (R_j)_{m'n'} = \frac{1}{\dim_j} \delta_{ij} \delta_{mm'} \delta_{nn'}. \tag{D.3}$$

This equality can be conveniently expressed graphically if one represents a matrix element $(R_i)_{mn}$ by a line labelled by i with the two open ends corresponding each to one of the indices m, n. Let us symbolically denote the integration over the group by a circle going around the lines representing the group elements. Then (D.3) takes the following form:

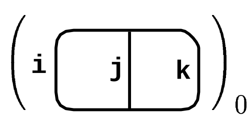


(D.4)

To find a graphical representation of the result of the integrals involving more than two matrix elements we introduce a trivalent vertex – analog of the Clebsch-Gordan symbol – normalized in a special way. But first, let us define the so-called *spin networks.*

An SU(2) spin network functional is defined by the following data: (i) an oriented graph Γ; (ii) a map $\mathbf{j}$ from the set of edges of Γ to the set of irreducible representations of SU(2); (iii) a map $\mathbf{i}$ from the set of vertices V to the set of intertwiners, which assigns to each vertex v an intertwiner from the tensor product of representations labelling the incoming edges to the tensor product of representations labelling the outgoing edges. A spin network is a function on a number of copies of SU(2), more precisely, on $(\mathrm{SU}(2))^E$, where E is the number of edges in Γ. To find the value of this function one takes for each edge the matrix element of the group element on that edge in the representation that labels the edge, and contracts the matrix elements corresponding to all edges using intertwiners at vertices. The function obtained this way is gauge invariant.

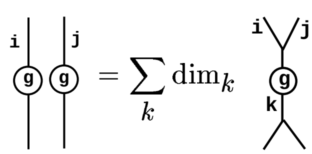
In this paper we will use the normalized trivalent vertex, defined in such a way that the θ-symbol constructed from 3 spins is always equal to unity:



$= 1,$ (D.5)

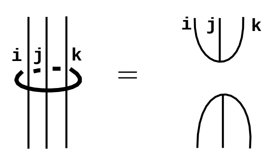
where the operation $(\cdot)_0$ denotes the evaluation of the corresponding spin network on all group elements equal to the unity in the group.

With this normalization of the trivalent vertex intertwiner, one can show that the following relation holds:

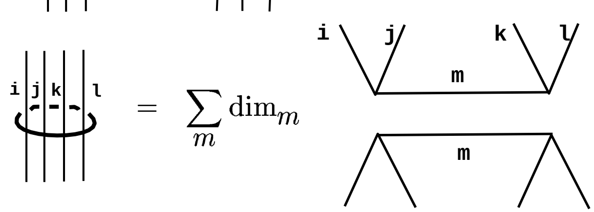


(D.6)

Using this relation one can find the result of the integral of a product of 3 and 4 group elements. The corresponding formulas are:

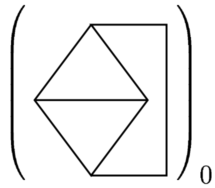


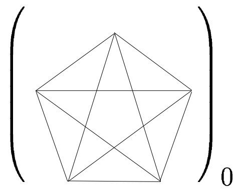
(D.7)



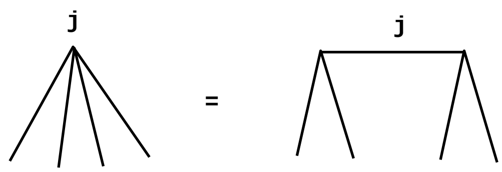
(D.8)

Also, we use the same trivalent vertex to define the normalized (6j),(15j)-symbols used in the body of the paper. Both symbols are given by evaluations of the corresponding spin networks, where the intertwiner used in a trivalent vertex is always the one normalized as in (D.5). Thus, we get:

$$(6j) =$$  (D.9)

$$(15j) =$$  , (D.10)

where the resolution of the 4-valent vertex in (D.10) is given by:



(D.11)

Appendix E

Quantum 6j symbol

Let X be a one dimensional oriented compact manifold, or more generally an oriented graph. A chord diagram (usually referred as Chinese character chord diagram) with support X is the union $D = \bar{D} \cup X$ where $\bar{D}$ is a graph with univalent and trivalent vertices, together with a cyclic orientation of trivalent vertices and such that univalent vertices lie in X. Trivalent vertices are referred to as internal vertices, and the degree of D denoted by $d^{\circ}(D)$ is half the number of vertices of the graph $\bar{D}$. Let $\tilde{\mathcal{A}}_n$ be the **Z** module freely generated by chord diagrams of degree n. We define the **Z** module of Vassiliev diagrams of degree n denoted by $\mathcal{A}_n$ as being the quotient of $\tilde{\mathcal{A}}_n$ by the relations (STU, IHX, AS) shown in Figure E.1. In the figures we always represent the support X with bold lines and the graph $\bar{D}$ with dashed chords.

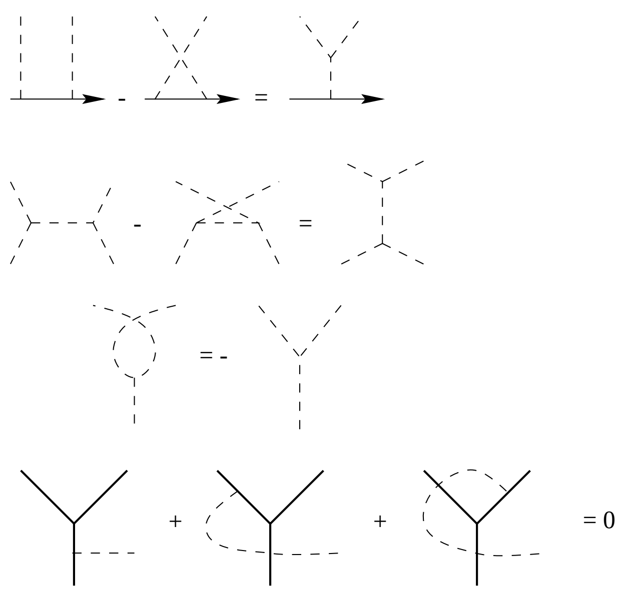


Fig. E.1.

Given a Lie group G, and a coloring C of the graph it is possible to define an evaluation of chord diagrams that is usually called weight system and denoted by $\omega_{G,C}$. Here C, the coloring of the graph X, is a map which assigns a representation of G to any edge of the graph and an intertwiner to any vertex of the graph X. Given a chord diagram D with support X and a coloring C of X, we define $\omega_{G,C}(D)$ as shown in Figure E.2. Here X_a denotes a basis of the Lie algebra of G, f^{abc} denotes the structure constants associated with each internal vertices and $t = t^{ab} X_a \otimes X_b$ is the quadratic casimir $t = t^{ab} X_a \otimes X_b$ associated to each chord.

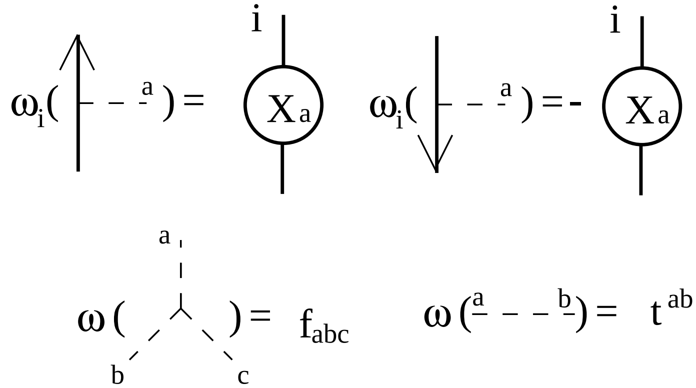
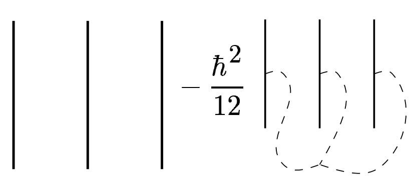
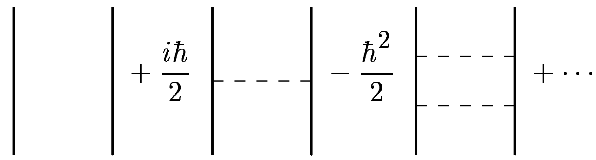
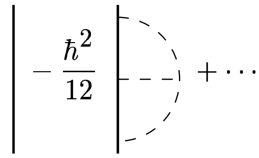


Fig. E.2.

The space of chord diagrams has been used to define the now famous universal invariant of framed link Z [39], which assigns to any link L in R^3 a formal power series in $\hbar$ $Z(L) = \sum_n \hbar^n Z_n(L)$, with $Z_n(L) \in \mathcal{A}_n$. We are not going into the details of this construction. However we must say that this construction uses three main building blocks: The braiding $R \in \mathcal{A}(I^2)$, the associator $\phi \in \mathcal{A}(I^3)$, which should satisfy the pentagon and hexagone identity, and the value of the unknot $\nu \in \mathcal{A}(I)$, where I denotes the unit interval (E.1) [27, 21].

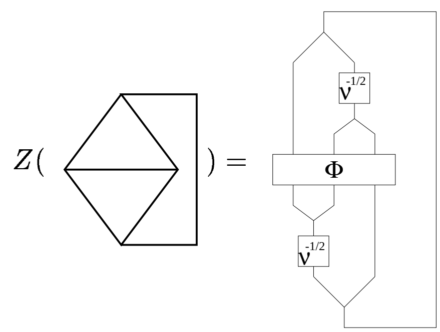
$\phi \;=$  $+ \cdots$ (E.1)

$R = \; =$  (E.2)

$\nu \; =$  (E.3)

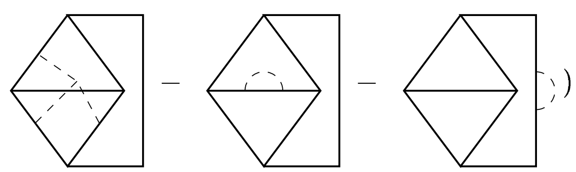
One important property of the universal invariant is that $\omega_{G,C}(Z(L))$ is equal to the Reshetikhin-Turaev evaluation of the link L associated with the quantum group $U_q(G)$ [1, 21, 41], where $q = \exp i\hbar$ and the casimir is normalized such that the norm of long roots is 2.

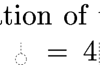
It has been shown recently [43, 32], that the universal invariant Z can be extended to an invariant of oriented, framed trivalent graphs embedded into R^3. The evaluation of the tetrahedral graph with the universal invariant is given by equation (E.4).

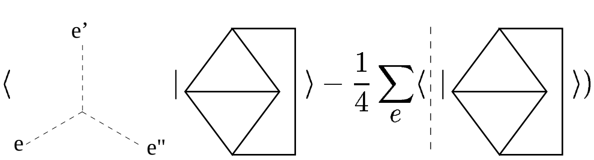


(E.4)

If we evaluate the result of Fig. E.4 using the weight system $\omega_{su(2),C}$ we get the normalized $6j$ symbol expanded in terms of $\hbar$ (Reshetikhin-Turaev evaluation of the tetrahedral graph for $U_{e^{i\hbar}}(su(2))$). The term proportional to $\hbar^0$ is just the classical $6j$ symbol. The next term, proportional to $\hbar^2$, is given by :

$V = \frac{1}{24}($  $).$ (E.5)

Here we have used the fact that, in the case of $su(2)$ and for the normalization of the casimir given by the trace in the fundamental representation, the identity  is satisfied. This term can be written in a symmetric form as:

$tet^{(1)} = \frac{1}{16}(\frac{1}{24} \sum_{e,e',e''} \langle$  $\rangle),$ (E.6) where the first sum is over all triples of distinct edges of the tetrahedra and the second sum is over all edges of the tetrahedra.

Appendix F

Simple representations of SO(D)

What is referred to in this paper as simple representations of SO(D) are the usual spherical harmonics representations. They are irreducible representations of SO(D) of class 1 with respect to the subgroup SO($D-1$) and, therefore, can be realized in the space of functions on S^{D-1}. This partially explains their relevance for quantum gravity in D dimensions, where the ($D-1$)-sphere has the geometrical meaning of the boundary of the D-simplex. In this Appendix we review some basic properties of these representations. For more information see, e.g., [61].

The spherical harmonics representations of SO(D) are the most obvious ones: they can be realized in the space of homogeneous polynomials of degree N. Let us denote the space of such polynomials by $V_N^{(D)}$. Then

$$\mathrm{dim} V_N^{(D)} = \frac{(N+D-1)!}{N!(D-1)!}.$$

It turns out, however, that the representation in this space is not irreducible. The invariant subspace in $V_N^{(D)}$ is given, as usual, by the space of polynomials satisfying the Laplace equation in $\mathbb{R}^D$. Thus, the irreducible representations of this type are realized in the space of homogeneous harmonic polynomials of degree N. Let us denote this space by $\mathcal{H}_N^{(D)}$. As one can show,

$$\mathrm{dim}\mathcal{H}_N^{(D)} = \frac{(2N+D-2)(D+N-3)!}{(D-2)!N!}. \tag{F.1}$$

As we have mentioned, these representations are of the class 1 with respect to SO($D-1$). Choosing the upper-left corner embedding of SO($D-1$) into SO(D), the vector in $\mathcal{H}_N^{(D)}$ that is invariant under the action of SO($D-1$) is given (up to normalization) by $C_N^p(x_D)$ for $x = (x_1, \ldots, x_D)$. Here $p = (D-2)/2$ and $C_N^p(x)$ is the so-called Gegenbauer polynomial defined in the next Appendix.

An explicit basis in $\mathcal{H}_N^{(D)}$ can be constructed by choosing a string of embeddings

$$\mathrm{SO}(2) \subset \mathrm{SO}(3) \subset \cdots \mathrm{SO}(D-1) \subset \mathrm{SO}(D).$$

Then $H_N^{(D)}$ decomposes into subspaces irreducible with respect to the action of the subgroup SO($D-1$). The later again decompose into the irreducible subspaces with respect to the action of SO($D-2$) etc. Finally, one arrives at SO(2) whose irreducible representations are 1-dimensional. Thus, we have:

$$\mathcal{H}_N^{(D)} = \oplus_{k_1=0}^{N} \oplus_{k_2=0}^{k_1} \cdots \oplus_{k_{D-2}=-k_{D-3}}^{k_{D-3}} V_{k_{D-2}}.$$

Here V_k are 1-dimensional representation spaces of SO(2). Note that k_{D-2} in the last sum runs over both positive and negative values. Thus, a basis in $\mathcal{H}_N^{(D)}$ can be labelled by a string of integers:

$$K := (k_1, k_2, \ldots, k_{D-2}), \qquad N \geq k_1 \geq k_2 \geq \cdots \geq |k_{D-2}|.$$

Appendix G

Properties of Gegenbauer polynomials

Gegenbauer polynomials are orthogonal polynomials satisfying many different properties. In this Appendix we review some of them. For more information of Gegenbauer polynomials see, e.g., [61, 36].

Let p be denote a quantity related to the dimension D according to $p = (D-2)/2$, or $D = 2p+2$. A generating functional for Gegenbauer polynomial is given by:

$$(1 - 2xr + r^2)^{-p} = \sum_{N=0}^{+\infty} C_N^p(x) r^N. \tag{G.1}$$

Gegenbauer polynomials satisfy the Rodriguez formula:

$$C_N^p(x) = \frac{(-1)^N (N+2p-1)(N+2p-2)\cdots(2p)}{2^N N!(N+p-\frac{1}{2})(N+p-\frac{3}{2})\cdots(p+\frac{1}{2})} \times \\ (1-x^2)^{-p+\frac{1}{2}} (\frac{d}{dx})^N (1-x^2)^{N+p-\frac{1}{2}}, \tag{G.2}$$

where the prefactor can also be written as

$$\frac{(-1)^N}{2^N N!} \frac{\Gamma(N+2p)\Gamma(p+\frac{1}{2})}{\Gamma(2p)\Gamma(N+\frac{1}{2}+p)}$$

The recurrence formula is given by:

$$(N+1)C_{N+1}^p(x) - 2(N+p)xC_N^p(x) + (N+2p-1)C_{N-1}^p(x) = 0, \tag{G.3}$$

with $C_0^p(x) = 1$ and $C_1^p(x) = x$. The polynomials satisfy the following differential equation:

$$\left\{(1-x^2)(\frac{d}{dx})^2 - (2p+1)x\frac{d}{dx} + N(N+2p)\right\} C_N^p(x) = 0.$$

A change of variable $x = \cos\theta$ puts this in the following form:

$$\left\{(\frac{d}{d\theta})^2 + 2p\frac{\cos\theta}{\sin\theta}\frac{d}{d\theta} + N(N+2p)\right\} C_N^p(\cos\theta) = 0. \tag{G.4}$$

The polynomials are normalized as:

$$C_N^p(1) = \frac{\Gamma(2p+N)}{\Gamma(2p)N!} = \dim\mathcal{H}_N^{(D)} \frac{D-2}{2N+D-2}.$$

where $\dim\mathcal{H}_N^{(D)}$ is given by (F.1). The polynomials satisfy the following orthogonality condition:

$$\int_{-1}^{+1} dx(1-x^2)^{p-\frac{1}{2}} C_N^p C_M^p = \delta_{N,M} \frac{\pi\Gamma(2p+N)}{2^{2p-1}N!(N+p)\Gamma^2(p)}. \tag{G.5}$$

Appendix H

Asymptotics of the Gegenbauer polynomial

To get the asymptotics of the Gegenbauer polynomial for large N we use the differential equation (G.4). It can be put into a form similar to that of a wave equation by setting

$$C_N^{(p)}(\cos\theta) = f(\theta)\sin^{-p}\theta.$$

One gets:

$$\frac{d^2 f}{d\theta^2} + f\left[(N+p)^2 - \frac{p(p-1)}{\sin^2\theta}\right] = 0.$$

For large N one can neglect the second term in the square brackets and p as compared to N in the first term. Thus, the large N asymptotics is given by

$$C_N^{(p)}(\cos\theta) \sim \frac{A}{\sin^p\theta}\sin[(N+p)\theta+\phi],$$

where ϕ is a phase and A is a normalization factor, both arbitrary at this stage. It can be constrained by using symmetry properties of C_N. From the expression for the generating functional one sees that

$$C_N(-x) = (-1)^N C_N(x).$$

Thus,

$$C_N^{(p)}(\cos(\pi-\theta)) = (-1)^N C_N^{(p)}(\cos\theta).$$

A simple analysis shows that this restricts ϕ to be

$$\phi = \frac{(1-p)}{2}\pi + \pi k,$$

where k is an arbitrary integer. Thus, the ambiguity in k is just the overall sign ambiguity. The constant A can be determined from the normalization condition (G.5). One gets:

$$\frac{\pi}{2}A^2 = \frac{\pi\Gamma(2p+N)}{2^{2p-1}N!(N+p)\Gamma^2(p)},$$

or

$$A = \pm\frac{1}{2^{p-1}\Gamma(p)}\left[\frac{\Gamma(2p+N)}{N!(N+p)}\right]^{1/2}.$$

For large N this behaves as

$$A \sim \pm \frac{N^{p-1}}{2^{p-1}\Gamma(p)}.$$

Using the fact that

$$C_{2N}^{(p)}(0) = \frac{(-1)^N}{N!}\frac{\Gamma(N+p)}{\Gamma(p)} \sim \frac{(-1)^N}{\Gamma(p)} N^{p-1},$$

and the expression for the derivative of the Gegenbauer polynomial

$$\frac{d}{d\theta} C_N^p = -2p \sin\theta \, C_{N-1}^{p+1},$$

we can fix the overall sign to be plus. Thus, finally, we get:

$$C_N^{(p)}(\cos\theta) \sim \frac{N^{p-1}}{2^{p-1}\Gamma(p)} \frac{1}{\sin^p \theta} \sin[(N+p)\theta + (1-p)\pi/2]. \tag{H.1}$$

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