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2+1 GENERAL RELATIVITY:
CLASSICAL AND QUANTUM

A Thesis in

Physics

by

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Abstract

2+1 general relativity is a useful toy model to test certain features of the full 3+1-dimensional theory. This thesis consists of two major parts - classical and quantum. In the classical part we provide boundary conditions on the variables of the theory with a negative cosmological constant. We consider two boundaries, one at spatial infinity and one internal boundary. The internal boundary conditions are chosen to represent a horizon of a stationary black hole. One of the interesting results in this part are explicit quasi-local formulae for energy and angular momentum of a black hole as well as for the whole space-time.

In the second part, we investigate the canonical quantization of 2+1 gravity. We begin with analysis of different classical formulations as points of departure for quantization. Next, we address the issue of constructing the physical Hilbert space of states. For this purpose we construct certain important operators in quantum gravity, including a new regularization of the Hamiltonian constraint. Finally we discuss two strategies for finding its solutions.

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Chapter 1

Introduction

The theory of 2+1 dimensional general relativity in vacuum is based on the analog of 3+1 dimensional Einstein-Hilbert action

$$S = \frac{1}{16\pi G} \int_{\mathcal{M}} (R - 2\Lambda) \sqrt{-g} d^3x \quad (1.1)$$

where \mathcal{M} is the space-time manifold, g - its metric, R - the curvature scalar and Λ - the cosmological constant. As is well known, the equations of motion arising from such action are

$$R_{ab} = 2\Lambda g_{ab} \quad (1.2)$$

which means that the Ricci tensor is a constant multiple of the space-time metric. In this form they are very similar to the 3+1 dimensional equations of motion. It is also known, however, that due to dimensionality of our space-time \mathcal{M} , the Riemann curvature tensor is completely determined by the Ricci tensor. This is a crucial difference from the 3+1 dimensions. It implies, in particular, that all the vacuum solutions are such that the metric is everywhere locally Minkowski, de Sitter or anti-de Sitter (depending on the cosmological constant Λ). The theory has no local degrees of freedom which means, for example, that there are no gravitational waves in 2+1 dimensions.

This characteristic feature of 2+1 dimensions makes the theory seem rather trivial as compared to 3+1 dimensional general relativity with its wealth of local dynamics. Nevertheless, this simpler theory serves as a useful toy model to test many ideas applicable to higher dimensions. In fact, a surprisingly large class of interesting features is shared by the 2+1 and higher dimensional theories. Also, lower dimensional gravity is interesting to study from the mathematical physics perspective. It possesses qualitatively new features which can be used to test robustness of the mathematical tools used in higher dimensions. Staruszkiewicz [57] was one of the first to study vacuum solutions. Even though they are locally the same, the differences in the global topology proved to be sufficient to characterize one- and two-body static solutions. Later Deser, Jackiw and 't Hooft [37] found solutions corresponding to N point-like masses as well as a solution with a spinning source.

Many other papers have been written on 2+1 dimensional theory since then. The purpose of this thesis is to analyze various aspects of 2+1 gravity. More precisely, we will concentrate on the following topics:

1. Black hole physics in the classical domain.
2. Dynamics of quantum geometry in the canonical quantum gravity.
3. Relation between Euclidean and Lorentzian theories.

Below we discuss in more details which issues in each of the three topics are discussed in this thesis.

The interest in 2+1 dimensional general relativity has increased when Bañados, Teitelboim and Zanelli [21] found a vacuum black hole solution in the case $\Lambda < 0$. This came somewhat as a surprise, since there can not be a curvature singularity in 2+1

dimensions (in vacuum). The singularity encountered in BTZ solution is in fact a causal singularity. This discovery motivated us to extend the higher dimensional framework of isolated horizons [5] to the 2+1 theory. The framework describes black holes in equilibrium [6, 7, 9, 11, 14] and was motivated by the following considerations. The zeroth and first laws of black hole mechanics apply to equilibrium situations and small departures therefrom. In older formulations of these laws, black holes in equilibrium are represented by stationary space-times with regular event horizons (see, e.g., [22, 29, 31, 42, 66]). While this idealization is a natural starting point, from a physical perspective it seems quite restrictive. (See [6, 7, 11] for a detailed discussion.) The isolated horizons framework generalizes this paradigm in two ways. First, one replaces the notion of an event horizon with that of an isolated horizon. While the former are defined only retroactively using the fully evolved space-time geometry, the latter are defined quasi-locally by suitably constraining the geometry of the horizon surface itself. Second, one drops the requirement that the space-time be stationary and asks only that the horizon be isolated. That is, the requirement that the black hole be in equilibrium is incorporated by demanding only that no matter or radiation fall through the horizon although the exterior space-time region may well admit radiation. Consequently, the generalization in the class of allowed space-times is enormous. In particular, space-times admitting isolated horizons need not possess *any* Killing vector field; although event horizons of stationary black holes are isolated horizons, they are a very special case. A recent series of papers [6, 7, 9, 14, 26] has generalized the laws of black hole mechanics to this broader context. The notion of isolated horizons has proved to be useful also in other contexts in 3+1 dimensions, ranging from numerical relativity to background

independent quantum gravity: i) it plays a key role in an ongoing program for extracting physics from numerical simulations of black hole mergers [5, 10, 8, 38]; ii) it has led to the introduction [12, 14, 35, 36] of a physical model of hairy black holes, systematizing a large body of results on properties of these black holes which has accumulated from a mixture of analytical and numerical investigations; and, iii) it serves as a point of departure for statistical mechanical entropy calculations in which all black holes (extremal or not) *and* cosmological horizons are incorporated in a single stroke [3, 4, 11].

In recent years, 2+1-dimensional stationary black holes have drawn a great deal of attention as simplified models for analyzing conceptual issues surrounding black hole thermodynamics (see, e.g., [29]). It is therefore natural to ask if the isolated horizon framework can be constructed and used to extend the standard 2+1-dimensional treatments. In Chapter 2 we provide such a framework, analyze the resulting horizon geometry in detail and use it to generalize the zeroth and first laws of black hole mechanics. The results described in this chapter have been obtained in collaboration with Abhay Ashtekar and Olaf Dreyer (see also [13]).

The 2+1 dimensional general relativity has also been often used to test various ideas on quantization of the theory. A comprehensive account of such works is given in [30]. In this thesis we are interested in the canonical quantization in connection variables [1, 52, 54, 65]. A central issue in this quantization program is a construction of the Hilbert space of physical states. This is done by imposing certain quantum constraints on functions of (generalized) connections [16]. Solving Gauss and diffeomorphism constraints provides us with the so called kinematical Hilbert space [18]. On functions from this space one then imposes the scalar (also called Hamiltonian) constraint. While all of

them have successfully been implemented [59, 63, 64, 61, 62], there are still remaining questions concerning the implementation of the scalar constraint. First of all, there exist in literature several different regularization choices which lead to different quantum operators having the same classical limit [40, 25, 23, 24]. At present we have no criteria that would allow us to favor one over another. Secondly, the techniques developed for construction of the kinematical Hilbert space are only valid for compact gauge groups. In the Lorentzian theories, however, naturally arising groups are often non-compact. Therefore it is interesting to ask if one can use similar techniques to construct scalar product in the non-compact case. Finally, although the Hamiltonian constraint can in principle be solved completely, the solutions are not known explicitly. In particular, we do not know if, among them, there are semi-classical states. As we will see in this thesis, all of those questions and problems can be addressed in 2+1 dimensions. They show up in a very analogous way as in higher dimensions, but due to dimensionality of space-time they are easier to handle.

In Chapter 3 we discuss different formulations of the Lorentzian theory. In order to address issues in quantum dynamics (topic 2 above), we compare them from the point of view of canonical quantization. Each formulation has certain advantages in a close analogy with higher dimensions. In particular, we introduce a new formulation based on $SU(2)$ gauge group having the advantage of the gauge group being compact but also a disadvantage of having a relatively complicated scalar constraint. Also, we discuss in this chapter the relations between those formulations and the Euclidean theory (topic 3). In analogy with Wick rotation in field theory, we introduce here two maps from Euclidean gravity to Lorentzian gravity. Depending on which formulation is a result of such a map,

we call them Wick rotation or Wick transform. The results on the Wick transform and Wick rotation discussed in Chapters 3 and 5 have been obtained in collaboration with Abhay Ashtekar and Bruno Hartmann (see also [41]).

In Chapter 4 we address some issues in canonical quantum gravity using methods closely analogous to those used in 3+1 dimensions. The main motivation is quantum dynamics - topic 2 above. In particular, we use the new $SU(2)$ formulation to construct the length operator as well as the Hamiltonian operator for the Lorentzian 2+1 theory. Due to well known constructions with $SU(2)$ group in 3+1 dimensions, the constructions here are a fairly straightforward applications of those ideas. In this chapter we also discuss a new alternative regularization of the Euclidean Hamiltonian constraint. The advantage of this new construction is that the result is a relatively simple operator. Again, a closely analogous construction can be applied in higher dimensions. It turns out that here, in contrary to higher dimensions, one can easily show that all the semi-classical states are the solutions to all the known Hamiltonian operators.

Chapter 5 discusses two different strategies for a construction of the Lorentzian physical Hilbert space. One is the use of a quantum Wick transform, which uses the knowledge of Euclidean space (topic 3). Here we implement rigorously ideas proposed for 3+1 dimensions which have, so far, remained only at the formal level. In order to do that we will use simplifications characteristic only to 2+1 dimensions. Second strategy (belonging to topic 2 above) relies on the reduction of the problem to certain finite dimensional subspaces. This strategy preserves full analogy with higher dimensions and allows a systematic construction of all solutions to Hamiltonian constraint.

As a general rule, arguments and proofs which are parallel to those in higher dimensions are only sketched and differences are emphasized. Throughout the thesis, we will set $8\pi G = c = 1$ (unless G appears explicitly in a formula). Since Newton's constant has dimensions of inverse mass in 2+1 dimensions, now mass and charge are dimensionless while angular momentum has dimensions of length. Also, we assume everywhere the signature $(-, +, +)$ (or $(-, +, +, +)$, depending on the number of dimensions) for the space-time metric. ∇_a denotes the derivative operator compatible with the metric, which treats internal indices as scalars, while D_a is always the derivative operator which is compatible with the orthonormal dyad (two dimensional analog of the triad and tetrad).

Chapter 2

Isolated Horizons

In this chapter we present the results of application of the isolated horizons framework to 2+1 dimensional theory with a negative cosmological constant.

In Section 2.1, we introduce the definitions of non-expanding and weakly isolated horizons and derive their main consequences, including the generalized zeroth law. While the basic definitions are the same as in higher dimensions, some of their consequences are different because of special features of the 3-dimensional Riemannian geometry. In particular, the Weyl tensor, which plays an important role in higher dimensions, now vanishes identically. Similarly, since in 2+1 dimensions black holes exist only if the cosmological constant is non-zero, there is now an inherent length scale in the problem. However, the spirit of the analysis is the same as in higher dimensions: We extract, from the notion of Killing horizons, the minimal structure that is needed to generalize the laws of black hole mechanics. As in 3+1 dimensions, some of the structure becomes more transparent in terms of null-triads. Therefore, in the Appendix we construct the 2+1-dimensional analog of the Newman-Penrose framework [47, 49, 58] and use it to elucidate the meaning and consequences of our horizon boundary conditions.

In Section 2.3 we introduce the action principle, and in Section 2.4, the phase space and the associated Hamiltonian framework. While the overall procedure is the same as in higher dimensions [9, 14], there is a significant technical complication in the choice of

boundary conditions at infinity. In particular, while the electromagnetic potential falls off as $1/r^n$ in $n + 2$ spatial dimensions for $n > 0$, one must now allow it to *blow up* logarithmically. Since the treatment of these boundary conditions is perhaps the most difficult technical part of our analysis, they are spelled out in detail separately in Section 2.2. The end result is that there do exist boundary conditions which suffice to make the action principle, the symplectic structure and the Hamiltonians well-defined. Using these structures, we establish the generalized first law. As in higher dimensions, it arises as a consistency condition for the evolution to be generated by a Hamiltonian; there is thus an infinite family of first laws, one for each time-like vector field in space-time, the evolution along which is Hamiltonian.

In Section 2.5, we consider the issue of introducing a canonical notion of horizon energy. In the Hamiltonian framework, this problem reduces to that of selecting (for each point in the phase space) a preferred time-evolution vector field at the horizon. One expects this choice to vary from one point in the phase space to another; in the non-rotating case, one would expect the preferred vector field to point along the null normal to the horizon, while in the rotating case, one would expect it to have a non-zero component also along the space-like, rotational direction at the horizon. As in 3+1 dimensions, we resolve this problem by making use of known stationary solutions [28, 20, 21, 33, 32]. However, we encounter a new subtlety, again because the electromagnetic potential is now allowed to diverge at infinity: when the horizon charge and angular momentum are both non-zero, there is an additive ambiguity in the definition of this ‘canonical’ energy which can not be removed without some extra input. Thus, the situation differs from that in higher dimensions in this respect.

Sections 2.3-2.5 focus on the infinite dimensional space of histories and the phase space in presence of weakly isolated horizons. In Section 2.6, by contrast, we consider individual space-times and analyze the geometrical structures and their interplay with field equations at the horizon. Specifically, we first show that non expanding horizons admit a natural derivative operator \mathcal{D} and study the geometrical information it encodes, beyond the natural degenerate metric, and then use field equations to isolate the freely specifiable parts of \mathcal{D} on weakly isolated horizons. As in higher dimensions [14] we introduce the notion of isolated horizons using the derivative operator \mathcal{D} . In contrast to higher dimensions, every non-expanding horizon can be equipped with an isolated horizon structure simply by selecting an appropriate null normal and, generically, this can be achieved in a unique fashion.

2.1 Definitions and geometrical structures

In this Section we will define *weakly isolated horizons* and analyze their geometric properties. It is convenient to proceed in two steps since certain preliminary results are needed to state the final definition.

Let \mathcal{M} be a three dimensional manifold with metric tensor g_{ab} of signature $(- + +)$. For simplicity, we will assume that all manifolds and fields are smooth. Let Δ be a null hypersurface in (\mathcal{M}, g_{ab}) . A *future directed* null normal to Δ will be denoted by ℓ . The *expansion* $\theta_{(\ell)}$ of ℓ is defined by $\theta_{(\ell)} \hat{=} m^a m^b \nabla_a \ell_b$, where ∇ is the derivative operator on (\mathcal{M}, g_{ab}) and m^a is any unit, space-like vector field tangent to Δ .¹ It is easy

¹Throughout this Chapter, $\hat{=}$ will denote equality restricted to the null surface Δ .

to check that the expansion is insensitive to the choice of m^a . However, as the notation suggests, it does depend on the choice of the null normal ℓ ; if $\ell' \hat{=} f\ell$, then $\theta_{(\ell')} \hat{=} f\theta_{(\ell)}$.

2.1.1 Non-expanding horizons

Definition 1: A 2-dimensional sub-manifold Δ of a space-time (\mathcal{M}, g_{ab}) is said to be a *non-expanding horizon* if it satisfies the following conditions:

- (i) Δ is topologically $S^1 \times \mathbb{R}$ and null;
- (ii) The expansion $\theta_{(\ell)}$ of ℓ vanishes on Δ for any null normal ℓ ;
- (iii) All equations of motion hold at Δ and the stress-energy tensor T_{ab} of matter fields at Δ is such that $-T_b^a \ell^b$ is future directed and causal for any future directed null normal ℓ .

Note that if conditions (ii) and (iii) hold for one null normal ℓ they hold for all.

The role of these conditions is as follows. The first condition just ensures that the cross-sections of Δ are compact which will in turn ensure that various integrals—defining, e.g., the symplectic structure and various Hamiltonians—over these cross-sections are well-defined. The second condition is the crucial one. It directly implies that all horizon cross-sections have the same length, which, following the terminology in the literature [29] we will call the ‘horizon area’ and denote by a_Δ . It will also be convenient to introduce the notion of the *horizon radius* R_Δ , defined by $a_\Delta = 2\pi R_\Delta$. Finally, as we will see below, condition (ii) also implies that there is no flux of matter-energy across the horizon and thus captures the intuitive notion that the black hole is isolated. The last condition, (iii), is analogous to the dynamical conditions one imposes at spatial infinity.

While at infinity one requires that the metric (and other fields) approach a specific solution to the field equations (namely, ‘the classical vacuum’), at the horizon one only asks that the field equations be satisfied. The energy condition involved is very weak; it follows from the much stronger dominant energy condition normally imposed. All these conditions are satisfied on any Killing horizon (with a S^1 cross-section) if gravity is coupled to physically reasonable matter (including perfect fluids, Klein-Gordon fields, Maxwell fields possibly with dilatonic coupling and Yang-Mills fields).

We will now present three examples of non-expanding horizons:

Example 1: The paradigmatic example of a non-expanding horizon in 2+1 dimensions is provided by the BTZ black holes[21]. We begin by showing that the horizons of these space-times trivially satisfy our Definition 1.

In Eddington-Finkelstein-like coordinates, the space-time metrics of these black holes are given by

$$ds^2 = -(N)^2 dv^2 + 2dvdr + r^2 (d\phi + N^\phi dv)^2, \quad (2.1)$$

where

$$N = \left(f(r) + \frac{J^2}{4\pi^2 r^2} \right)^{1/2} \quad \text{and} \quad N^\phi = -\frac{J}{2\pi r^2}, \quad (2.2)$$

and

$$f(r) = -\frac{M}{\pi} + \frac{r^2}{l^2}, \quad (2.3)$$

the length l being related to the cosmological constant Λ through

$$\Lambda = -\frac{1}{l^2}. \quad (2.4)$$

The metric coefficient N vanishes at $r = r_{\pm}$, where

$$r_{\pm}^2 = \frac{Ml^2}{2\pi} \left\{ 1 \pm \left[1 - \left(\frac{J}{Ml} \right)^2 \right]^{1/2} \right\}. \quad (2.5)$$

The 2-surface $r = r_+$ is the horizon of interest to us. Sometimes it is convenient to have the mass M and the angular momentum J expressed in terms of r_+ and r_- :

$$M = \pi \frac{r_+^2 + r_-^2}{l^2} \quad J = \frac{2\pi r_+ r_-}{l}. \quad (2.6)$$

The surface $r = r_+$ is null with normal

$$\ell = \partial_v - N^\phi(r_+) \partial_\phi. \quad (2.7)$$

Since it is coordinatized by v, ϕ , it has the required topology, $S^1 \times \mathbb{R}$. Since ℓ is a restriction to the horizon of a space-time Killing field $\partial_v - N^\phi(r_+) \partial_\phi$ it follows that $\theta_{(\ell)}$ vanishes. Finally, the third condition in the definition is trivially satisfied because BTZ metrics are vacuum solutions to Einstein's equation. \square

Example 2: The first two conditions in our definition are satisfied by the more general class of metrics (2.1), without the restriction (2.3) on the form of the function $f(r)$. If the function $f(r)$ is chosen to satisfy the weak condition $(\partial_r f)|_{r=r_+} \leq (r_+/l^2)$,

the third condition in the definition is also satisfied. Thus, we have a very large class of *generalized BTZ metrics* which admit a non-expanding horizon. This class includes, in particular, the metrics introduced in [28]. \square

Example 3: Our final example is the charged, rotating black hole solution first discovered by Clément [33]. (It was later independently found by Martinez, Teitelboim and Zanelli (MTZ) [46], who also analyzed its physical properties.) It is again a stationary axi-symmetric solution and is expressed in terms of three parameters, \bar{r}_0, ω, Q . As in higher dimensional dilatonic black holes, the dependence of the mass M and angular momentum J on the parameters appearing explicitly in the solution is quite complicated (see Section 2.5). Furthermore, in this case, the electro-magnetic fields and the metric coefficients *diverge* (logarithmically) at infinity. Hence the very meaning of mass and angular momentum is not a priori transparent. Finally, if one simply sets $Q = 0$, one obtains the BTZ metric with $M = 0$ and $J = 0$; to obtain the non-trivial solutions in the BTZ family, a more subtle limit has to be taken.

Clément gave the metric in the form:

$$ds^2 = -N^2 dt^2 + K^2 (d\phi + N^\phi dt)^2 + \frac{r^2}{K^2} \frac{dr^2}{N^2}, \quad (2.8)$$

where the functions N, N^ϕ and K are given by

$$N^2 = \frac{r^2}{K^2} \left(\frac{r^2}{l^2} - \frac{\bar{l}^2}{2\pi l^2} Q^2 \ln(r/\bar{r}_0) \right), \quad (2.9)$$

$$N^\phi = -\frac{\omega}{2\pi K^2} Q^2 \ln(r/\bar{r}_0), \quad (2.10)$$

$$K^2 = r^2 + \frac{1}{2\pi} \omega^2 Q^2 \ln(r/\bar{r}_0), \quad (2.11)$$

$$\bar{l}^2 = l^2 - \omega^2 \quad (2.12)$$

The Maxwell field is given by:

$$\mathbf{F} = \frac{Q}{r} dr \wedge (dt - \omega d\phi). \quad (2.13)$$

In the Eddington-Finkelstein-like coordinates the metric becomes

$$ds^2 = -N^2 dv^2 + \frac{2r}{K} dv dr + K^2 \left(d\phi + N^\phi dv \right)^2. \quad (2.14)$$

(As is usual in the passage to the Eddington-Finkelstein type coordinates in the *stationary* context, the angle ϕ in (2.14) is not the same as the one in (2.8). In the analysis of the horizon structure, we will use (2.14).) It is straightforward to check that the 2-surfaces $N = 0$, co-ordinatized by v, ϕ , are non-expanding horizons. \square

Although the conditions imposed in Definition 1 seem rather weak, they have a number of interesting consequences. To explore them it is often convenient to introduce, as in the Newman-Penrose framework, a triad consisting of the vectors ℓ^a, n^a , and m^a in the neighborhood of the horizon Δ . The vectors ℓ^a and n^a are null and m^a , space-like.

We choose ℓ^a to be a future pointing null normal of the horizon and then normalize n^a by requiring $\ell^a n_a \hat{=} -1$ and m^a by requiring $m^a m_a \hat{=} 1$. All other contractions vanish. (Thus, in contrast to the 3+1 dimensional NP framework, m^a is now real and space-like rather than complex and null.) Given such a triad, we can introduce NP-like coefficients as in 3+1 dimensions. Appendix gives the corresponding definitions and a summary of important relations for these coefficients. It is often convenient to use the triad so that the pull-back to Δ of the 1-form n is orthogonal to S^1 cross-sections of Δ , i.e., $\underline{dn} \hat{=} 0$ (so that, in the NP-like framework of Appendix, $\alpha \hat{=} \pi$).² We will explicitly specify when this restriction is made.

(a) *Intrinsic metric of Δ* : Denote by q_{ab} the pull-back of the space-time metric g_{ab} to Δ ; $q_{ab} \hat{=} \underline{g_{ab}}$. Since Δ is a 2-dimensional, null sub-manifold of (\mathcal{M}, g_{ab}) , and ℓ a null-normal to it, it follows that

$$q_{ab} \ell^b \hat{=} 0; \quad q_{ab} \hat{=} \underline{m_a m_b} \tag{2.15}$$

for a unique 1-form $\underline{m_a}$, defined *intrinsically* on Δ . Furthermore, as the explicit calculation of spin coefficients in Appendix shows, $\underline{dm} \hat{=} 0$. We will choose out NP triad such that $m_{\underline{a}} = \underline{m_a}$.

(b) *Properties of ℓ* : Since ℓ^a is a null normal to Δ , it is automatically twist-free and geodesic. We will denote the *acceleration* of ℓ^a by $\kappa_{(\ell)}$

$$\ell^a \nabla_a \ell^b \hat{=} \kappa_{(\ell)} \ell^b. \tag{2.16}$$

²Throughout the thesis, an under-arrow will denote pull-back.

Note that the acceleration is a property not of the horizon Δ itself, but of a specific null normal to it: if we replace ℓ by $\ell' \hat{=} f\ell$, then the acceleration changes via

$$\kappa_{(\ell')} \hat{=} f\kappa_{(\ell)} + \mathcal{L}_\ell f. \quad (2.17)$$

(In the NP-type notation of Appendix, $\kappa_{(\ell)}$ is denoted by ϵ .)

(c) *A natural connection 1-form on (Δ, ℓ)* : Since the expansion $\theta_{(\ell)}$ (or, in the framework of Appendix, the NP-type coefficient ρ) vanishes, and since in 2+1 dimensions there is no analog of the 3+1 dimensional shear, we conclude that given any vector field X^a tangential to Δ , we have:

$$X^a \nabla_a \ell^b \hat{=} X^a \omega_a \ell^b$$

for some (ℓ -dependent) 1-form ω_a on Δ . In particular, we have $\kappa_{(\ell)} = \ell^a \omega_a$. Thus, there exists a one-form ω_a intrinsic to Δ such that

$$\nabla_{\underline{a}} \ell^b \hat{=} \omega_a \ell^b. \quad (2.18)$$

ω_a will play an important role in this paper. (In the NP-type framework of Appendix, ω can be expressed in terms of spin coefficients: $\omega_a = \alpha \underline{m}_a - \epsilon n_a \equiv \alpha \underline{m}_a - \kappa_{(\ell)} n_a$.)

Under the rescaling $\ell \rightarrow f\ell$, the 1-form ω transforms as a connection:

$$\omega_a \rightarrow \omega_a + \nabla_{\underline{a}} \ln f. \quad (2.19)$$

A particular consequence of (2.18) is:

$$\mathcal{L}_\ell q_{ab} \hat{=} 2 \nabla_a \ell_b \hat{=} 0;$$

every null normal ℓ to Δ is a ‘Killing field’ of the degenerate metric on Δ . Thus, our key condition in the Definition —that ℓ be expansion-free— implies that non-expanding horizons are Killing horizons of the *intrinsic geometry* to ‘first order’.

Example 1: What is the expression of ω_a in the case of the BTZ black hole? On the horizon, let us choose the triad vector m^a as $m \hat{=} (1/r_+) \partial_\phi$. Then, a direct calculation yields:

$$\omega_a = N^\phi m_a - \kappa_{(\ell)} n_a \tag{2.20}$$

where the acceleration $\kappa_{(\ell)}$ is given by

$$\kappa_{(\ell)} = \frac{r_+}{l^2} - r(N^\phi)^2. \tag{2.21}$$

(See the discussion of this example in Appendix.) As in higher dimensions [7, 9], the angular momentum information is contained in the spatial component of ω . \square

(d) *Conditions on the Ricci tensor:* As in higher dimensions [14], we can use the Raychaudhuri equation to obtain conditions satisfied by the 3-dimensional Ricci tensor at the horizon. Thus, by calculating $\mathcal{L}_\ell \theta_{(\ell)}$ for a general null congruence ℓ in terms of the derivatives of ℓ and the Ricci tensor and applying it to any normal of a non-expanding horizon, we obtain:

$$R_{ab} \ell^a \ell^b \hat{=} 0. \tag{2.22}$$

(For a derivation in the NP-type framework, see equation (A.33) in Appendix.) Next, let us use the energy condition required in the Definition: $P^a : \hat{=} -T_b^a \ell^b$ is future pointing, and time-like or null on Δ . Using the field equations

$$R_{ab} - \frac{1}{2}R g_{ab} + g_{ab}\Lambda = T_{ab} \quad (2.23)$$

and (2.22), we obtain $P_a \ell^a \hat{=} 0$, whence, at the horizon, P^a is of the form $P^a \hat{=} f \ell^a + g m^a$.

The energy condition now implies $g \hat{=} 0$, i.e., the component $T_b^a \ell^b$ is proportional to ℓ^a .

The field equations then imply:

$$R_{ab} \ell^a m^b \hat{=} 0. \quad (2.24)$$

This constraint on the Ricci curvature has an important consequence. Using the expression of the 3-dimensional Riemann tensor in terms of the Ricci tensor in the equality $2\nabla_{[a} \nabla_{b]} \ell^d \hat{=} R_{abc}{}^d \ell^c$, and (2.18), it is straightforward to express $d\omega$ in terms of $R_{ab} \ell^a m^b$. (2.24) now implies that ω is exact:

$$d\omega \hat{=} 0. \quad (2.25)$$

(In the NP-type framework of Appendix, $R_{ab} m^a \ell^b$ can be expressed in terms of spin-coefficients as $R_{ab} m^a \ell^b \hat{=} \mathcal{L}_\ell \pi - \mathcal{L}_m \kappa_{NP}$ and $d\omega \hat{=} (\mathcal{L}_\ell \pi - \mathcal{L}_m \kappa_{NP}) m \wedge n$, whence (2.24) implies that ω is exact.) By contrast, in 3+1 dimensions, $d\omega$ is essentially determined by the imaginary part of the Weyl tensor component Ψ_2 , which encodes the angular momentum information [9]. We will see that angular momentum information continues

to reside in ω ; it is just that, since the Weyl tensor vanishes identically in 3 dimensions, we can no longer further simplify that expression and rewrite it in terms of the Ψ_2 .

(e) *Projective space*: Since ℓ Lie-drags the intrinsic metric q_{ab} of Δ , it is natural to pass to the space $\hat{\Delta}$ of orbits of ℓ . We will conclude the discussion of non-expanding horizons with a discussion of $\hat{\Delta}$.

It follows from our topological restriction in Definition 1 that $\hat{\Delta}$ has the topology of S^1 . Denote by $\hat{\Pi}$ the canonical projection map from Δ to $\hat{\Delta}$. Then, since $q_{ab}\ell^b \hat{=} 0$ and $\mathcal{L}_\ell q_{ab} \hat{=} 0$, it follows that there exists a metric \hat{q}_{ab} on $\hat{\Delta}$ such that $q_{ab} \hat{=} \hat{\Pi}_\star \hat{q}_{ab}$. The metric \hat{q}_{ab} on $\hat{\Delta}$ can be uniquely expressed as $\hat{q}_{ab} = \hat{m}_a \hat{m}_b$ and $\underline{m}_a = \hat{\Pi}_\star \hat{m}_a$.

2.1.2 Weakly isolated horizons

Although non-expanding horizons already have a rather rich structure, the notion is not sufficiently strong to be directly useful to black hole mechanics. In particular, as we have seen, there is a freedom to rescale the null normal via $\ell^a \rightarrow \ell'^a = f\ell^a$ for any positive function f on Δ under which the acceleration of ℓ transforms via $\kappa_{(\ell')} = f\kappa_{(\ell)} + \mathcal{L}_\ell f$. Since $\kappa_{(\ell)}$ is the obvious candidate for surface gravity, because of this rescaling freedom, $\kappa_{(\ell)}$ can not be constant for all null normals ℓ . Thus, on a general non expanding horizon, we can not hope to establish the zeroth law. In this sub-section, we will introduce a stronger definition by adding the minimal requirements needed for a natural generalization of black hole mechanics.

Let us begin by introducing an equivalence relation on the space of null normals to a non-expanding horizon Δ . The transformation property (2.19) of ω_a under rescalings of ℓ^a shows that ω_a remains unaltered if and only if ℓ^a is rescaled by a constant. Therefore

it is natural to regard two null normals as equivalent if they differ only by a constant rescaling. We will denote each of these equivalence classes by $[\ell]$. In what follows we will be interested in non-expanding horizons Δ , *equipped with* such an equivalence class $[\ell]$ of null normals.

Definition 2: A *weakly isolated horizon* $(\Delta, [\ell])$ consists of a non-expanding horizon Δ , equipped with an equivalence class $[\ell]$ of null normals to it satisfying

$$\mathcal{L}_\ell \omega \hat{=} 0 \text{ for all } \ell \in [\ell]. \quad (2.26)$$

As pointed out above, if this last equation holds for one ℓ , it holds for all ℓ in $[\ell]$. Condition (2.26) strengthens the notion that Δ has ‘reached equilibrium’: where as the intrinsic metric q_{ab} is ‘time-independent’ on any non-expanding horizon, on a weakly isolated horizon, the connection 1-form ω is also ‘time-independent’. Since ℓ^a is normal to Δ , one can regard $K_a{}^b := \nabla_{\underline{a}} \ell^b$ as an analog of the extrinsic curvature of the null surface Δ . In this sense, on a weakly isolated horizon, not only the intrinsic metric q_{ab} but also the extrinsic curvature $K_a{}^b$ is ‘time independent’; while a non-expanding horizon approximates a Killing horizon only to ‘first order’, an isolated horizon approximates it to ‘first’ *and* ‘second’ order.

We will first make a few remarks to elucidate this Definition and then work out some of its consequences, including the zeroth law.

(a) *Remaining rescaling freedom:* A Killing horizon (with S^1 -cross-sections) is automatically a weakly isolated horizon (provided the matter fields satisfy the energy condition of Definition 1). Furthermore, given a non-expanding horizon Δ , one can always find

an equivalence class $[\ell]$ of null-normals such that $(\Delta, [\ell])$ is a weakly isolated horizon. However, condition (2.26) does not by itself single out the appropriate equivalence class $[\ell]$ uniquely. As indicated in Section 2.6.4, one *can* further strengthen the boundary conditions and provide a specific prescription to select the equivalence class $[\ell]$ uniquely. However, for mechanics of isolated horizons, these extra steps are unnecessary. In particular, our analysis will not depend on how the equivalence class $[\ell]$ is chosen. The adverb ‘weakly’ in Definition 2 emphasizes this point.

(b) *Surface gravity*: In the case of Killing horizons $\Delta_{\mathbf{K}}$, surface gravity is defined as the acceleration of the Killing field ξ normal to $\Delta_{\mathbf{K}}$. However, if $\Delta_{\mathbf{K}}$ is a Killing horizon for ξ , it is also a Killing horizon for $c\xi$ for any positive constant c . Hence, surface gravity is not an intrinsic property of $\Delta_{\mathbf{K}}$, but depends also on the choice of a specific Killing field ξ . (Of course the result that the surface gravity is constant on $\Delta_{\mathbf{K}}$ is insensitive to this rescaling freedom.) This ambiguity is generally resolved by selecting a preferred normalization in terms of the structure at infinity. However, in absence of a *global* Killing field this strategy does not work and we simply have to accept the constant rescaling freedom in the definition of surface gravity. In the context of isolated horizons, then, it is natural to keep this freedom.

A weakly isolated horizon is similarly equipped with a preferred family $[\ell]$ of null normals, unique up to constant rescalings. Therefore, it is natural to interpret $\kappa_{(\ell)}$ as the surface gravity associated with ℓ . Under permissible rescalings $\ell \mapsto \tilde{\ell} = c\ell$, the surface gravity transforms via: $\kappa_{(\tilde{\ell})} = c\kappa_{(\ell)}$. Thus, while ω is insensitive to the rescaling freedom in $[\ell]$, $\kappa_{(\ell)}$ captures this freedom fully. One can, if necessary, select a specific ℓ in $[\ell]$ by demanding that $\kappa_{(\ell)}$ be a specific function of the horizon parameters which

are insensitive to this freedom, e.g., by setting $\kappa(\ell) = (R_\Delta/l^2)$, where R_Δ is the horizon radius and $\Lambda = -(1/l^2)$, the cosmological constant.

(c) *Zeroth law*: We will now show that the surface gravity $\kappa(\ell)$ is constant on Δ . Applying the Cartan identity to (ω, ℓ) we have:

$$0 = \mathcal{L}_\ell \omega = d(\ell \cdot \omega) + \ell \cdot d\omega. \quad (2.27)$$

However, we have already seen that ω is curl-free on any non-expanding horizon. Hence $d(\ell \cdot \omega)$ is zero, i.e.,

$$\kappa(\ell) \hat{=} \text{const.} \quad (2.28)$$

Thus, weakly isolated horizons have constant surface gravity; the zeroth law holds on all weakly isolated horizons $(\Delta, [\ell])$. However, as noted above, the precise value of surface gravity $\kappa(\ell)$ depends on the choice of a specific normal ℓ in $[\ell]$, unless $\kappa(\ell)$ vanishes, i.e., $(\Delta, [\ell])$ is an extremal weakly isolated horizon.

2.1.3 Symmetries of weakly isolated horizons

Let us now analyze the symmetries of a weakly isolated horizon. This analysis will play a key role in the construction of the horizon angular momentum and energy.

By its definition, a weakly isolated horizon is equipped with three basic fields: i) the equivalence class $[\ell]$ of null-normals; ii) the intrinsic (degenerate) metric q_{ab} of signature $(0,+)$, and, iii) the one-form ω_a . Therefore it is natural to define symmetries of a given weakly isolated horizon as diffeomorphisms of Δ which preserve these three fields.

At an infinitesimal level, then, a vector field ξ^a on a weakly isolated horizon $(\Delta, [\ell])$ will be called a symmetry if

$$\mathcal{L}_\xi \ell^a = C\ell^a; \quad \mathcal{L}_\xi q_{ab} \hat{=} 0; \quad \text{and} \quad \mathcal{L}_\xi \omega \hat{=} 0; \quad (2.29)$$

for some (possibly vanishing) constant C . Now, any vector field ξ^a on the horizon can be written as a linear combination of the fields ℓ^a and m^a

$$\xi^a = A\ell^a + Bm^a. \quad (2.30)$$

To qualify as a symmetry, the coefficients A, B have to be constrained appropriately. A simple calculation shows that, if the surface gravity $\kappa_{(\ell)}$ is non-zero, these conditions reduce to

$$A = \text{const.} \quad B = \text{const.}, \quad (2.31)$$

while if $\kappa_{(\ell)}$ is non-zero, the condition on A is weakened to

$$A = C(\phi) + Dv, \quad (2.32)$$

where ϕ, v are given by $\ell = \partial/\partial v$, $m = (1/R_\Delta)\partial/\partial\phi$, and D is a constant.

Note that, by the definition of weak isolation, ℓ^a is always an infinitesimal symmetry of $(\Delta, [\ell])$. In the generic, non-extremal case, the only other possible symmetry is the rotational one. Thus, in this case there are only two possibilities:

i) The symmetry group is two dimensional and Abelian. In this case metric q_{ab} and

the connection 1-form ω_a on the horizon are stationary, axi-symmetric. We will refer to these as type I horizons. In this case, we will be able to introduce a natural notion of angular momentum. The event horizons of all known stationary black hole solutions are of type I.

ii) The symmetry group is 1-dimensional and corresponds only to ‘time’ translations along $[\ell]$. In this case, at least one of these fields fails to be axi-symmetric. These are type II horizons.

In the special, extremal case, the group can be infinite dimensional.

2.1.4 The Maxwell field.

So far, we have focused only on gravitational fields at the horizon. Let us now allow Maxwell fields and analyze the implications of the conditions in Definitions 1 and 2.

Recall first that the stress-energy tensor of a Maxwell field \mathbf{F}_{ab} is given by³

$$T_{ab} = \frac{1}{2\pi} \left[\mathbf{F}_{ac} \mathbf{F}_b{}^c - \frac{1}{4} g_{ab} \mathbf{F}_{cd} \mathbf{F}^{cd} \right]. \quad (2.33)$$

Since $R_{ab} \ell^a \ell^b \hat{=} 0$, using field equations at Δ , we conclude $T_{ab} \ell^a \ell^b \hat{=} 0$ which in turn implies $\mathbf{F}_{ab} \ell^a m^b \hat{=} 0$. The condition $R_{ab} \ell^a m^b \hat{=} 0$ does not constrain \mathbf{F}_{ab} any further. Thus, the boundary conditions imply that the Maxwell field is constrained on Δ via:

$$\underline{\mathbf{F}} \equiv \underline{d\mathbf{A}} \hat{=} 0 \quad (2.34)$$

³The numerical factor $1/2\pi$ —rather than $1/4\pi$ —is essential to ensure that the first law has the familiar numerical coefficients even within the family of known solutions.

As in higher dimensions, the electric charge is defined as a surface integral and conserved because of Maxwell's equations. The horizon charge Q_Δ is given by

$$Q_\Delta = \frac{-1}{2\pi} \oint_{S_\Delta} \star \mathbf{F} \quad (2.35)$$

and is well-defined because $\star \mathbf{F}$, the Hodge-dual of \mathbf{F} , is a 1-form. By contrast, since \mathbf{F} is a 2-form, we can not integrate it on a cross-section to obtain a horizon magnetic charge. (One might imagine integrating \mathbf{F} over the whole horizon but this integral vanishes because $\underline{\mathbf{F}} \hat{=} 0$.)

Finally, let us analyze the electromagnetic scalar potential $\Phi_{(\ell)} := -\mathbf{A}_a \ell^a$. Since ω_a is the gravitational analog of \mathbf{A} , $\Phi_{(\ell)}$ can be regarded as the electromagnetic analog of the surface gravity $\kappa_{(\ell)} \equiv \omega_a \ell^a$. Let us first note that since $\underline{\mathbf{F}} \hat{=} 0$, we can always choose a gauge in which the vector potential \mathbf{A} satisfies $\mathcal{L}_\ell \underline{\mathbf{A}} \hat{=} 0$. The standard analysis of Killing horizons strongly suggests that this is a natural gauge choice on the horizon. A vector potential \mathbf{A} satisfying this condition will be said to be *in a gauge adapted to* (Δ, ℓ) . In this gauge, we have:

$$\underline{d} \Phi_{(\ell)} \hat{=} \mathcal{L}_\ell \underline{\mathbf{A}} \hat{=} 0; \quad (2.36)$$

$\Phi_{(\ell)}$ is constant on Δ . This is the electromagnetic counterpart of the zeroth law established above.

2.2 Asymptotic behavior at spatial infinity

In this section we will specify the asymptotic fall-off of our field variables. As mentioned earlier, the conditions at infinity turn out to be rather subtle because of

peculiarities associated with 2+1 dimensions. As usual, the conditions should be weak enough so that a large class of interesting space-times is admissible and strong enough for the action principle, the phase space and Hamiltonians generating interesting canonical transformations to be well-defined. We present such a choice here. We will consider two cases: i) there are no matter fields near infinity; and ii) the only matter field near infinity is the Maxwell case. We separate these cases because, in presence of charges, the second involves additional, significant complications which do not arise in the first case. For both, we will assume that in the neighborhood of spatial infinity

$$A \sim \overset{\circ}{A} + \tilde{A}, \quad e \sim \overset{\circ}{e} + \tilde{e}, \quad (2.37)$$

$$\mathbf{A} \sim \overset{\circ}{\mathbf{A}} + \tilde{\mathbf{A}}, \quad \star \mathbf{F} \sim \overset{\circ}{\star \mathbf{F}} + \star \tilde{\mathbf{F}}, \quad (2.38)$$

where the quantities with the circle on top are certain background fields (which we specify explicitly below) and the ones with tilde are ‘smaller’ quantities with specific fall-off (specified below) in a radial coordinate r defined by the background metric. Our choice of asymptotic conditions is dictated by the following stringent requirements: i) All explicitly known stationary black hole solutions (that we are aware of) belong to the phase-space defined by these conditions; ii) For fields satisfying these asymptotic conditions, the action is finite (on- and off-shell) and differentiable; iii) On the full phase-space, Hamiltonian is finite (on- and off-shell) and differentiable; iv) The symplectic structure is well-defined; and, v) The boundary conditions are preserved by the infinitesimal evolution.

2.2.1 Vacuum space-times.

In this case, the BTZ solutions naturally provide the required background fields. Thus, we assume that a neighborhood of infinity of every space-time of interest is diffeomorphic to a neighborhood of infinity of the BTZ space-time. Then, in terms of the BTZ coordinates t, r, ϕ , we can specify the background co-triads and connection:

$$\overset{\circ}{\ell} = \frac{1}{2}\Lambda r^2 dt + \frac{1}{2}dr, \quad (2.39)$$

$$\overset{\circ}{n} = -dt + \frac{dr}{\Lambda r^2}, \quad (2.40)$$

$$\overset{\circ}{m} = r d\phi, \quad (2.41)$$

$$\overset{\circ}{A}^I = d\phi \ell^I - \frac{1}{2}\Lambda r^2 d\phi n^I - \Lambda r^2 dt m^I + dr m^I \quad (2.42)$$

where ℓ^I, n^I, m^I is a constant internal triad, satisfying our orthogonality and normalization conditions with the fixed internal metric η_{IJ} . An appropriate set of fall-off conditions on the deviations \tilde{e} and \tilde{A} is given by:

$$\tilde{\ell}_t \sim 1, \quad \tilde{\ell}_r \sim 1/r^2, \quad \tilde{\ell}_\phi \sim 1/r, \quad (2.43)$$

$$\tilde{n}_t \sim 1/r^2, \quad \tilde{n}_r \sim 1/r^4, \quad \tilde{n}_\phi \sim 1/r, \quad (2.44)$$

$$\tilde{m}_t \sim 1/r, \quad \tilde{m}_r \sim 1/r^2, \quad \tilde{m}_\phi \sim 1/r, \quad (2.45)$$

$$l_I \tilde{A}_t^I \sim 1/r, \quad l_I \tilde{A}_r^I \sim 1/r, \quad l_I \tilde{A}_\phi^I \sim 1/r, \quad (2.46)$$

$$n_I \tilde{A}_t^I \sim 1/r^2, \quad n_I \tilde{A}_r^I \sim 1/r^2, \quad n_I \tilde{A}_\phi^I \sim 1/r^3, \quad (2.47)$$

$$m_I \tilde{A}_t^I \sim 1/r^2, \quad m_I \tilde{A}_r^I \sim 1/r^2, \quad m_I \tilde{A}_\phi^I \sim 1/r. \quad (2.48)$$

2.2.2 Electro-vacuum space-times.

To accommodate non-zero angular momentum *and* charge, a considerably more complicated choice of the background fields is needed. A natural strategy would be to replace the BTZ background with that provided by the Clément's solution. Thus, for the co-triad we are led to choose

$$\overset{\circ}{\ell} = \left(\frac{1}{2}\Lambda r^2 + \frac{Q^2}{4\pi} \ln \frac{r}{\bar{r}_0}\right) dt + \frac{1}{2} dr, \quad (2.49)$$

$$\overset{\circ}{n} = -dt + (\Lambda r^2)^{-1} dr, \quad (2.50)$$

$$\overset{\circ}{m} = -\frac{\omega Q^2}{2\pi r} \ln \frac{r}{\bar{r}_0} dt + \frac{\omega Q^2 \ln \frac{r}{\bar{r}_0}}{2\pi \Lambda r^3} dr + \left(r + \frac{\omega^2 Q^2}{4\pi r} \ln \frac{r}{\bar{r}_0}\right) d\phi, \quad (2.51)$$

$$\begin{aligned} \overset{\circ}{A} &= l^I \left(\frac{1}{8\pi} \Lambda \omega Q^2 dt + \frac{Q^2 \omega}{8\pi r^2} dr + \left(1 + \frac{Q^2 \omega^2}{4\pi r^2}\right) d\phi \right) + \\ &+ n^I \left(\frac{1}{4\pi} \Lambda \omega Q^2 \ln \frac{r}{\bar{r}_0} dt + \frac{Q^2 \omega}{8\pi r^2} dr - \left(\frac{1}{2} \Lambda r^2 + \frac{1}{8\pi} \Lambda Q^2 \omega^2 + \frac{1}{4\pi} Q^2 \ln \frac{r}{\bar{r}_0}\right) d\phi \right) + \\ &+ m^I \left(\left(-r\Lambda - \frac{\Lambda Q^2 \omega^2}{4\pi r} \ln \frac{r}{\bar{r}_0}\right) dt + \frac{1}{r} dr + \frac{\omega Q^2}{4\pi r} \left(1 - 2 \ln \frac{r}{\bar{r}_0}\right) d\phi \right), \end{aligned} \quad (2.52)$$

and, for the Maxwell field,

$$\overset{\circ}{\mathbf{A}} = Q \ln \frac{r}{\bar{r}_0} (dt - \omega d\phi) + \frac{1}{2} \Lambda Q \omega^2 dt - F(Q) dt, \quad (2.53)$$

$$\star \overset{\circ}{\mathbf{F}} = Q \omega \Lambda dt + Q d\phi, \quad (2.54)$$

where $F(Q)$ is an arbitrary function of the parameter Q such that it vanishes in the limit $Q \rightarrow 0$. Leaving this freedom in the asymptotic conditions is motivated by the analysis of boundary conditions at the horizon as explained in Section 2.5. Although the form of $\star \overset{\circ}{\mathbf{F}}$ is determined by that of the background connection, we have displayed it explicitly

for convenience. The fall off conditions on the permissible deviations are given by

$$\tilde{\mathbf{A}}_t \sim \frac{\ln r}{r}, \quad \tilde{\mathbf{A}}_r \sim \frac{\ln r}{r^2}, \quad \tilde{\mathbf{A}}_\phi \sim \frac{\ln r}{r}, \quad (2.55)$$

$$\star \tilde{\mathbf{F}}_t \sim \frac{1}{r}, \quad \star \tilde{\mathbf{F}}_r \sim \frac{\ln r}{r^2}, \quad \star \tilde{\mathbf{F}}_\phi \sim \frac{1}{r}, \quad (2.56)$$

$$\tilde{l}_t \sim \ln r/r, \quad \tilde{l}_r \sim \ln r/r^2, \quad \tilde{l}_\phi \sim 1/r, \quad (2.57)$$

$$\tilde{n}_t \sim 1/r^3, \quad \tilde{n}_r \sim \ln r/r^4, \quad \tilde{n}_\phi \sim \ln r/r^2, \quad (2.58)$$

$$\tilde{m}_t \sim \ln r/r^2, \quad \tilde{m}_r \sim \ln r/r^4, \quad \tilde{m}_\phi \sim \ln r/r^2, \quad (2.59)$$

$$l_I \tilde{A}_t^I \sim \ln r/r, \quad l_I \tilde{A}_r^I \sim \ln r/r^3, \quad l_I \tilde{A}_\phi^I \sim 1/r, \quad (2.60)$$

$$n_I \tilde{A}_t^I \sim \ln r/r, \quad n_I \tilde{A}_r^I \sim \ln r/r^3, \quad n_I \tilde{A}_\phi^I \sim 1/r^3, \quad (2.61)$$

$$m_I \tilde{A}_t^I \sim \ln r/r^2, \quad m_I \tilde{A}_r^I \sim \ln r/r^3, \quad m_I \tilde{A}_\phi^I \sim \ln r/r^2. \quad (2.62)$$

In these conditions, the parameters Q , ω and \bar{r}_0 do not depend on the coordinates (t, r, ϕ) and we consider only such histories in the Lagrangian formulation for which the variations of these parameters vanish at infinity at the rate $1/r$.

2.3 Action principle.

Fix a manifold \mathcal{M} , topologically $M \times \mathbb{R}$, with an inner boundary Δ which is topologically $S^1 \times \mathbb{R}$, and future and past space-like boundaries M^\pm , which are partial Cauchy surfaces. We will denote the cylinder serving as the boundary at infinity by τ_∞ . We will assume that the complement of a compact set of M is diffeomorphic to the complement of a compact set in \mathbb{R}^2 ; topological complications, if any are confined to a compact set. We equip the inner boundary Δ with an equivalence class of vector fields

$[\ell]$ which are transversal to the S^1 -cross-sections of Δ (and where, as before, $\ell \sim \ell'$ if and only if they are related by a constant rescaling). Finally, we fix on Δ an internal triad (ℓ^I, n^I, m^I) and raise and lower its internal indices with a fixed Minkowskian metric η_{IJ} on the internal space.

We will use a first order framework based on (orthonormal) co-triads e_I and $SO(2, 1)$ connections A_a^I where I takes values in the Lie algebra of $SO(2, 1)$. These fields will be subject to certain boundary conditions. On the inner boundary, we will require: i) $\ell^a \cong \ell^I e_I^a$ belong to $[\ell]$ on Δ ; ii) $(\Delta, [\ell])$ is a weakly isolated horizon; and, iii) \mathbf{A} is in an adapted gauge. At infinity we will impose the conditions specified in the Section 2.2.

The action for 2+1-dimensional Einstein-Maxwell theory is given by:

$$\begin{aligned}
S(e, A, \mathbf{A}) &= \int_{\mathcal{M}} \left(e^I \wedge F_I - \frac{\Lambda}{6} \varepsilon^{IJK} e_I \wedge e_J \wedge e_K \right) - \frac{1}{2} \int_{\tau_\infty} e^I \wedge A_I \\
&- \frac{1}{4\pi} \int_{\mathcal{M}} \mathbf{F} \wedge \star \mathbf{F} + \frac{1}{4\pi} \int_{\tau_\infty} \star \mathbf{F} \wedge \mathbf{A} \\
&+ \int_{\Delta} C_{\Delta} - \int_{\tau_\infty} C_{\infty}.
\end{aligned} \tag{2.63}$$

Here, F_I is the curvature of the gravitational connection A^I , \mathbf{F} the curvature of the electromagnetic connection \mathbf{A} and $\star \mathbf{F}$ its Hodge dual. C_{∞} and C_{Δ} are 2-forms on the world tube at infinity and on the horizon, respectively, which are functionals of asymptotic values of fields on these boundaries which are *constants of motion due to boundary conditions*. (Thus, for example, C_{Δ} can be a function of the horizon parameters such as area, charge, and angular momentum.) In the variational principle we keep values

of fields on the initial and final surface fixed. Under such variations, C_Δ and C_∞ do not change. They are thus irrelevant for the action principle itself and are therefore not specified here. However, when we perform the Legendre transform to pass to the phase space, their values will enter the resulting Hamiltonian. The requirements that the symplectic structure be well defined and conserved and that the Hamiltonian be a well-defined, differentiable function on the phase space will fix the forms of these terms in the next Section. (This subtle difference between Lagrangian and Hamiltonian frameworks is also encountered in higher dimensions [7].) Finally, all integrals should be understood as suitable limits of integrals evaluated on finite regions of \mathcal{M} and their boundaries as the regions expand to fill M and boundaries tend to $\tau_\infty = S_\infty \times \mathbb{R}$. Then, with our boundary conditions the action is finite and its variations are well-defined on the entire space of histories under consideration. In contrast to the asymptotically flat situation (in 3+1 dimensions) considered in earlier papers [9, 14], here the surface terms at infinity are essential to ensure that the action is finite.

Let us vary the action keeping fields fixed on initial and final surfaces M_\pm . Since the calculation is closely analogous to that in 3+1 dimensions [7], we will only sketch the main steps. We have:

$$\delta S_{\text{grav}} = \text{bulk terms} + \int_{\Delta} e^I \wedge \delta A_I, \quad (2.64)$$

where the bulk terms just provide the equations of motion, provided the surface terms vanish. There is no surface term at infinity because of the asymptotic conditions of Section 2.2. Let us examine the surface term at the horizon. It can be further simplified:

our boundary conditions imply that the pull-back \underline{A}^I to the horizon of the gravitational connection A^I is necessarily of the form

$$\underline{A}^I \hat{=} \omega m^I + C \ell^I \quad (2.65)$$

where C is a 1-form on Δ which is annihilated by ℓ^a . (For a Newman-Penrose type derivation, see Appendix.) Hence,

$$\delta S_{\text{grav}} = \text{bulk terms} + \int_{\Delta} e^I \wedge (\delta\omega) m_I, \quad (2.66)$$

where we used the fact that the *internal* triad is kept fixed on Δ . Now, since $\delta\ell^a \hat{=} c_{\delta} \ell^a$ for some constant c_{δ} , $\mathcal{L}_{\ell}\omega \hat{=} 0$ in each history, and the variation $\delta\omega$ vanishes on the initial and final cross-sections of Δ (i.e., on the intersections of Δ with M_{\pm}), we conclude $\delta\omega \hat{=} 0$ on all of Δ . Thus, all the gravitational surface terms vanish under permissible variations.

The situation with the electromagnetic terms is analogous. We have:

$$\delta S_{\text{Maxwell}} = \text{bulk terms} - \frac{1}{2\pi} \int_{\Delta} \delta \mathbf{A} \wedge \star \mathbf{F} \quad (2.67)$$

Since \mathbf{A} is assumed to be in an adapted gauge, $\mathcal{L}_{\ell} \underline{\mathbf{A}} \hat{=} 0$. Again, since $\delta\ell^a \hat{=} c_{\delta} \ell^a$ for some constant c_{δ} and the variation $\delta \underline{\mathbf{A}}$ vanishes on the initial and final cross-sections of Δ , the surface term vanishes.

Thus the variations of the action $S(e, A, \mathbf{A})$ are well-defined and just yield the Einstein-Maxwell equations.

2.4 Legendre transform, phase-space and the first law

In this section we will construct the phase space, use it to introduce the notion of angular momentum and energy on Δ , and obtain the first law. In contrary to previous discussions of black hole mechanics [9, 13, 14], we will arrive at the Hamiltonian description by performing the Legendre transform of the action (2.63). Similarly, however, as in that discussion, our first law will arise as a consistency condition for the time-evolution to be Hamiltonian.

2.4.1 Legendre transform.

To be able to define angular momentum, we will now restrict ourselves to type I isolated horizons of section 2.1.3. Thus, in addition to the structures introduced in the beginning of Section 2.3, we now equip the inner boundary Δ with a vector field φ^a such that its affine parameter φ runs from 0 to 2π . Since Δ is assumed to be of type I, the intrinsic metric q_{ab} and the 1-form ω are Lie-dragged by φ^a . (Note that this condition is imposed *only* at Δ ; we do *not* ask that there be an axial Killing field outside, even in a neighborhood of Δ .) For simplicity, we will also assume that φ^a is tangential to the intersections S_{Δ}^{\pm} of Δ with the past and future surfaces M^{\pm} . Finally, it is convenient to introduce two scalar fields ψ and χ on Δ which serve as ‘potentials’ for the surface gravity $\kappa_{(\ell)}$ and its electro-magnetic analog $\Phi_{(\ell)}$ via: i) $\mathcal{L}_{\ell} \psi \hat{=} \kappa_{(\ell)}$ and $\mathcal{L}_{\ell} \chi \hat{=} \Phi_{(\ell)}$; and, ii) ψ and χ vanish on S_{Δ}^{-} .

In each history, introduce on \mathcal{M} a foliation of space-like 3-manifolds M which are level surfaces of a time coordinate t , and introduce a time-like evolution field t^a

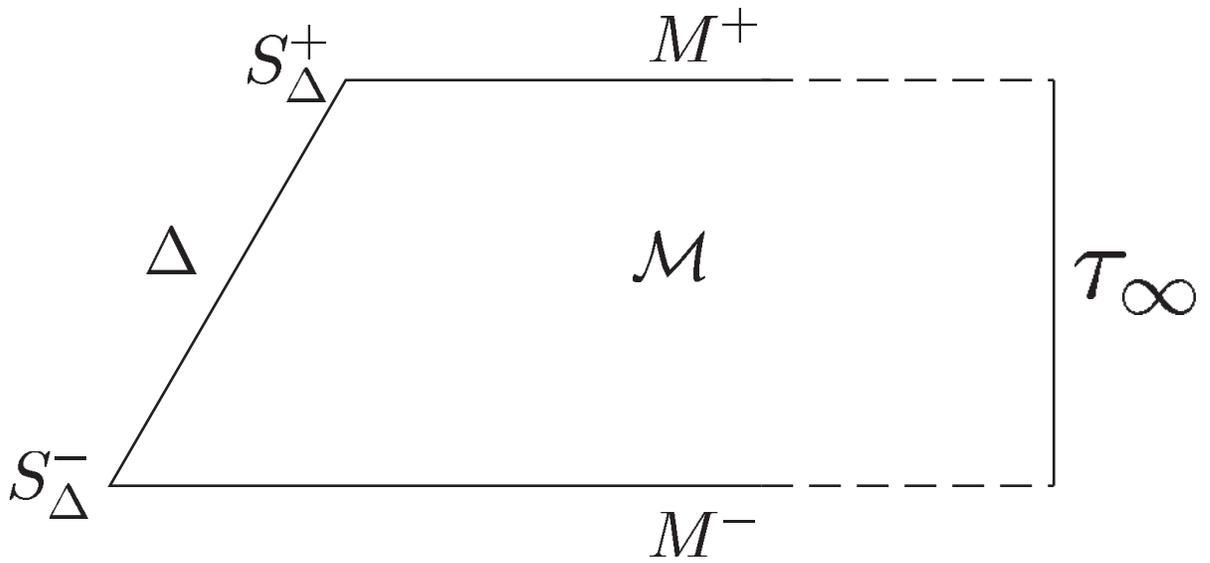


Fig. 2.1. The region of space-time \mathcal{M} under consideration has an internal boundary Δ and is bounded by two partial Cauchy surfaces M^{\pm} which intersect Δ in the 2-spheres S_{Δ}^{\pm} and extend to spatial infinity where they cross the cylinder τ_{∞} .

transverse to this foliation, satisfying $\mathcal{L}_t t \hat{=} 1$. We will assume that t^a is a symmetry field at the two boundaries: it approaches fixed time translation at spatial infinity and has the form $t^a = c_{(t)}\ell^a - \Omega_{(t)}\varphi^a$ on Δ where $c_{(t)}$ and $\Omega_{(t)}$ are constants on Δ . As usual, we can decompose this evolution vector field t^a into a lapse and a shift part: $t^a = Nn^a + N^a$, where n^a is the unit normal to the $t = \text{const}$ slices and N^a is the projection of t^a into these slices. We allow t^a to vary from one space-time to another; in the numerical relativity terminology, the vector field, and hence the corresponding lapse and shift fields, are “live”, particularly at the horizon. The asymptotic value of t^a at spatial infinity will be universal, it will define a fixed time translation Killing field of the asymptotic metric. On the horizon, on the other hand, $c_{(t)}$ and $\Omega_{(t)}$ are allowed to vary from one space-time to another. For example, for physical reasons, in the non-rotating BTZ solution, we would like $\Omega_{(t)}$ to vanish, while in the rotating case we would like it to be non-zero. As we will see, this generalization is essential to obtain a well-defined Hamiltonian as well as the first law.

For notational convenience, from now onwards we will primarily deal with spatial fields defined intrinsically on the $t = \text{const}$ slices. To distinguish these from the space-time, 3-dimensional fields of Section 2.3, the space-time fields will now carry a superscript 3. Then, a simple calculation yields the following 3+1 decomposition of the action:

$$\begin{aligned}
& S(e, A, \mathbf{A}) \\
&= \int_{t_-}^{t_+} dt \left\{ \int_{M_t} [-e^I \wedge \mathcal{L}_t A_I + \frac{1}{2\pi} \star \mathbf{E} \wedge \mathcal{L}_t \mathbf{A}] - \oint_{S_\Delta} [(\mathcal{L}_t \psi) m - \frac{1}{2\pi} (\mathcal{L}_t \chi) \star \mathbf{E}] \right. \\
&+ \int_{M_t} [(t \cdot A^I) D e_I + (t \cdot \mathfrak{A}^I) (F_I - \frac{1}{2} \Lambda \varepsilon_{IJK} e^J \wedge e^K)] \\
&+ \left. \frac{1}{4\pi} \int_{M_t} [N(\star \mathbf{E} \wedge \mathbf{E} - (\star \mathbf{B}) \mathbf{B}) + 2 \star \mathbf{E} \wedge (\vec{N} \cdot \mathbf{B}) + 2(t \cdot \mathfrak{A}) d\mathbf{E}] \right\} \\
&- \int_{t_-}^{t_+} dt \oint_{S_\infty} [(t \cdot \mathfrak{A}^I) A_I + (t \cdot \mathfrak{A})^I e_I] + \frac{1}{4\pi} [(t \cdot \mathfrak{A} \star \mathbf{F}) \mathbf{A} + (t \cdot \mathfrak{A}) \star \mathbf{E}] - [C_\infty^{(t)}] \\
&+ \int_{t_-}^{t_+} dt \oint_{S_\Delta} [\Omega_{(t)}(\varphi \cdot \mathfrak{A}) e_I] - \frac{\Omega_{(t)}}{2\pi} [\varphi \cdot \mathfrak{A} m] - [C_\Delta^{(t)}]. \tag{2.68}
\end{aligned}$$

Here, \star denotes the Hodge-dual on the *spatial* 2-manifolds M_t when applied to *spatial* fields; \mathbf{E} is a pull-back of $n \cdot \mathbf{F}$ to M_t , \mathbf{B} is a pull-back of \mathbf{F}_{ab} to M_t and $C_\Delta^{(t)}$ and $C_\infty^{(t)}$ are 1-forms on the two boundaries given by $C_\Delta^{(t)} = t \cdot C_\Delta$ and $C_\infty^{(t)} = t \cdot C_\infty$.

2.4.2 The phase space.

Our next task is to construct the phase space, introduce a symplectic structure from the appropriate ‘ $p \cdot q$ terms’ in (2.68) and interpret the rest as the (negative of the) Hamiltonian. This is to be achieved so that: i) the symplectic structure is well-defined and ‘conserved on shell’ in spite of the logarithmic divergences at infinity and presence of internal boundaries, and, ii) the Hamiltonian is finite and differentiable on the whole phase space.

As the Legendre transform (2.68) suggests, the phase space Γ will consist of fields on a 2-dimensional manifold M such that $\mathcal{M} = M \times \mathbb{R}$. Topologically, the boundary of

M consists of two circles, spatial infinity S_∞ and a cross section of the horizon S_Δ . Our basic variables will be 1-forms $(A_I, e^I; \mathbf{A}, \mathbf{E})$ on M (and functions $(\psi; \chi)$ on S_Δ). In the space-time language, the 1-forms are all just pull-backs of the respective 3-dimensional fields to Σ . These fields are subject to the isolated horizon conditions on S_Δ and the asymptotic fall-off conditions at S_∞ .

Our next task is to specify the symplectic structure Ω on Γ . Eq (2.68) suggests an obvious candidate, whose the action on any two tangent vectors δ_1 and δ_2 is given by

$$\begin{aligned} \Omega|_{(A,e,\mathbf{A})}(\delta_1, \delta_2) = & - \int_M \left(\delta_1 A^I \wedge \delta_2 e_I - \delta_2 A^I \wedge \delta_1 e_I \right) - \oint_{S_\Delta} (\delta_1 \psi \delta_2 m - \delta_2 \psi \delta_1 m) \\ & + \frac{1}{2\pi} \int_M (\delta_1 \mathbf{A} \wedge \delta_2 \star \mathbf{E} - \delta_2 \mathbf{A} \wedge \delta_1 \star \mathbf{E}) + \frac{1}{2\pi} \oint_{S_\Delta} (\delta_1 \chi \delta_2 \star \mathbf{E} - \delta_2 \chi \delta_1 \star \mathbf{E}) \end{aligned} \quad (2.69)$$

One can verify that, with our boundary conditions, the integral converges in spite of the logarithmic divergences at infinity and, because of field equations, it is conserved in spite of the presence of internal boundaries. More-precisely, given a general solution to the field equations and for general solutions δ_1 and δ_2 to the linearized equations on \mathcal{M} , the integral (2.69) evaluated on a partial Cauchy slice M is well-defined and independent of the choice of that slice [9, 14]. Note that this conservation would *not* hold had we left out the boundary term. The integral of the ‘symplectic current’ constructed from the bulk terms across Δ is compensated by the difference between the boundary terms evaluated at M_\pm [7].

2.4.3 Angular momentum.

Let ϕ^a be a vector field on M with closed orbits whose affine parameter runs between 0 and 2π such that it is a rotational Killing vector of the asymptotic metric at infinity and coincides with the fixed rotational symmetry vector field φ on Δ . Diffeomorphisms generated by ϕ^a naturally induce a vector field

$$\delta_\phi = (\mathcal{L}_\phi A, \mathcal{L}_\phi e, \mathcal{L}_\phi \mathbf{A}, \mathcal{L}_\phi \mathbf{E}) \quad (2.70)$$

on the phase space Γ . It is natural to ask whether it preserves the symplectic structure. The answer is in the affirmative if and only if the 1-form $X^{(\phi)}$ on Γ defined by

$$X^{(\phi)}(\delta) = \Omega(\delta, \delta_\phi) \quad (2.71)$$

is exact, where δ is an arbitrary vector field on Γ . A direct calculation of the right hand side of (2.71) shows that this is indeed the case: $X^{(\phi)} = dJ^{(\phi)}$ where the phase space function $J^{(\phi)}$ is given, up to an additive constant, by:

$$\begin{aligned} J^{(\phi)} &= \oint_{S_\Delta} \left[(\varphi \cdot \omega) m - \frac{1}{2\pi} (\varphi \cdot \mathbf{A}) \star \mathbf{E} \right] + \oint_{S_\infty} \left[(\varphi \cdot e_I) A^I - \frac{1}{2\pi} (\varphi \cdot \mathbf{A}) \star \mathbf{E} \right] \\ &= J_\Delta - J_\infty \end{aligned} \quad (2.72)$$

The requirement that $J^{(\phi)}$ must vanish in the non-rotating BTZ solution implies that the undetermined constant must be zero.

The integral J_∞ at infinity is the total angular momentum of the system, including contributions from matter fields outside Δ . Note that, in contrast to the situation in 3+1 dimensions, this surface term contains a contribution from the Maxwell field. The evaluation of the surface term is delicate. As before, one has to first evaluate the integral on the exterior boundary of a finite region of a partial Cauchy surface and then take the limit to infinity. In the limit, the contributions from the Maxwell and the gravitational parts diverge individually but the sum is finite.⁴ Finally, it is natural to interpret the horizon integral J_Δ as the horizon angular momentum. As in higher dimensions [9], this interpretation is supported by various properties. In particular, if the space-time admits a rotational Killing field ϕ^a in a neighborhood of Δ , then

$$J_\Delta^{\text{grav}} \equiv J_\Delta^{\text{Komar}} = -\frac{1}{2} \oint_{S_\Delta} \star d\phi, \quad (2.73)$$

where J_Δ^{grav} is the gravitational part of the horizon angular momentum J_Δ in (2.72). As one might expect from the presence of a *global* Killing field ϕ , in the Clément solution $J_\Delta = J_\infty$ so that the Hamiltonian J^ϕ generating the diffeomorphism along ϕ vanishes identically. From general symplectic geometry considerations, it follows that this result holds also on the entire connected component of *axi-symmetric* solutions containing the Clément solution. In the general non axi-symmetric case, on the other hand J^ϕ is non-zero and represents the angular momentum in the Maxwell field outside the horizon. Finally, in the Clément solution, one can express $J_\Delta = J_\infty$ in terms of the original

⁴It is because of such subtleties that the expression for the total mass and angular momentum in the general charged, rotating case had been unavailable in 2+1 gravity [29].

parameters as:

$$J_{\Delta}^{\text{Cle}} = J_{\infty}^{\text{Cle}} = -\frac{1}{2\pi} \Lambda \omega a_{\Delta}^2 - \frac{1}{2} \omega Q^2 - \omega Q^2 \ln \frac{a_{\Delta}}{a_0} \quad (2.74)$$

where $a_0 = 2\pi l$.

2.4.4 Energy and the first law.

The Legendre transform of Section 2.4.1 provides us with the expression of the Hamiltonian $H^{(t)}$ generating evolution along t^a :

$$\begin{aligned} H^{(t)} &= \int_M -[(t \cdot {}^3A^I) D e_I + (t \cdot {}^3e^I)(F_I - \frac{1}{2} \Lambda \varepsilon_{IJK} e^J \wedge e^K)] \\ &- \frac{1}{4\pi} [N(\star \mathbf{E} \wedge \mathbf{E} - (\star \mathbf{B})\mathbf{B}) + 2\star \mathbf{E} \wedge (\vec{N} \cdot \mathbf{B}) + 2(t \cdot {}^3\mathbf{A})d\mathbf{E}] \\ &- \oint_{S_{\infty}} [(t \cdot {}^3e)^I A_I + (t \cdot {}^3A)^I e_I] + \frac{1}{8\pi} [(t \cdot {}^3\star \mathbf{F})\mathbf{A} + (t \cdot {}^3\mathbf{A})\mathbf{E} - C_{\infty}^{(t)}] \\ &+ \oint_{S_{\Delta}} [(\varphi \cdot {}^3A) - \frac{1}{2\pi} \varphi \cdot {}^3\mathbf{A} - C_{\Delta}^{(t)}]. \end{aligned} \quad (2.75)$$

So far, the terms $C_{\Delta}^{(t)}$ and $C_{\infty}^{(t)}$ are arbitrary ‘time-independent’ 1-forms, defined in terms of fields at the horizon and infinity, respectively, which can vary from one phase space point to another. Now, we will constrain them by demanding that the Hamiltonian be differentiable on the full phase space. To obtain Hamiltonian’s equations of motion, we have to allow variations along arbitrary vector fields in the phase space, *including those which change the values of the conserved quantities*, such as $\kappa_{(\ell)}$ and $\Phi_{(\ell)}$. Hence the relevant requirement of differentiability on the Hamiltonian is stronger than that on the

action. It is this difference that imposes restrictions on C_Δ and C_∞ which were left unconstrained in the Lagrangian framework.

Using the asymptotic fall-off conditions, one can show that C_∞ is restricted to be:

$$C_\infty^{(t)} = \frac{1}{4}\Lambda\omega^2 Q^2 d\phi. \quad (2.76)$$

where ω and Q are constants that can be read off from the asymptotic behavior in the Section 2.2. Therefore we can also evaluate the boundary term at infinity in the Hamiltonian E_∞^t . It can be expressed in terms of parameters ω , Q and \bar{r}_0 (appearing in the fall-off conditions) as follows

$$E_\infty^t = \frac{1}{4}\Lambda Q^2 \omega^2 - \frac{1}{2}Q^2 \ln \frac{\bar{r}_0}{l} (1 - \Lambda\omega^2) \quad (2.77)$$

As in the case of angular momentum, the actual evaluation of this surface term is somewhat delicate: the gravitational and the electromagnetic terms diverge individually; it is only the sum that is finite.

Analogously, variation of C_Δ is now not zero in general. In fact this term is needed if we want the Hamiltonian framework to make sense. We will show in the next paragraphs that the requirement of differentiability of the Hamiltonian restricts the form of this term.

We can evaluate the horizon surface term in the variation of the Hamiltonian using (2.75) and simplify it using equations of motion, conditions(2.65) and (2.34) on the gravitational and electromagnetic potentials, and the fact that, on the horizon, $\delta\ell \doteq c_\delta \ell$

for some constant c_δ . The resulting condition for differentiability of the Hamiltonian is:

$$\delta E_\Delta^t - \kappa_{(t)} \delta a_\Delta - \Omega_{(t)} \delta J_\Delta - \Phi_{(t)} \delta Q_\Delta = 0, \quad (2.78)$$

where $\kappa_{(t)}$ and $\Phi_{(t)}$ are, respectively, the surface gravity and electric potential on Δ , both associated with $c_{(t)}\ell$. From (2.78) and (2.77) we conclude that the evolution along t^a is Hamiltonian if and only if there exists a function E_Δ^t on the phase space, constructed from fields at the horizon, such that (2.78) holds.

It is natural to identify E_Δ^t as the *horizon energy*. Remarkably, (2.78) is precisely the statement of the first law. Thus, *the first law (2.78) is the necessary and sufficient condition that the time evolution generated by the live vector field t^a on \mathcal{M} is Hamiltonian*. Not every live vector field satisfies this condition. A vector field which does will be said to be *admissible*. We will show in the next section that there exists an *infinite* number of admissible vector fields, whence there is an infinite family of first laws. A natural question is whether one can make a canonical choice, using our knowledge of known exact solutions. We will show that the answer is in the affirmative. The horizon energy defined by this canonical live vector field will be called the *horizon mass*.

Remark: In contrast to the asymptotically flat case treated in higher dimensions [9, 14], in stationary space-times, the expression (2.77) of the energy at infinity does not agree with the Komar integral. In fact, the Komar integral now diverges, while our expression is finite.

2.5 Horizon mass

In this section, we will first introduce a systematic procedure to construct admissible vector fields and then use our knowledge of stationary, axi-symmetric black hole solutions to introduce preferred admissible vector fields on *all* space-times in the phase space Γ .

2.5.1 Admissible vector fields

Note first that (2.78) implies that t^a is an admissible vector field only if $E_{\Delta}^t, \kappa_{(t)}, \Omega_{(t)}$ and $\Phi_{(t)}$ are all functions *only* of horizon parameters $(a_{\Delta}, J_{\Delta}, Q_{\Delta})$. Furthermore, following rather stringent condition must be met at the horizon:

$$\frac{\partial \kappa_{(t)}}{\partial J_{\Delta}} = \frac{\partial \Omega_{(t)}}{\partial a_{\Delta}}. \quad (2.79)$$

We will turn the argument around and use this equation to construct admissible vector fields. Let us begin by fixing a ‘suitably regular’ function $\kappa_0(a_{\Delta}, J_{\Delta}, Q_{\Delta})$ of the horizon parameters. Now, given a general solution, the surface gravity $\kappa_{(\ell)}$ of the null generator ℓ will not equal κ_0 . However, there will be a unique constant c such that $\kappa_{(c\ell)} = \kappa_0$. Next, we find a constant $\Omega_{(t)}$ by ‘integrating’ (2.79):

$$\Omega_t = \int_{\infty}^a \frac{\partial \kappa_0}{\partial J_{\Delta}} da_{\Delta} + F(Q_{\Delta}, J_{\Delta}), \quad (2.80)$$

where F is an arbitrary function of the two parameters. (The qualification ‘suitably regular’ above is meant to ensure that the integral on the right is well-defined.) Finally,

we can fix the arbitrariness in $\Omega_{(t)}$ by imposing the following physical requirement:

$$\lim_{\substack{J_\Delta, Q_\Delta = \text{const.} \\ a_\Delta \rightarrow \infty}} \Omega_{(t)} = 0.$$

Now, in any given solution in the phase space, we choose any evolution vector field t^a such that it tends to the fixed asymptotic time-translation at infinity and satisfies $t^a \hat{=} c\ell^a - \Omega_{(t)}\varphi^a$ on Δ . It is straightforward to check that, by construction, this evolution vector field is admissible if the Maxwell field of the solution under consideration vanishes on Δ .

If the Maxwell field on Δ is non-zero, we must also ensure that the Maxwell gauge is fixed appropriately for (2.78) to hold. Recall first that in an adapted gauge, the Maxwell potential \mathbf{A} is such that $\Phi_{(\ell)} \hat{=} -\mathbf{A}_a \ell^a$ is constant on Δ . However, the *value* of the constant, i.e., its possible dependence on the horizon parameters, is still completely unconstrained. (2.78) imposes severe restrictions on this choice: $\Phi_{(t)} \equiv \Phi_{(c\ell)}$ must satisfy

$$\begin{aligned} \frac{\partial \Phi_{(t)}}{\partial a_\Delta} &= \frac{\partial \kappa_{(t)}}{\partial Q_\Delta}, \\ \frac{\partial \Phi_{(t)}}{\partial J_\Delta} &= \frac{\partial \Omega_{(t)}}{\partial Q_\Delta}. \end{aligned} \tag{2.81}$$

Again, we can just *use* this condition to constrain $\Phi_{(t)}$: setting $\kappa_{(t)} = \kappa_0$ and using $\Omega_{(t)}$ determined above, we can simply ‘integrate these equations’ to determine $\Phi_{(t)}$ up to an additive function $F(Q)$ of the charge, Q_Δ . In 3+1 dimensions, there was a natural way to fix this freedom [7, 9]: One could just impose the physical requirement that

$\Phi_{(t)}$ should vanish in the limit of large areas, with fixed charge and angular momentum. Unfortunately, in 2+1 dimensions this strategy is *not* viable because now, in presence of a non-zero charge, the potential *diverges* at spatial infinity! Therefore now $\Phi_{(t)}$ is not completely determined on Δ . The only physical restriction we impose on $F(Q)$ is through

$$\lim_{\substack{J_{\Delta}, a_{\Delta} = \text{const.} \\ Q_{\Delta} \rightarrow 0}} \Phi_{(t)} = 0, \quad (2.82)$$

which only determines the value of $F(Q)$ at $Q = 0$.

Note, however, that the remaining freedom is irrelevant for the purpose of defining ‘admissible vector fields’: the vector fields t^a constructed above are admissible for *every* choice of $\Phi_{(t)}$ satisfying (2.81). However, the choice of $\Phi_{(t)}$ will, in general, enter the expression of the horizon energy E_{Δ}^t which is obtained by ‘integrating’ (2.78).

2.5.2 Preferred admissible vector fields

In this section, we will indicate how one can use the known solutions to fix κ_0 and $\Phi_{(t)}$ in a ‘canonical fashion’. The resulting E_{Δ}^t can be naturally interpreted as the horizon mass. Several subtleties arise in presence of a non-zero charge and angular momentum. Therefore we will divide the discussion into three cases.

2.5.2.1 The case with $\mathbf{F}_{ab} \hat{=} 0$

Let us suppose that the Maxwell field vanishes on the horizon. Then we only have to choose a function κ_0 of the horizon parameters a_{Δ}, J_{Δ} . However, in this case, there is a *unique* BTZ black hole solution for each choice of these two parameters. Therefore,

it is natural to set $\kappa_0 = \kappa_{(t)}^{\text{BTZ}}$, where t^a is the ‘canonical’ time-translation Killing field of the BTZ black hole:

$$\kappa_0 \equiv \kappa_{(t)} = -\frac{\Lambda a_\Delta}{2\pi} - \frac{2\pi J_\Delta^2}{a_\Delta^3}. \quad (2.83)$$

Our construction of Section 2.5.1 implies

$$\Omega_{(t)} = \frac{2\pi J_\Delta}{a_\Delta^2}. \quad (2.84)$$

We can now integrate out the first law to obtain the expression of the horizon energy up to an undetermined additive constant. We eliminate this freedom through the physical requirement: $\lim_{a_\Delta \rightarrow 0} E_\Delta^t = 0$ for non-rotating isolated horizons. The resulting horizon mass is given as a function of the horizon parameters as:

$$M_\Delta = -\frac{1}{4\pi}\Lambda a_\Delta^2 + \frac{\pi J_\Delta^2}{a_\Delta^2}. \quad (2.85)$$

The functional form of M_Δ is the same as that in the BTZ family. However, the (2.85) was not simply postulated but derived systematically from Hamiltonian considerations and applies to *all* isolated horizons including those which may admit electromagnetic radiation in the exterior region, away from Δ . In presence of such radiation, M_Δ will not equal the mass at infinity.

2.5.2.2 Charged, non-rotating horizons

Next, let us consider non-rotating horizons with electric charge. In this case $\Omega_{(t)} \hat{=} 0$. To fix $\kappa_{(t)}$ and $\Omega_{(t)}$, we again use the facts that there is a unique BTZ [29]

solution for each value of a_Δ and Q_Δ , and the required κ_0 and $\Omega_{(t)}$ depend *only* on a_Δ and Q_Δ . Expressed in Eddington-Finkelstein-like coordinates space-time metrics of these solutions take the form

$$ds^2 = -N^2 dv^2 + 2dvdr + r^2 d\phi^2, \quad (2.86)$$

where

$$N = -\frac{M}{\pi} + \frac{r^2}{l^2} - \frac{Q^2}{2\pi} \ln \frac{r}{r_0} \quad (2.87)$$

and the electro-magnetic potentials \mathbf{A} are given by

$$\mathbf{A} = Q \ln\left(\frac{r}{r_0}\right) dv. \quad (2.88)$$

Using the Killing field $t = \partial_v$ as the evolution field, it is again natural to set $\kappa_0 = \kappa(t)$ and $\Phi_{(t)} \hat{=} -\mathbf{A}_a t^a$. Thus, we now have:

$$\kappa_0 \equiv \kappa(t) = -\Lambda \frac{a_\Delta}{2\pi} - \frac{Q_\Delta^2}{2a_\Delta}. \quad (2.89)$$

and

$$\Phi_{(t)} = -Q_\Delta \ln \frac{a_\Delta}{a_0}, \quad (2.90)$$

where $a_0 = 2\pi r_0$ is an undetermined constant. In fact, even its dependence on the horizon parameters is not completely fixed at this point. As explained in Section 2.5.1, while it does not depend on the area a_Δ , it can be an arbitrary function of the charge

Q_Δ subject to the condition (2.82). The horizon mass is therefore given by:

$$M_\Delta = -\frac{\Lambda a_\Delta^2}{4\pi} - \frac{1}{2}Q_\Delta^2 \ln \frac{a_\Delta}{a_0}. \quad (2.91)$$

Again, this formula now holds for arbitrary non-rotating weakly isolated horizons Δ . However, in striking contrast to the situation in the 3+1 Einstein-Maxwell theory, now the expression contains a new undetermined constant.

2.5.2.3 Charged rotating black hole.

Finally, let us consider the general case. We can now use our Example 3 —the Clément solution— to fix κ_0 and $\Omega_{(t)}$. However, before we can introduce admissible vector fields, we have to address a technical issue. Although the electromagnetic potential used by Clément is in an adapted gauge on Δ , the resulting $\Phi_{(t)}$ does *not* satisfy conditions (2.81) which are necessary for the first law to hold. Moreover, the potential \mathbf{A} in this solution contains a free scale parameter and each choice provides a new gauge inequivalent solution. Therefore, we have to choose a suitable parameter and make a suitable gauge transformation. In this modified gauge \mathbf{A} is given by:

$$\mathbf{A} = Q \ln \frac{r}{l} (dv - \omega d\phi) + \frac{1}{2}Q\omega^2 \Lambda dv - F(Q)dv, \quad (2.92)$$

where we again encounter arbitrary function of charge $F(Q)$, subject only to the condition (2.82).

Let us now express the horizon parameters a_Δ , Q_Δ and J_Δ in terms of the parameters ω , Q , \bar{r}_0 that appear in the solution:

$$\begin{aligned} a_\Delta &= \frac{2\pi r_\Delta}{\sqrt{1 + \omega^2 \Lambda}} \\ Q_\Delta &= Q \\ J_\Delta &= -\omega \left(\frac{\Lambda a_\Delta^2}{2\pi} + \frac{1}{2} Q_\Delta^2 + Q_\Delta^2 \ln \frac{a_\Delta}{2\pi l} \right), \end{aligned} \quad (2.93)$$

where r_Δ is given by $N(r = r_\Delta) = 0$. (The parameter \bar{r}_0 enters the expression of the area through this condition.)

Now we can calculate the surface gravity $\kappa_{(t)}$ and the electric potential $\Phi_{(t)}$ corresponding to the stationary Killing field t^a of the Clément solution:

$$\kappa_{(t)} = - \left(\frac{\Lambda a_\Delta}{2\pi} + \frac{Q_\Delta^2}{2a_\Delta} \right) (1 + \omega^2 \Lambda), \quad (2.94)$$

$$\Phi_{(t)} = -Q_\Delta \ln \frac{a_\Delta}{a_0} (1 + \omega^2 \Lambda) - \frac{1}{2} Q_\Delta \omega^2 \Lambda + F(Q_\Delta), \quad (2.95)$$

where $a_0 = 2\pi l$.

In a general solution in the phase space, then, we set $\kappa_0 = \kappa_{(t)}$ given above. Our procedure of Section 2.5.1 provides the required $\Omega_{(t)}$:

$$\Omega_{(t)} = -N^\phi(r = r_\Delta) = -\omega \Lambda. \quad (2.96)$$

The triplet $\kappa_{(t)}$, $\Omega_{(t)}$, $\Phi_{(t)}$ can now be used to construct preferred admissible vector fields and by integrating the corresponding first law, we obtain the expression of the horizon

mass:

$$M_{\Delta} = -\frac{\Lambda a_{\Delta}^2}{4\pi} - \frac{1}{2}Q_{\Delta}^2 \ln \frac{a_{\Delta}}{a_0} + \frac{1}{2} \frac{\Lambda J_{\Delta}^2}{\frac{\Lambda a_{\Delta}^2}{2\pi} + \frac{1}{2}Q_{\Delta}^2 + Q_{\Delta}^2 \ln \frac{a_{\Delta}}{a_0}} + \tilde{F}(Q_{\Delta}) \quad (2.97)$$

where we have again eliminated an undetermined constant by requiring that every non-rotating, uncharged horizon should have vanishing mass in the limit of vanishing area. Also, $\tilde{F}(Q_{\Delta}) = \int F(Q_{\Delta})dQ_{\Delta}$. This is our *general* expression of the horizon mass.

Finally, we can compare our formula for the energy of the horizon with the energy at infinity —Eq.(2.77). It is easy to check that in the case of Clément’s solution, the two expressions are equal to each other, just as one might expect from results in 3+1 dimensions. General symplectic arguments [4, 9] now imply that this equality between our horizon mass and the mass at infinity must continue to hold for *all* stationary space-times in the connected component of the phase-space containing Clément (or, equivalently, BTZ) solutions.

2.6 Horizon geometry

In this section, we examine geometrical structures on Δ and analyze their interplay with the field equations.

The section is divided into four parts. In the first, we will show that every non-expanding horizon is naturally equipped with an intrinsic derivative operator \mathcal{D} . In the second, we will turn to weakly isolated horizons and, using field equations, isolate the freely specifiable data on Δ . In the third, we will show that every *non-extremal* weakly isolated horizon admits a natural foliation (irrespective of whether it is axi-symmetric,

i.e., type I in the terminology of section 2.1.3). In the last sub-section we strengthen the definition of weak isolation to introduce the notion of isolated horizons. While a non-expanding horizon Δ can be made weakly isolated by suitably choosing $[\ell]$ in infinitely many inequivalent ways, generically, it admits a unique $[\ell]$ which makes it isolated.

2.6.1 A natural derivative operator

Let Δ be a non-expanding horizon. Had it been space-like or time-like, its intrinsic metric would have selected a (torsion-free) derivative operator uniquely. However, since it is null, there are infinitely many derivative operators which are compatible with it. Nonetheless, because Δ is expansion and shear-free, as in higher dimensions, the full space-time derivative operator ∇ induces a preferred intrinsic derivative \mathcal{D} on it. Given a vector field X^a or a 1-form f_a , on Δ , we have:

$$Y^a \mathcal{D}_a X^b \hat{=} Y^a \nabla_a \bar{X}^b, \quad \text{and} \quad Y^a Z^b \mathcal{D}_a f_b \hat{=} Y^a Z^b \nabla_a \bar{f}_b, \quad (2.98)$$

where Y^a, Z^a are arbitrary vector fields tangential to Δ and \bar{X}^a and \bar{f}_a are arbitrary smooth extensions of X^a and f_a to a space-time neighborhood of Δ . It is easy to check that \mathcal{D} is well-defined: the right hand sides of the two equations are independent of the choice of extension and the right hand side of the first equation is again tangential to Δ . Since $\nabla_a g_{bc} = 0$ in space-time, $\mathcal{D}_a q_{bc} \hat{=} 0$ on Δ ; as expected, \mathcal{D} is compatible with q_{bc} .

What information does \mathcal{D} have beyond that contained in the degenerate metric q_{ab} on Δ ? The action of \mathcal{D} on tensors is completely determined by that on all 1-forms defined intrinsically on Δ . Let f be a 1-form satisfying $f \cdot \ell \hat{=} 0$ and $\mathcal{L}_\ell f \hat{=} 0$. Then it is

easy to verify that the action of \mathcal{D} on f can be expressed just in terms of exterior and Lie derivatives:

$$2\mathcal{D}_a f_b = 2\mathcal{D}_{[a} f_{b]} + \mathcal{L}_{\hat{f}} q_{ab} \quad (2.99)$$

where the vector field $\hat{f}^a = m^a m^b f_b$ is independent of the choice of the unit space-like vector m^a tangential to Δ . Thus, the action of \mathcal{D} on these 1-forms is determined by q_{ab} . Therefore, \mathcal{D} is completely determined by its action $\mathcal{D}_a n_b =: S_{ab}$ on 1-forms n satisfying $n \cdot \ell = -1$. Without loss of generality, we can assume that n satisfies, in addition,

$$\mathcal{L}_\ell n \hat{=} 0 \quad \text{and} \quad dn \hat{=} 0 \quad (2.100)$$

on Δ . Then S_{ab} is symmetric. Since $dn \hat{=} 0$, we have $n = -dv$ for some function v on Δ satisfying $\mathcal{L}_\ell v = 1$. The $v = \text{const}$ cross-sections will be assumed to be topologically S^1 and denoted $\tilde{\Delta}$.

Now,

$$\ell^b S_{ab} \hat{=} -n_b \mathcal{D}_a \ell^b \hat{=} -n_b \nabla_{\underline{a}} \ell^b \hat{=} \omega_a \quad (2.101)$$

Thus, part of the ‘new’ information in \mathcal{D} is contained in the 1-form ω of Section 2.1.1. The rest is contained in the projection $\tilde{\mu}$ of S_{ab} on $\tilde{\Delta}$: $\tilde{\mu} : \hat{=} \tilde{m}^a \tilde{m}^b \mathcal{D}_a n_b$ where \tilde{m}^a is the unit vector field tangential to $\tilde{\Delta}$. This function $\tilde{\mu}$ is the ‘transversal expansion’ of n (see Appendix).

Following the terminology used in higher dimensions [10], we will refer to the pair (q, \mathcal{D}) as the *intrinsic geometry* of Δ . Thus, the intrinsic geometry is determined by a triplet $(m^a, \omega_a, \tilde{\mu})$ on Δ for any choice of n satisfying (2.100).

2.6.2 Field equations and ‘free-data’ on a weakly isolated horizon

Consider a weakly isolated horizon $(\Delta, [\ell])$. In this sub-section we will analyze the restrictions imposed by field equations on the intrinsic geometry of Δ and extract the free-data that suffices to determine this geometry.

We already know that the pair (g, \mathcal{D}) satisfies

$$q_{ab}\ell^b \hat{=} 0; \quad \mathcal{L}_\ell q_{ab} \hat{=} 0; \quad \mathcal{D}_a q_{bc} \hat{=} 0; \quad \mathcal{D}_a \ell^b \hat{=} \omega_a \ell^b; \quad \mathcal{L}_\ell \omega \hat{=} 0; \quad (2.102)$$

and Eqs (2.99) and (2.101). We now want to analyze the further constraints imposed by the full field equations: $E_{ab} := R_{ab} - \frac{1}{2}Rg_{ab} + \Lambda g_{ab} - T_{ab} \hat{=} 0$. We already saw in Section 2.1 that weak isolation implies $R_{ab}\ell^a\ell^b \hat{=} 0$ and $R_{ab}\ell^a m^b \hat{=} 0$. Hence these projections of the field equations do not further constrain the horizon geometry; they only restrict the matter fields at the horizon. It turns out that the projections $E_{ab}n^a\ell^b \hat{=} 0$ and $E_{ab}n^a m^b \hat{=} 0$ dictate the propagation of ω (or, the Newman-Penrose spin coefficients ϵ and π of Appendix) ‘off’ Δ while $E_{ab}n^a n^b \hat{=} 0$ dictates the propagation of $\tilde{\mu}$ ‘off’ Δ . Thus, these equations do not constrain the intrinsic horizon geometry in any way. (For details, see Appendix)

The only new constraint comes from the equation $E_{ab}\tilde{m}^a\tilde{m}^b \hat{=} 0$. Had Δ been a space-like surface, the analogous equations would have given the evolution equations. In the present case they also dictate an ‘evolution’ —that of $\tilde{\mu}$ — but now *within* Δ . We have:

$$\mathcal{L}_\ell \tilde{\mu} \hat{=} -\kappa_{(\ell)} \tilde{\mu} + \tilde{m}^a \tilde{m}^b \mathcal{D}_{(a} \tilde{\omega}_{b)} + \tilde{\omega}_a \tilde{\omega}_b + \frac{1}{2} \tilde{m}^a \tilde{m}^b t_{ab}. \quad (2.103)$$

where $\tilde{\omega}_a = \tilde{m}_a \tilde{m}^b \omega_b$ is the projection of ω on $\tilde{\Delta}$ and $t_{ab} = (T_{ab} + (2\Lambda - T)g_{ab})$. This exhausts the field equations.

We can now specify the freely specifiable part of the horizon geometry. Fix a 2-manifold Δ , topologically $S^1 \times \mathbb{R}$, and equip it with a vector field ℓ along the \mathbb{R} direction. Fix a foliation by circles labeled by $v = \text{const}$, where $\mathcal{L}_\ell v = 1$. On any one cross-section, $\tilde{\Delta}$, fix a function $\tilde{\mu}$ and 1-forms m and ω such that m is nowhere vanishing, $m \cdot \ell = 0$ and $\omega \cdot \ell = \kappa(\ell)$, where $\kappa(\ell)$ is a constant. This is the free data. ‘Evolve’ it to all of Δ through $\mathcal{L}_\ell m = 0$, $\mathcal{L}_\ell \omega = 0$ and (2.103), for a given t_{ab} on Δ . Then the triplet $(m, \omega, \tilde{\mu})$ on Δ provides us with the intrinsic geometry of a weakly isolated horizon.

Finally, under the mild assumption, $\mathcal{L}_\ell R_{\underline{q}b} \hat{=} 0$, we can integrate (2.103) to obtain:

$$\tilde{\mu} = e^{-\kappa(\ell)v} \tilde{\mu}^0 + \frac{1}{\kappa(\ell)} \left[\tilde{m}^a \tilde{m}^b \mathcal{D}_{(a} \tilde{\omega}_{b)} + \tilde{\omega}_a \tilde{\omega}_b + \frac{1}{2} \tilde{m}^a \tilde{m}^b t_{ab} \right] \quad (2.104)$$

if $\kappa(\ell) \neq 0$ and

$$\tilde{\mu} = \tilde{\mu}^0 + \left[\tilde{m}^a \tilde{m}^b \mathcal{D}_{(a} \tilde{\omega}_{b)} + \tilde{\omega}_a \tilde{\omega}_b + \frac{1}{2} \tilde{m}^a \tilde{m}^b t_{ab} \right] v \quad (2.105)$$

if $\kappa(\ell) \hat{=} 0$, where $\mathcal{L}_\ell \tilde{\mu}^0 \hat{=} 0$. These solutions bring out the generalization entailed in considering weakly isolated horizons in place of Killing horizons: now, even the *intrinsic geometry* on the horizon (as defined above) can be time-dependent. In spite of this, the zeroth and first laws hold on any weakly isolated horizon.

2.6.3 Good cuts of non-extremal weakly isolated horizon

As in higher dimensions [10], every non-extremal weakly isolated horizon admits a natural foliation. However, because the Weyl tensor vanishes in 3 dimensions, and the first homology of Δ is now non-trivial, the construction is now somewhat different.

Recall first that every non-expanding horizon Δ carries a natural closed 1-form \underline{m}_a . Since $d\underline{m} \hat{=} 0$, \underline{m} generates a class in $H^1(\Delta)$, the first cohomology of Δ . Since Δ is isomorphic to $S^1 \times \mathbb{R}$, we have:

$$H^1(\Delta) = H^0(S^1) = \mathbb{R}. \quad (2.106)$$

Since the integral of \underline{m} over any cross section yields $-2\pi R_\Delta$, \underline{m} is not in the zero class of $H^1(\Delta)$. Next, recall that the 1-form ω on Δ is also closed. Therefore, it must be of the form

$$\omega_a = C \underline{m}_a + \partial_a \psi, \quad (2.107)$$

for some constant $C \in \mathbb{R}$ and some function ψ on Δ . The function ψ can now be used to define a preferred foliation of the horizon. Since

$$\mathcal{L}_\ell \psi = \ell^a \omega_a = \kappa_{(\ell)}, \quad (2.108)$$

with $\kappa_{(\ell)}$ constant on the horizon, the lines $\psi = \text{const}$ define a foliation of Δ provided $\kappa_{(\ell)}$ is non-zero. If the vector fields $[\ell]$ are complete, as for example in stationary black holes, the leaves of the foliation are guaranteed to be topologically S^1 .

In the Newman-Penrose type framework of Appendix, the preferred foliation is characterized by the fact that the spin-coefficient π is constant on each leaf. Therefore, if the underlying space-time is axi-symmetric in a neighborhood of Δ , the foliation coincides with the integral curves of the rotational Killing vector. In BTZ space-times, ω is given by (2.20), $d\psi \hat{=} -\kappa_{(\ell)} n$, whence $\psi \hat{=} \kappa_{(\ell)} v$.

Example 1: Let us return to our BTZ example. What is the function ψ in this case? Recall first the expression (2.20) of ω_a . Since N^ϕ is constant on Δ , we have

$$\partial_a \psi = -\kappa_{(\ell)} n_a. \quad (2.109)$$

We have chosen n_a to be $-dv$ where v is one of the coordinates of the BTZ metric. Thus, ψ is simply given by $\psi = \kappa v$, where v is the Eddington-Finkelstein-like coordinate (see (2.1)).

2.6.4 Isolated horizons and uniqueness of $[\ell]$

Let Δ be a non-expanding horizon. Fix any cross-section, choose any null normal to Δ on the cross-section and propagate it by the geodesic equation to obtain a null normal ℓ_0 on Δ . Then $(\Delta, [\ell_0])$ is an extremal weakly isolated horizon. Denote by v_0 its affine parameter. Set

$$\ell^a \hat{=} \kappa_{(\ell)} (v_0 - B) \ell_0^a \quad (2.110)$$

where $\kappa_{(\ell)}$ is a non-zero constant and $\mathcal{L}_\ell B \hat{=} 0$. It is straightforward to check that ℓ is a null normal with surface gravity $\kappa_{(\ell)}$ and every null normal with surface gravity $\kappa_{(\ell)}$

arises in this way. Similarly, any null normal with zero surface gravity is given by

$$\ell'^a \cong \left(\frac{1}{A}\right) \ell_0^a \quad (2.111)$$

for some function A satisfying $\mathcal{L}_{\ell_0} A \cong 0$. To summarize, simply by restricting the null normals ℓ to lie in a suitably chosen equivalence class $[\ell]$, from any given non-expanding horizon Δ , we can construct a weakly isolated horizon $(\Delta, [\ell])$ which is either extremal or non-extremal. However, because of the arbitrary functions involved in (2.111) and (2.110), *there is an infinite dimensional freedom in this construction.*

It is natural to ask if this freedom can be reduced by strengthening the notion of isolation. The answer is in the affirmative.

Definition 3 An *isolated* horizon $(\Delta, [\ell])$ consists of a non-expanding horizon Δ equipped with an equivalence class $[\ell]$ of null normals satisfying

$$(\mathcal{D}_a \mathcal{L}_\ell - \mathcal{L}_\ell \mathcal{D}_a) X^b \cong 0 \quad (2.112)$$

for all vector fields X tangential to Δ . As before, ℓ is equivalent to ℓ' if and only if $\ell' = c\ell$ for some positive constant c and if condition (2.112) holds for one null normal ℓ , it holds for all null normals in $[\ell]$. If a non-expanding horizon Δ admits a normal ℓ satisfying (2.112), we will say its geometry *admits an isolated horizon structure.*

Before analyzing the remaining freedom in the choice of $[\ell]$, let us examine the difference between weakly isolated and isolated horizons. Note first that the weak isolation

condition can be written as

$$(\mathcal{D}_a \mathcal{L}_\ell - \mathcal{L}_\ell \mathcal{D}_a) \ell^b \hat{=} 0 .$$

Thus, the present strengthening of that notion asks that the commutator of \mathcal{D} and \mathcal{L}_ℓ vanish on *all* vector fields on Δ , not just on ℓ . Since the information in \mathcal{D} (beyond q_{ab}) is contained in the pair $(\omega, \tilde{\mu})$, the additional condition is *precisely* $\mathcal{L}_\ell \tilde{\mu} \hat{=} 0$. (While $\tilde{\mu}$ depends on the choice of cross sections $\tilde{\Delta}$, $\mathcal{L}_\ell \tilde{\mu}$ does not.) Next, it is straightforward to check that, on any isolated horizon, the pull-back of the full space-time curvature is time-independent: $\mathcal{L}_\ell R_{\underline{q}\underline{b}} \hat{=} 0$. (Since $R_{\underline{a}\underline{b}} \ell^b \hat{=} 0$, it follows that $R_{\underline{q}\underline{b}}$ is Lie-dragged by *every* null normal, $f\ell^a$, to Δ .) Thus, on an isolated horizon, the restriction that led us to the solution (2.104) and (2.105) for $\tilde{\mu}$ is automatically satisfied. Therefore, in the non-extremal case, $\tilde{\mu}^0$ of (2.104) vanishes while in the extremal case the quantity in the square brackets in (2.105) must vanish. In both cases, the freely specifiable data of Section 2.6.2 is restricted; $\tilde{\omega}$ and $\tilde{\mu}$ can not be specified freely on a cross-section $\tilde{\Delta}$, but are constrained.

Finally, let us analyze the issue of existence and uniqueness of $[\ell]$. Let Δ be a non-expanding horizon. We can always choose a null normal ℓ such that (Δ, ℓ) is a non-extremal, weakly isolated horizon. Let us further suppose that $(\Delta, [\ell])$ is not already an isolated horizon, i.e., $\mathcal{L}_\ell \mu \not\hat{=} 0$ and ask if we can find another null normal $\ell' = f\ell$ such that $(\Delta, [\ell'])$ is isolated. Now, using the definition of weak isolation, it is straightforward to check:

$$[\mathcal{L}_\ell, \mathcal{D}]_a K_b \hat{=} C_{ab}{}^c K_c \quad \text{where} \quad C_{ab}{}^c \hat{=} -N q_{ab} \ell^c \quad (2.113)$$

for any 1-form K_b on Δ . The function N is given by $N \hat{=} \mathcal{L}_\ell \mu$. Under the rescaling $\ell' \hat{=} f \ell$, we have

$$(N' - N)q_{ab} \hat{=} 2\omega_{(a} \mathcal{D}_{b)} f + \mathcal{D}_a \mathcal{D}_b f \quad (2.114)$$

By transvecting this equation with ℓ^b we obtain

$$\mathcal{D}_a (\mathcal{L}_\ell f + \kappa_{(\ell)} f) \hat{=} 0 \quad (2.115)$$

which implies

$$f \hat{=} B e^{-\kappa_{(\ell)} v} + \frac{\kappa_{(\ell')}}{\kappa_{(\ell)}}, \quad \text{with } \mathcal{L}_\ell B \hat{=} 0. \quad (2.116)$$

Thus, the key question now is: Does there exist a function B such that $N' \hat{=} 0$? Substituting for f in (2.114) and using the expression (2.104) of μ , we conclude that N' vanishes if and only if B satisfies

$$\mathbf{M} \cdot B := [\tilde{\mathcal{D}}^2 + 2\tilde{\pi} \tilde{\mathcal{D}} + \tilde{\mathcal{D}} \tilde{\pi} + \tilde{\pi}^2 + \tilde{m}^a \tilde{m}^b R_{ab}] B \hat{=} \kappa_{(\ell')} \tilde{\mu}^0 \quad (2.117)$$

on any cross-section $\tilde{\Delta}$ of Δ , where \tilde{m}^a is the unit vector field tangential to $\tilde{\Delta}$, $\tilde{\mathcal{D}} = \tilde{m}^a \mathcal{D}_a$ and $\tilde{\pi} = \tilde{m}^a \omega_a$. Note that, given any cross-section $\tilde{\Delta}$, the operator is completely determined by the non-expanding horizon geometry (q_{ab}, \mathcal{D}) .

We will say that the horizon geometry is *generic* if the operator \mathbf{M} has trivial kernel. In this case, $B = \mathbf{M}^{-1}(\kappa_{(\ell')} \tilde{\mu}^0)$ is the unique solution to (2.117), where, without loss of generality, we have assumed that (the v -dependence of) f was so chosen that $\kappa_{(\ell')}$

is non-zero. Thus, every generic non-expanding horizon admits a unique $[\ell]$ such that $(\Delta, [\ell])$ is isolated horizon. Furthermore, this isolated horizon is non-extremal.

What happens if the horizon geometry is non-generic? In this case, Eq (2.117) implies that if we choose B to belong to the kernel of \mathbf{M} , then $(\Delta, [\ell'])$ is an *extremal* isolated horizon. Thus, in contrast to the situation in higher dimensions, *every* non-expanding horizon admits an isolated horizon structure. However, in the non-generic case, uniqueness is not assured a priori; it may be possible to choose another null normal ℓ'' such that $(\Delta, [\ell''])$ is an isolated horizon. However, assuming that Δ admits an extremal isolated horizon structure and repeating the analysis starting from (2.114), it is easy to verify that: i) Δ can not admit a distinct extremal isolated horizon structure; and, ii) If it also admits a non-extremal isolated horizon structure, then it admits a foliation on which *both* null normals (ℓ^a and n^a) have zero expansions. This is an extremely special situation.

To summarize, in contrast to higher dimensions, *every* non-expanding horizon admits an isolated horizon structure which furthermore is unique except in extremely special cases.

Chapter 3

Relations between various classical theories

In this chapter we present various formulations of 2+1 dimensional general relativity and relations among them. In Section 3.1 we introduce a canonical Lorentzian theory with the gauge group $SU(2)$. The important feature of this formulation is compactness of its gauge group. In Section 3.2 we show that there exists a 1-parameter family of formulations with this gauge group which are related by the 2+1 dimensional analog of the canonical transformation called Barbero-Immirzi transform. In Section 3.3 we construct a map on the phase-space of the Euclidean theory which has an image in the phase-space of the Lorentzian theory with a non-compact gauge group (this is Wick rotation), while in Section 3.4 we map the same Euclidean theory to the formulation with the compact gauge group (this is Wick transform). Finally, in Section 3.5, we summarize the situation with two schematic figures.

3.1 Lorentzian gravity with compact gauge group

The purpose of this section is to introduce a new formulation of Lorentzian general relativity in which the internal gauge group is compact. In order to make the introduction of this new formulation easier and to highlight relations with the standard (non-compact) formulation, in this section we describe both formulations (with non-vanishing cosmological constant Λ). Both descriptions are Hamiltonian, where a pair of

canonically conjugate variables is a connection one-form A_a^i and a densitized dyad E_j^b . The difference between the two descriptions lies in the gauge group choice: we obtain one formulation with $SU(1,1)$ connections and another one with $SU(2)$ connections. Naturally the form of constraints depends on the variables chosen. The difference between the two sets of constraints is very similar to the one encountered in 3+1 gravity between the $SL(2, \mathbb{C})$ and $SU(2)$ Lorentzian theories.

Below, we begin with a more detailed description of the two formulations. Sometimes it is convenient to think about them in terms of suitable Killing reductions of a 3+1 theory. In this case we shall always assume that there exists a space-like Killing vector field ξ^a , such that any three-dimensional spatial slice Σ is topologically equal to $M \times S^1$, where S^1 is the orbit of ξ^a with affine length equal to one and M is a two dimensional submanifold. Moreover, we assume that the Killing vector field is constant or in other words $\nabla_a \xi_b = 0$ and is normalized to one $\xi^a \xi_a = 1$.

3.1.1 Standard $SU(1,1)$ formulation

A natural way of deriving 2+1 dimensional general relativity is to start with the 3+1 theory and perform Killing reduction with respect to ξ^a . Let us briefly outline the procedure without going into details. One can start for example from the Lagrangian formulation with the $SO(3,1)$ gauge group. Let $(^4)e_i^a$ be a tetrad corresponding to the four-dimensional metric. We can define an internal vector $\xi^i \equiv \xi^a (^4)e_a^i$. In order to obtain the 2+1 theory with the $SO(2,1)$ gauge group we need to make a partial gauge fixing. Namely, we impose the conditions: $(\xi_i) = (0, 0, 0, 1)$ in an orthonormal basis in the Lie algebra of $SO(3,1)$, as well as $\mathcal{L}_\xi (^4)A_a^i = 0$, $\xi_i (^4)A_a^i = 0$ and $\xi^a (^4)A_a^i = 0$.

In other words, we reduce the gauge freedom to a $SO(2,1)$ subgroup of $SO(3,1)$ and effectively reduce the connection to a connection one-form on the Killing reduced 3-manifold. We can then define the 2+1 dimensional variables as the projection of ${}^{(4)}e_a^i$ and the projection of ${}^{(4)}A_a^i$ to the manifold of orbits of ξ^a . The four dimensional action reduces in this way to the standard 2+1 action:

$$S = \frac{1}{8\pi G} \int (e_i \wedge F^i - \frac{\Lambda}{6} \varepsilon^{ijk} e_i \wedge e_j \wedge e_k) \quad (3.1)$$

The Hamiltonian description of this theory is well known. The Legendre transform can be performed *without* any further gauge fixing. The pair of canonically conjugate variables is (A_a^i, E_j^b) , which can be defined as:

$$A_a^i = \Gamma_a^i - K_a^i \quad (3.2)$$

$$E_j^b = -\sqrt{h} \varepsilon_c^b e_j^c \quad (3.3)$$

where $\Gamma_a^i \equiv \frac{1}{2h} \varepsilon_{jk}^i E^{bj} \nabla_a E_b^k$, $K_a^i \equiv \frac{1}{\sqrt{h}} E_b^i K_a^b$, h is the determinant of the intrinsic metric h_{ab} on the two-dimensional spatial slice and ∇ is its intrinsic derivative operator. It can also be shown that E_i^a can be regarded as square roots of the 2-metric: $hh^{ab} = E_i^a E^{bi}$.

The constraints of this theory can be written in the following form:

$$\mathcal{G}_i = D_a E_i^a = 0 \quad (3.4)$$

$$\mathcal{V}_b = E_i^a F_{ab}^i = 0 \quad (3.5)$$

$$\mathcal{S} = \varepsilon_{ijk} E^{ia} E^{jb} F_{ab}^k + 2h\Lambda = 0 \quad (3.6)$$

In the case of vanishing cosmological constant, the above equations simply restrict the reduced configuration space to the space of flat connections modulo gauge transformations. Let us also note that using the Lie algebra isomorphism $\mathfrak{so}(2, 1) \cong \mathfrak{su}(1, 1)$ and the group homomorphism $SO(2, 1) \cong SU(1, 1)/\mathbb{Z}_2$, we can equivalently describe our theory in terms of $SU(1, 1)$ connection. Furthermore, this extension is essential to incorporate spinor fields in 2+1 gravity. Therefore, we will generally use the second description from now on. For future reference, let us also describe the metric on $\mathfrak{su}(1, 1)$. In any orthonormal basis, the Cartan-Killing metric has the form $(\eta_{ij}) = \text{diag}(-1, 1, 1)$. We can also write it as $\eta_{ij} = h_{ij} - n_i n_j$, where $h_{ij} = \frac{1}{\hbar} E_i^a E_{aj}$ and $n_i n^i = -1$, $n^i E_i^a = 0$ (n^i can be thought of as the ‘internal version’ of the unit normal to the spatial slice).

3.1.2 $SU(2)$ formulation

This formulation can be obtained from the 3+1 dimensional $SO(3, 1)$ theory in the following way. First, we construct the 3+1 canonical formulation with $SO(3)$ connection. This has been done by Barbero in [39]. He used the following observation. Let us partially fix the internal gauge freedom in such a way that the tetrad is adapted to the space-time slicing, i.e. $n_a^{(4)} e_0^a = 1$ and $n_a^{(4)} e_i^a = 0$ for $i = 1, 2, 3$. We can therefore replace the gauge group $SO(3, 1)$ with the group $SO(3)$. Now we can perform the Legendre transform. It turns out that the canonically conjugate pair of variables is $(({}^{(3)}E_i^a, {}^{(3)}K_b^j)$, where ${}^{(3)}E_a^i = \sqrt{q}^{(4)} e_{\underline{a}}^i$ (q is the determinant of the induced metric on 3-dimensional slice Σ and tetrad is pulled-back from \mathcal{M} to Σ) and ${}^{(3)}K_a^i = \frac{1}{\sqrt{q}} {}^{(3)}E_b^i {}^{(3)}K_a^b$ (${}^{(3)}K_{ab}$ is the extrinsic curvature of Σ). Finally, Barbero noticed that one can perform

a canonical transformation on the phase-space which leads us to the $SO(3)$ connection

$${}^{(3)}A_i^a = {}^{(3)}\Gamma_i^a - {}^{(3)}K_i^a.$$

In order to obtain our 2+1 dimensional GR, we now just need to perform Killing reduction with respect to ξ^a . While doing this, we choose *not to* restrict gauge freedom any further. In this way, we obtain the pair of canonically conjugate variables on the two-dimensional spatial slice (A_a^i, E_j^b) , which can be defined as

$$A_a^i = \Gamma_a^i - K_a^i = {}^{(3)}A_{\underline{a}}^i \quad (3.7)$$

$$E_b^j = {}^{(3)}E_{\underline{b}}^j \quad (3.8)$$

where $K_a^i \equiv \frac{1}{\sqrt{h}} E_b^i K_a^b$ and $\Gamma_a^i \equiv -\frac{1}{2h} \varepsilon_{jk}^i E^{bj} \nabla_a E_b^k$.

The constraints that those variables must satisfy are the Killing reduced version of the 3+1 dimensional ones and have a completely analogous form, given by:

$$\mathcal{G}_i = D_a E_i^a = 0 \quad (3.9)$$

$$\mathcal{V}_b = E_i^a F_{ab}^i = 0 \quad (3.10)$$

$$\mathcal{S} = \varepsilon_{ijk} E^{ia} E^{jb} F_{ab}^k - 4E_i^{[a} E_j^{b]} K_a^i K_b^j + 2h\Lambda = 0 \quad (3.11)$$

Finally let us mention that, analogously as before, we have the following relations: $\mathfrak{so}(3) \cong \mathfrak{su}(2)$ and $SO(3) \cong SU(2)/\mathbb{Z}_2$. We will often use the equivalent $SU(2)$ formulation (rather than $SO(3)$). The internal metric on the Lie algebra $\mathfrak{su}(2)$ is given by $(\eta_{ij}) = \text{diag}(1, 1, 1) = (h_{ij}) + (\xi_i \xi_j)$ (in any orthonormal basis in the Lie algebra), where

as before, $h_{ij} = \frac{1}{\hbar} E_i^a E_{aj}$ and $\xi^i \xi_i = 1$, $\xi^i E_i^a = 0$ (i.e. ξ^i is the internal version of the Killing vector field ξ^a).

Let us finish this section with a remark. In [43], Holst has proposed a new action principle from which one can derive Barbero's Hamiltonian. One could imagine therefore an alternative derivation of the $SU(2)$ formulation. Namely, one could naively expect that it should arise from the Killing reduction of the Holst action followed by the Legendre transform. This, however, turns out not to be the case. Killing reduction of the Holst action results in the *standard* action (3.1) for the 2+1 Lorentzian theory.

3.2 Barbero-Immirzi transform

This section is an application of the ideas introduced by Barbero in [39]. While he worked with 3+1 gravity, we would like to show here that analogous arguments work in 2+1 dimensional case. Namely, we would like to show that the complex connection (discussed below) and the real $SU(2)$ connection formulations are related by an analog of the Barbero canonical transformation.

Let us consider the following pair of phase-space coordinates:

$${}^{(\beta)} E_i^a = \frac{1}{\beta} E_i^a \tag{3.12}$$

$${}^{(\beta)} A_a^i = \Gamma_a^i + \beta K_a^i \tag{3.13}$$

Since we already know that for $\beta = -1$ these are canonically conjugate variables, it is easy to show that this is a canonically conjugate pair for arbitrary real number $\beta > 0$. For the convenience, from now on we will skip the index (β) in the upper left corner of

E and A . In case $\beta = -1$, these are the $SU(2)$ real variables. We will also consider the choice $\beta = -i$. As we will see below, the latter choice results in a simple polynomial form of the scalar constraint. Since the connection becomes complex in this case, we are naturally led to complexified phase-space supplied with appropriate reality conditions. Because of those features this choice of variables is an analog of the self-dual variables in 3+1 dimensions [1] even though there is no analogous notion of self-duality here.

It remains to check behavior of the constraints under this map (we restrict ourselves here to Lorentzian signature space-times). For an arbitrary value of the parameter β we have

$$D_a E_i^a = \partial_a E_i^a + \epsilon_{ijk} \Gamma_a^j E^{ka} + \beta \epsilon_{ijk} K_a^j E^{ka} = \beta \epsilon_{ijk} K_a^j E^{ka} \quad (3.14)$$

where we used compatibility of E with Γ . Thus, we conclude that the form of the Gauss constraint is independent from which " β -variables" we use. Similarly, for an arbitrary value of β we have

$$\begin{aligned} E_i^a F_{ab}^i &= E_i^a (2\nabla_{[a} \Gamma_{b]}^i + \epsilon_{jk}^i \Gamma_a^j \Gamma_b^k) + \beta (2\nabla_{[a} K_{b]}^i + 2\epsilon_{jk}^i A_{[a}^j K_{b]}^k) E_i^a - \\ \beta^2 E_i^a \epsilon_{jk}^i K_a^j K_b^k &= \frac{1}{2} \epsilon_{jk}^i E_i^a e_c^j e_d^k R_{ab}^{cd} + 2\beta E_i^a D_{[a} K_{b]}^i + \beta^2 K_b^i \mathcal{G}_i \approx 2\beta E_i^a D_{[a} K_{b]}^i \end{aligned} \quad (3.15)$$

where we have used Bianchi identity to eliminate the term with curvature tensor R_{abcd} and \approx means modulo Gauss constraint. Therefore, as above, we come to the conclusion that the form of diffeomorphism constraint does not change. Finally, one can show that the general form of the scalar constraint does depend on the choice of β . We obtain

(when $\Lambda = 0$)

$$\mathcal{S} = \epsilon^{ijk} E_i^a E_j^b F_{abk} - 2(\beta^2 + 1) E_{[i}^a E_{j]}^b K_a^i K_b^j = 0 \quad (3.16)$$

Thus, we can see that the scalar constraint is very simple for the choice $\beta = \pm i$. The price one has to pay for it, however, is that the phase-space has to be complexified.

3.3 Wick rotation

The purpose of this section is to construct a map from the phase-space of Euclidean general relativity into the Lorentzian one. In this sense, the map is a generalization of the usual Wick rotation known from field theory. We will consider two different ways of implementing the rotation on the Euclidean variables (i.e. two different maps). It turns out that none of them leads to the Lorentzian $SU(2)$ formulation described above. One of them leads to the standard Lorentzian theory with $SU(1,1)$ gauge and the second one leads to the formulation with complex phase-space variables. Nevertheless, we will see in Chapter 5 that at the quantum level our classical Wick rotation is necessary to understand the relation between Euclidean quantum states and Lorentzian quantum states arising in the $SU(2)$ formulation. Mathematically more detailed introduction to the Wick rotation has also been given by Hartmann in [41].

3.3.1 Complex Wick rotation

In this subsection we will describe a way of defining a Wick rotation which will not be used in the remaining chapters. We include it only for completeness. We define below a map on the coordinates of the phase-space variables in some fixed internal basis.

The choice of the internal basis is naturally dictated by the dyad E_a^i . Moreover, the description of the complex Wick rotation given below is valid for a large class of such choices. We call this way of implementing Wick rotation complex because as result of this map, the coordinates of connection A and dyad E in a fixed internal basis become complex (they have non-vanishing imaginary part). This has to be contrasted with the map constructed in the next paragraph, where the coordinated remain real while the internal basis is complexified.

Let us define an internal vector $n^i \in \mathfrak{su}(2)$ by the following conditions:

$$n^i E_i^a = 0 \quad \text{and} \quad n^i n_i = 1 \quad (3.17)$$

In order to define complex Wick rotation we will fix the internal basis to be any orthonormal basis containing n^i . The map is then defined in the following way. The coordinates of phase-space variables in this internal basis which are perpendicular to n are multiplied by complex unit i and the parallel components are multiplied by -1. In particular this implies:

$$E_I^a \mapsto iE_I^a \quad (3.18)$$

where we used the superscript I to indicate components of the internal vectors (as opposed to abstract index notation with index i). Notice that dyad becomes purely imaginary under the action of this Wick rotation.

The map can be naturally extended to the objects with no internal indices. For those objects the map is defined to be identity. The consistency of this definition follows

from the following observation

$$h_{IJ} = E_I^a E_{Ja} \mapsto -h_{IJ} \quad (3.19)$$

Furthermore, the definition of the complex Wick transform implies:

$$K_I^a \mapsto iK_I^a \quad (3.20)$$

Notice that components of the Cartan-Killing metric on the internal algebra are mapped according to

$$\eta_{IJ} = h_{IJ} + n_I n_J \mapsto -h_{IJ} + n_I n_J = -\eta_{IJ}^{(L)} \quad (3.21)$$

i.e. they are mapped to components of a negative of the internal metric $\eta_{IJ}^{(L)}$ with Lorentzian signature $(-, +, +)$. This is the reason why we use the name Wick for our map.

Finally, we map components of the structure constants ϵ^i_{jk} as well as spin connection Γ_a^i . We have: $\epsilon^I_{JK} \mapsto \epsilon^{I(L)}_{JK}$ and $\Gamma_a^I \mapsto \Gamma_a^{I(L)}$. The superscript (L) stands for Lorentzian. Therefore we obtain complex connection:

$$A_a^I = \Gamma_a^I - K_a^I \mapsto \Gamma_a^{I(L)} + iK_a^I \quad (3.22)$$

The constraints preserve their form under the complex Wick rotation. In this construction we only chose to provide a map on the phase-space and the functions on it remain to be *the same* functions of the *new* variables. In Section 3.4 we will follow

another route, where it is the functions that are mapped and the phase-space remains the same. Since the phase-space coordinates here are complex, the theory should be supplied with suitable reality conditions. In this way we obtain the 2+1 analog of the self-dual 3+1 formulation.

3.3.2 Internal Wick rotation

In this section we will make use of the well known fact that the complexifications of the Lie algebra of $SU(2)$ and the Lie algebra of $SU(1, 1)$ are the same:

$$\mathbb{C} \otimes \mathfrak{su}(2) = \mathbb{C} \otimes \mathfrak{su}(1, 1) = \mathfrak{sl}(2, \mathbb{C}) \quad (3.23)$$

Thus, we can consider the $SU(2)$ and $SU(1, 1)$ phase-spaces as two different real sections of the same complex phase-space where the variables take values in the Lie algebra of $SL(2, \mathbb{C})$. The embeddings in the complexified space are not unique. However, our construction of the Wick rotation will unambiguously pick out the required subspace identifications. It will be defined as a rotation of the internal $\mathfrak{sl}(2, \mathbb{C})$ vectors contained in one real section into the other real section.

Given a dyad E^{ai} one can always find a unit internal $\mathfrak{su}(2)$ vector which is perpendicular to the dyad: $n^i E_i^a = 0$. Let v^i, w^i be *any* two internal $\mathfrak{su}(2)$ vectors, such that (n, v, w) is an orthonormal basis in $\mathfrak{su}(2)$. If we now complexify this Lie algebra, we can identify another real section of this $\mathfrak{sl}(2, \mathbb{C})$ space with the Lie algebra of $\mathfrak{su}(1, 1)$. This subspace is spanned by the basis vectors $(-n, iv, iw)$. Notice that the subspace identification does not depend on the specific choice of the basis vectors v, w . At the

same time, the choice of n is somewhat subtle: it depends on both, the phase-space and the space-time points. It is however natural, because there is no arbitrariness in this choice; it is an inherent feature of the theory.

This suggests the following definition of the internal Wick rotation on the phase-space variables:

$$W_R(E_a^i) = i \cdot E_a^i \quad (3.24)$$

$$W_R(A_a^i) = i \cdot h_j^i A_a^j - n^i n_j A_a^j \quad (3.25)$$

However, in order for the connection above to provide a well defined $SU(1, 1)$ covariant derivative, we also need to further specify the reference derivative operator ∂ .¹ An appropriate choice is $\partial n^i = 0$. This allows us to employ the following strategy. Let us extend linearly the $SU(2)$ reference derivative to the complexified internal group $SL(2, \mathbb{C})$. Then, we can define the $SU(1, 1)$ reference derivative as the restriction of this extension to the appropriate real section. Finally, let us also linearly extend the internal metric η_{ij} and structure constants ε_{jk}^i on $SU(2)$ to the whole complex group $SL(2, \mathbb{C})$ and denote them as $\eta_{ij}^{\mathbb{C}}$ and $(\varepsilon_{jk}^i)^{\mathbb{C}}$. The appropriate objects on the $SU(1, 1)$ section can now be identified with $-\eta_{ij}^{\mathbb{C}}|_{\mathfrak{su}(1,1)}$ and $-(\varepsilon_{jk}^i)^{\mathbb{C}}|_{\mathfrak{su}(1,1)}$.

We are ready now to check the action of the internal Wick rotation on the symplectic structure and the constraints. This action can be found by first extending the appropriate expressions to the complexified phase-space. Then one has to restrict those

¹This has been first noted by Hartmann in [41], where a more detailed discussion on this issue can be found.

expressions to the $SU(1, 1)$ real section and identify appropriate $SU(1, 1)$ objects according to the definitions above. Symplectic structure on both theories ($SU(2)$ Euclidean and $SU(1, 1)$ Lorentzian) is given by

$$\Omega(\delta_1, \delta_2) = \int_M d^2x \left[\delta_1 E^{ai} \delta_2 A_a^j - (1 \leftrightarrow 2) \right] h_{ij} \quad (3.26)$$

Using $W_R(h_{ij}) = -h_{ij}$, it is easy to show that the Euclidean symplectic structure is mapped to the Lorentzian one by the Wick rotation. Let us also illustrate the action of Wick rotation on constraints of the Euclidean theory with an explicit calculation for the Gauss constraint. Recall that Euclidean $SU(2)$ constraints have the same form as Lorentzian $SU(1, 1)$ constraints (3.4) - (3.6). Therefore for the Gauss constraint we have:

$$W_R \left(\mathcal{G}^i(A, E) \right) = \left(ih_j^i - n^i n_j \right) \left(\partial_a E^{aj} + \epsilon^j_{kl} A_a^k E^{al} \right) = \quad (3.27)$$

$$\partial_a^{\mathbb{C}} \Big|_{\mathfrak{su}(1,1)} iE^{ai} - (\epsilon^i_{jk})^{\mathbb{C}} \Big|_{\mathfrak{su}(1,1)} (ih_l^j A_a^l - n^j n_l A_a^l) iE^{ak} = \mathcal{G}^i(W_R(A, E)) \quad (3.28)$$

Thus, as expected, Wick rotation preserves the Gauss constraint. Similarly, it is a straightforward calculation to show that the diffeomorphism and scalar constraints of Euclidean theory are mapped to those of the standard $SU(1, 1)$ formulation of Lorentzian general relativity.

3.4 Wick transform

In this section we construct a map called Wick transform. The purpose of this map is to relate the Euclidean general relativity with the $SU(2)$ formulation of the

Lorentzian theory. Since the phase-spaces of the two theories are the same, only the constraint functionals are different, the appropriate map is an automorphism on the *space of functions* on the phase-space. The ideas presented here have been first applied in 3+1 dimensions in [2]. The purpose of this section is to show that analogous ideas can be applied in 2+1 dimensions.

Let us define the following map W_T on the functions of phase-space:

$$W_T(f) = f + \{f, T\} + \frac{1}{2!}\{\{f, T\}, T\} + \dots = \sum_{n=0}^{\infty} \frac{1}{n!}\{f, T\}_n \quad (3.29)$$

where $\{, \}$ denotes the Poisson bracket and T a fixed function on the phase-space. W_T is a Poisson bracket-preserving automorphism on the algebra of complex-valued functions on the phase-space. The 3+1 analog of the following function T has been first proposed in [60]:

$$T = \frac{i\pi}{2} \int_M d^2x K_a^i E_i^a \quad (3.30)$$

Let us denote by W the map corresponding to this choice of function T . Then, as in 3+1 dimensions, regarding E_i^a and K_a^i as coordinate functions on the phase-space we have

$$W(E_i^a) = iE_i^a \quad (3.31)$$

$$W(K_a^i) = -iK_a^i \quad (3.32)$$

Using the automorphism property $W\left(f(E_i^a, K_a^i)\right) = f(iE_i^a, -iK_a^i)$, one can then show that the Euclidean constraints are mapped into the Lorentzian ones (with a minus sign for the scalar constraint). A form of the constraints convenient for this calculation is

given by Equations (3.14)-(3.16). The strategy for construction of a quantum Wick transform operator will be described in Chapter 5.

3.5 Putting it all together

The situation is illustrated in the Figure 3.1. We have four different theories with relations denoted by arrows.

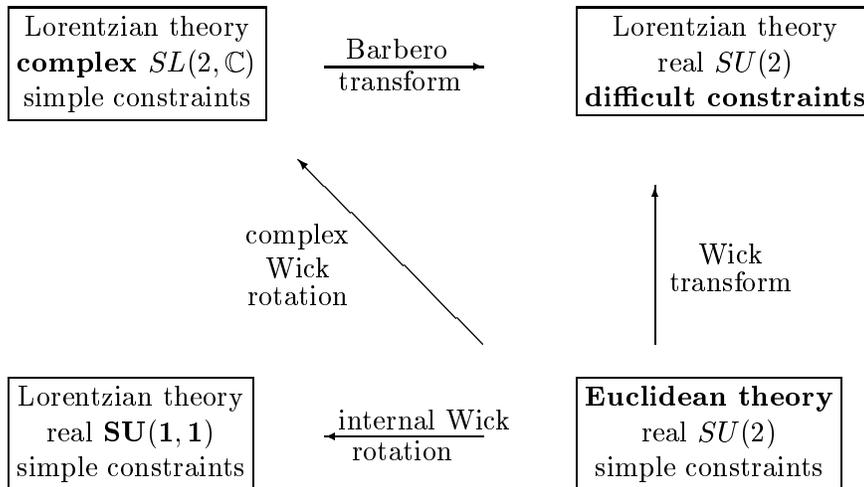


Fig. 3.1. Overview of the situation

Each of the theories included in the figure has a certain unwanted feature. These are marked in boldface. Complex formulation is inconvenient for quantization because of problems with incorporating suitable reality conditions. $SU(1, 1)$ is a non-compact group and standard techniques used to construct physical Hilbert space are only suited to treat compact gauge groups [16]. One can also quantize the $SU(1, 1)$ formulation

using a simplification which is characteristic to 2+1 dimensions only. Namely the diffeomorphism and scalar constraints are classically equivalent to the constraint $F = 0$. We know how to fully quantize such theory. Physical states turn out to be functions on finite dimensional moduli space of flat connections. Those simplifications do not occur, however, in the 3+1 dimensional theory. $SU(2)$ formulation, on the other hand, is very analogous to the $SU(2)$ formulation of 3+1 dimensional Lorentzian theory. It has non-polynomial constraints which are difficult to quantize. Despite this complication rigorous constructions of quantum operators corresponding to all constraints exist in literature and the 2+1 model can be used to test those ideas. Finally, the unwanted feature of the Euclidean theory is that it does not reflect physical reality.

In order to further clarify the relations between various variables discussed in this chapter we also provide a schematic description in the Figure 3.2.

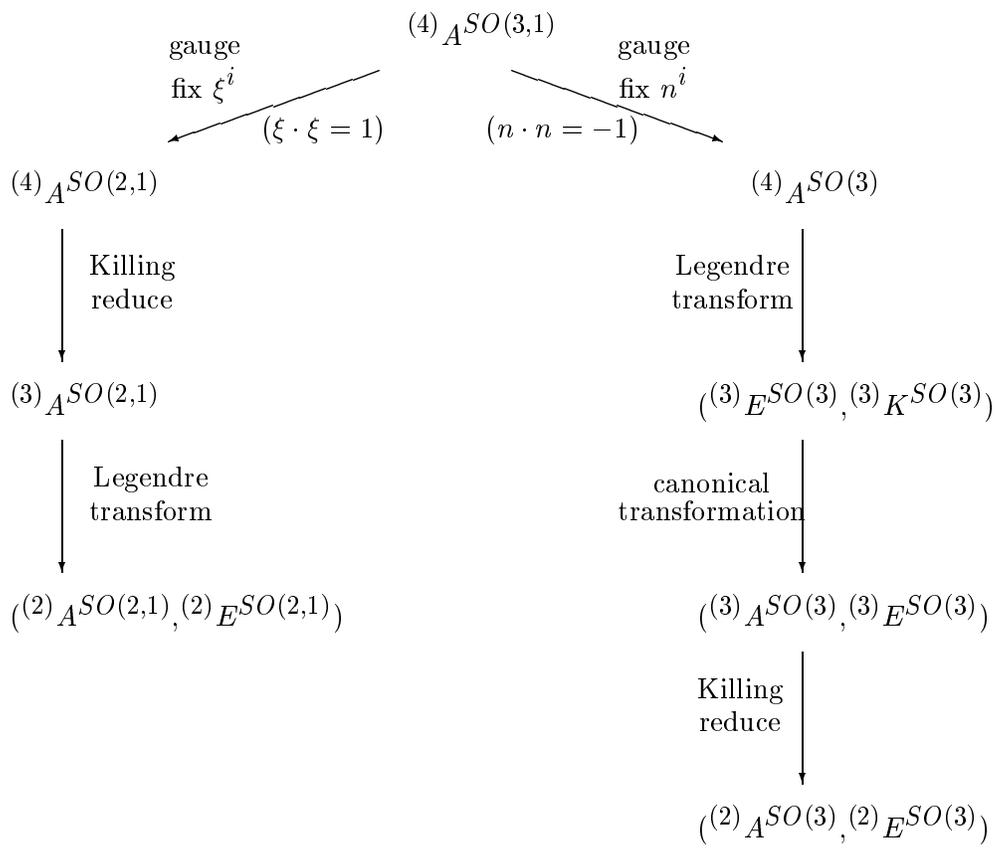


Fig. 3.2. Relations between different variables

Chapter 4

Canonical quantum gravity

In this chapter we present several new constructions of important operators in 2+1 dimensional general relativity with $SU(2)$ gauge group. In Section 4.1 we describe construction of the length operator. Due to the results of the previous chapter, this operator describes now lengths in the Lorentzian theory as well as Euclidean one. In Section 4.2 we construct the Hamiltonian operator for the new formulation from the previous chapter. Section 4.3 provides a new alternative construction of the Euclidean Hamiltonian constraint and finally in Section 4.4 we discuss some simple solutions of that constraint.

4.1 Length operator

This chapter is devoted to a construction of a quantum operator on the kinematical Hilbert space of $SU(2)$ connection formulation of general relativity. The goal is to construct the analog of the area operator of 3+1 dimensional theory in our 2+1 dimensional setting. It corresponds to lengths of curves but because of the analogy we often call it area operator. It is worth to notice here that, so far, the character of length spectrum in the 2+1 dimensional Lorentzian theory was unknown. It has been sometimes suggested that it is continuous, in contrary to 3+1 dimensional area. The results below, however, that the situation is completely analogous and the spectrum is discrete.

Since the gauge group $SU(2)$ is the same for both, 3+1 and 2+1 theories, we can use the techniques and ideas introduced in [17]. In fact the whole construction can be repeated with only very small changes (see also earlier construction with loop variables in [53]).

We start with the classical expression for length of a (space-like) curve S parameterized by $t \in \mathbb{R}$

$$A_S = \int_S dt \sqrt{\frac{dx^a(t)}{dt} \frac{dx^b(t)}{dt} g_{ab}} \quad (4.1)$$

For a given curve S , we can always choose coordinates (x^1, x^2) such that $x^2 = 0$ on S .

Then

$$A_S = \int_S [E_i^2 E^{i2}]^{1/2} dx^1 \quad (4.2)$$

In order to define corresponding operator, we need to smear the dyad E . Therefore we pick any function $f_\epsilon(x, y)$ which satisfies $\lim_{\epsilon \rightarrow 0} \int_S dy f_\epsilon(x, y) g(y) = g(x)$. Then we can define a smeared version of the dyad component

$$[E_i^2]_f(x) \equiv \int_S dy f_\epsilon(x, y) E_i^2(y) \quad (4.3)$$

and the smeared version of area functional

$$[A_S]_f \equiv \int_S dx^1 \left([E_i^2]_f(x) [E^{i2}]_f(x) \right)^{1/2} \quad (4.4)$$

We are now ready to construct the required quantum operator. We just need to replace the dyad E_i^2 with $\hat{E}_i^2 = -i\hbar \frac{\delta}{\delta A_i^2}$. Then the action of $[\hat{E}_i^2]_f(x)$ on a cylindrical

function $\Psi_\gamma(\bar{A}) = \psi(h_1[A], \dots, h_N[A])$ can be shown as in [17] to be

$$[\hat{E}_i^2]_f(x)\Psi_\gamma(\bar{A}) = i\hbar \int_S dy f_\epsilon(x, y) \sum_{k=1}^N \left[\int_0^1 dt \dot{e}_k^2(t) \delta(y, e_k^1(t)) \delta(0, e_k^2(t)) (h_k(1, t) \tau_i h_k(t, 0))_B^A \right] \frac{\partial \psi}{\partial (h_k)_B^A}(A) \quad (4.5)$$

where γ is a graph with edges $(e_k^a(t))_{k=1, \dots, N}$, $t = 0, 1$ correspond to vertices of γ and h_j is holonomy of the connection A along the j -th edge. Also τ_i is the (normalized) i -th generator of $SU(2)$ and \bar{A} is the generalized connection discussed in [16]. Using this, one can finally show that the area operator in 2+1 dimensions takes a completely analogous form to the one in 3+1 dimensions.

$$\hat{A}_S \Psi_\gamma = \frac{\hbar}{2} \sum_{v \in V(\gamma)} \left[- \sum_{j_v k_v} \kappa(j_v, k_v) X_{ij_v} X_{k_v}^i \right]^{1/2} \Psi_\gamma \quad (4.6)$$

Here j_v, k_v label edges incident at the vertex v and $\kappa(j_v, k_v)$ equals zero if either of the two edges e_j or e_k fails to intersect S or lies entirely in S , +1 if they lie on the same side and -1 if they lie on the opposite sides of S . Also, we define the vector fields on the group $SU(2)$ as follows

$$X_{j_v}^i \psi(h_1[A], \dots, h_N[A]) = \begin{cases} \left(h_{j_v}[A] \tau^i \right)_B^A \frac{\partial \psi}{\partial (h_{j_v}[A])_B^A} & \text{when } e_{j_v} \text{ is outgoing} \\ - \left(\tau^i h_{j_v}[A] \right)_B^A \frac{\partial \psi}{\partial (h_{j_v}[A])_B^A} & \text{when } e_{j_v} \text{ is incoming} \end{cases} \quad (4.7)$$

We can also give the complete spectrum of area (i.e. length). On the space of gauge-invariant states Ψ_γ the spectrum is given by

$$a_S = 8\pi G\hbar \sum_v \sqrt{j_v(j_v + 1)} \quad (4.8)$$

where j_v are arbitrary half-integers. Notice that in 2+1 dimensions G has units of inverse mass and therefore $G\hbar \sim L$. Let us define Planck length

$$l_P \equiv 8\pi G\hbar \quad (4.9)$$

Now we can clearly see that the length is quantized in units of Planck length, just like the 3+1 dimensional area was quantized in units of Planck length squared. In fact the above expression differs from the one found in [17] *only* in the power of l_P , even the numerical coefficients agree.

4.2 Hamiltonian operator for $SU(2)$ Lorentzian theory á-la Thiemann

In this section we will use $SU(2)$ formulation of Lorentzian gravity (with vanishing cosmological constant) to construct the quantum operator corresponding to the scalar constraint. This will be achieved using techniques analogous to those introduced in [59]. The scalar constraint of the 2+1 Euclidean theory has already been quantized in [62]. Thus we only need to construct the operator corresponding to "KKEE" term.

Let us write the scalar constraint as the sum of two terms

$$H = H^E - T \quad (4.10)$$

where $H^E = \frac{\epsilon_{ijk} E^{ia} E^{jb} F_{ab}^k}{\sqrt{h}}$ and $T = 4 \frac{E_i^{[a} E_j^{b]} K_a^i K_b^j}{\sqrt{h}}$. As in previous works, we divided the constraint by the determinant of the 2-metric in order to obtain density of weight one. Let us denote by K the integral over M of the trace of extrinsic curvature:

$$K \equiv \int_M d^2x K_a^i E_i^a \quad (4.11)$$

The key observation due to Thomas Thiemann is that K can be written as a Poisson bracket of quantities which we know how to quantize

$$K = -\{V, \int_M d^2x H^E(x)\} \quad (4.12)$$

where $V = \int_M d^2x \sqrt{h(x)}$ and the operator \hat{V} has been constructed in [62]. Then one can write

$$K_a^i = \{A_a^i, K\} \quad (4.13)$$

Also, like in [62], let us introduce the degeneracy vector $E^i \equiv \frac{1}{2} \epsilon^{ijk} \epsilon_{ab} E_j^a E_k^b$. We can use it to rewrite T in a different form

$$T = 2\epsilon_{ijk} \epsilon^{ab} \frac{E^i K_a^j K_b^k}{\sqrt{h}} \quad (4.14)$$

Finally, we use the expression of E^i in terms of Poisson brackets of A with V found in [62] to write the scalar constraint in the form suitable for quantization

$$T = \frac{4\epsilon^{ab} \epsilon^{cd}}{\sqrt{h}} \text{tr} (\{A_a, V\} \{A_b, V\} \{A_c, K\} \{A_d, K\}) \quad (4.15)$$

Next step in the quantization procedure is to regularize the classical expression by introduction of a triangulation. We will find the operator corresponding to the regulated constraint and take the limit of refining the triangulation at the end. Let us use the notation and the triangulation introduced in [62]. Let us denote by $T(\gamma)$ triangulation adapted to a particular graph γ . The strategy is to construct the operator acting on a cylindrical function f_γ and show at the end that the family of operators $\hat{T}_{T(\gamma)}$ gives rise to an operator on the kinematical Hilbert space $\mathcal{H} = L_2(\overline{\mathcal{A}/\mathcal{G}}, d\mu_0)$. We will not repeat here all the details of the construction. We will only give a brief explanation of the arguments which can be taken over directly from Thiemann's work.

First of all the integral of $T(x)$ over any triangle Δ can be approximated by

$$T_\Delta[N] \equiv \int_\Delta d^2x N(x) T(x) \cong 4N(v(\Delta)) \epsilon^{ij} \epsilon^{kl} \text{tr} \left(h_{s_i(\Delta)} \{h_{s_i(\Delta)}^{-1}, V\} h_{s_j(\Delta)} \{h_{s_j(\Delta)}^{-1}, V\} h_{s_k(\Delta)} \{h_{s_k(\Delta)}^{-1}, K\} h_{s_l(\Delta)} \{h_{s_l(\Delta)}^{-1}, K\} \right) \quad (4.16)$$

where $v(\Delta)$ is one of the vertices of the triangle and $s_i(\Delta)$ ($i = 1, 2$) are the edges outgoing from $v(\Delta)$. The approximation becomes an exact expression in the limit of the triangle Δ shrinking to $v(\Delta)$.

$V(\gamma)$ denotes the set of vertices of the graph γ . Let $v \in V(\gamma)$ and e_1, e_2 be two analytical edges incident and outgoing at $v = e_1 \cap e_2$. Also, let s be analytic extension of e_1 and \bar{s}_1 the half of s starting at v but not including e_1 . Finally, let U be a small neighborhood of v so that it is split in two halves by s . We define the upper half U^+ of U to be the half of U which one encounters turning e_1 counter-clockwise. Now we are ready to give an important definition which will be necessary to prescribe the

Hamiltonian constraint operator. The two edges e_1, e_2 will be called *right oriented* iff there exists a neighborhood U of v such that e_2 intersects the upper half U^+ (defined as above).

Repeating after [59], we will now specify our triangulation $T(\gamma)$ for each possible graph γ . Fix $v \in V(\gamma)$ and let n be its valence. We can label the edges of γ incident at v in such a way that the pairs $(e_1, e_2), (e_2, e_3), \dots, (e_{n-1}, e_n)$ are right oriented and such that e_i and e_{i+1} are neighboring edges. Let $E(v)$ equal n if the pair (e_n, e_1) is right oriented, otherwise set $E(v) = n - 1$. We can now construct $E(v)$ triangles $\Delta_i(v)$ ($i = 1, \dots, E(v)$) in the following way:

$$\Delta_i(v) \equiv s_1 \circ a_{12} \circ s_2^{-1} \tag{4.17}$$

where s_1 (s_2) is any segment of e_i (e_{i+1} respectively) which does not include the other endpoint of e_i (e_{i+1}) different from v and which starts at v ; a_{12} connects the endpoints of s_1 and s_2 with additional special property that the tangent to it is parallel to the tangent of s_1 where they meet and it is antiparallel to the tangent of s_2 at their common point.

We can now triangulate our manifold M by the triangles $\Delta_i(v)$ at each vertex $v \in V(\gamma)$ ($i = 1, \dots, E(v)$) with the remaining part $M - \bigcup_{v,i} \Delta_i(v)$ triangulated arbitrarily¹. It turns out that only the operators \hat{T}_Δ corresponding to $\Delta_i(v)$'s give a non-zero contribution to the total operator $\hat{T}_{T(\gamma)} = \sum_{\Delta \in T(\gamma)} \hat{T}_\Delta$. More precisely, one can define

¹Strictly speaking, we impose an additional condition that no basepoint (see [62]) of a triangle should lie on an edge of γ or that all tangents at the intersection are co-linear

our Hamiltonian constraint operator using

$$\hat{T}(N)f_\gamma \equiv \hat{T}_{T(\gamma)}(N)f_\gamma \equiv \frac{4}{\hbar^4} \sum_{v \in V(\gamma)} N(v) \frac{4}{E(v)} \sum_{\Delta_i(v)} \epsilon^{ij} \epsilon^{kl} \text{tr} \left(h_{s_i} [h_{s_i}^{-1}, \hat{V}_v] h_{s_j} [h_{s_j}^{-1}, \hat{V}_v] h_{s_k} [h_{s_k}^{-1}, \hat{K}_v] h_{s_l} [h_{s_l}^{-1}, \hat{K}_v] \right) f_\gamma \quad (4.18)$$

Here $\hat{K}_v = \frac{i}{\hbar} [\hat{V}_v, \hat{H}^E(N=1)]$, where $\hat{H}^E(N=1)$ is the operator corresponding to $\int_M d^2x H^E(x)$ and \hat{V}_v is the vertex contribution to the volume operator, both constructed in [62]. By the arguments given there it is also an easy observation that the operator $(\hat{T}(N))'$ (where $'$ denotes the vector space dual) is already well defined in the continuum limit on the diffeomorphism invariant states. Furthermore, using techniques of [63] and [62] one can show the cylindrical consistency and anomaly-freeness of the operator $\hat{H}(N) \equiv \hat{H}^E(N) - \hat{T}(N)$.

4.3 Alternative Hamiltonian operator for Euclidean theory

In this section we provide a construction of the Hamiltonian operator which is different from Thiemann's construction. Its starting point is a different classical form of the constraint. The advantage of this approach is a comparatively simple action of the final operator.

Let us begin by rewriting the Euclidean scalar constraint in the following form

$$H(\tilde{N}) := \int_M d^2x \tilde{N}(x) \epsilon_{ijk} E^{ia}(x) E^{jb}(x) F_{ab}^k(x) \quad (4.19)$$

where we divided the lapse function $N(x)$ by the square root of the determinant of the intrinsic metric on the spatial slice M . In this way we obtain a density of weight minus one $\tilde{N}(x)$. The reason for this is to form an expression under the integral which is a density of weight one.

Next we need to introduce a triangulation of M . We choose a family of triangulations in the same way as described in the previous section. Each triangulation is adapted to a certain graph γ . Therefore $M = \bigcup_i \Delta_i$, where we fix the coordinate area of the triangles $a_{\Delta_i} = \delta^2$. We will treat δ as a small parameter and approximate our classical expression by terms of the lowest order in δ . Therefore we will need to use:

$$N(x) = N(v) + O(\delta) \quad (4.20)$$

$$F(x) = F(v) + O(\delta) \quad (4.21)$$

if $x \in \Delta_v$ and v is a vertex of Δ_v . Also

$$\epsilon^{ab} F_{ab}^k(v) = -\frac{(h_{\alpha_v} - h_{\alpha_v}^{-1})^k}{ia_{\alpha_v}} + O(\delta^2) \quad (4.22)$$

where α_v is any loop which starts and ends in v such that its coordinate area $a_{\alpha_v} = \delta^2$.

The operator corresponding to $E^i := \frac{1}{2}\epsilon^{ijk}\epsilon_{ab}E^{ja}E^{kb}$ has been constructed in [62]. Its action on a cylindrical function $\Psi_\gamma(A)$ is given by

$$\hat{E}^i(x)\Psi_\gamma(A) = -\frac{\hbar^2}{8} \sum_{e \neq e'} \delta(x, v) \frac{\dot{e}^{[a(v)e'b]}(v)}{|\epsilon_{cd}\dot{e}^c(v)\dot{e}'^d(v)|} X_e^i X_{e'}^j \Psi_\gamma(A) \quad (4.23)$$

Therefore it is not zero only if x coincides with one of the vertices of the graph γ and X_e^i is defined as in Equation (4.7). Therefore we can now write the following expression for the operator corresponding to the classical expression under the integral in (4.19) (when acting on a cylindrical function):

$$\sum_{v \in V(\gamma)} \frac{\hbar^2}{8} \tilde{N}(v) \epsilon_{ijk} \delta(x, v) \sum_{e \neq e'} \text{sgn}(e, e') \frac{(h_{\alpha_v} - h_{\alpha_v}^{-1})^k}{ia_{\alpha_v}} X_e^i X_{e'}^j \quad (4.24)$$

where $\text{sgn}(e, e') := \frac{\epsilon_{ab} \dot{e}^a(v) \dot{e}'^b(v)}{|\epsilon_{cd} \dot{e}^c(v) \dot{e}'^d(v)|}$ and we only keep the terms with the lowest order in δ .

It is now straightforward to check that we obtain a simple Hamiltonian operator given by

$$\hat{H}_\delta(\tilde{N}) \Psi_\gamma(A) = -\frac{i\hbar^2}{8\delta^2} \sum_{v \in V(\gamma)} \tilde{N}(v) \sum_{e \neq e'} \text{sgn}(e, e') \epsilon_{ijk} (h_{\alpha_v} - h_{\alpha_v}^{-1})^i X_e^j X_{e'}^k \Psi_\gamma(A) \quad (4.25)$$

The density \tilde{N} evaluated at a vertex v depends on the coordinates chosen. Therefore the operator as given above is not well defined. Notice, however, that this operator has to vanish on physical states for any choice of lapse function N . This implies that each term in the sum over vertices has to vanish separately. Since such condition is independent of the coordinates chosen, the operator provides a well defined quantum constraint. It is important to notice that we have left a lot of freedom in the above definition. The loops α_v in this expression can be chosen arbitrarily as long as $a_{\alpha_v} = \delta^2$. In fact, we are even free to prescribe a different loop for each term in the sum. We will not attempt here to decide which choice is most appropriate. Let us also note that if we apply the operator \hat{H}_δ to a diffeomorphism invariant state Ψ then its vanishing does not depend on δ (as

long as δ is smaller than some finite δ_0 , which has to be fixed separately for each v).

Therefore from now on we will omit the subscript δ and consider the following operator:

$$\hat{H}(N)\Psi_\gamma(A) = \sum_{v \in V(\gamma)} N(v) \sum_{e \neq e'} \text{sgn}(e, e') \epsilon_{ijk} (h_{\alpha_v} - h_{\alpha_v}^{-1})^i X_e^j X_{e'}^k \Psi_\gamma(A) \quad (4.26)$$

Since it will be useful in Chapter 5, let us also describe here the action of the operator just constructed on an arbitrary spin network function $T_\gamma(A)$ [56]. In order to provide such a description we need to make certain choice for the loops α . We will not fix them completely and, in particular, Thiemann's regularization choice [60] is an example in the more general class described as follows. For each term in the sum over the edges $e \neq e'$, the loop α_v is the loop which overlaps with any two neighboring edges e_I, e_J meeting at the vertex v . Furthermore α_v overlaps with e_I and e_J on precisely two finite intervals s_I and s_J which are contained in the respective edges and $v \in s_I, s_J$. Recall also that we have previously fixed the area $a_\alpha = \delta^2$. With this choice of regularization, our Euclidean Hamiltonian operators act on a spin network function through multiplication by a certain spin dependent factor, creating a new edge e_{IJ} connecting s_I and s_J at their endpoints and subdividing e_I, e_J into two new edges each. The new edges in the subdivision are s_I, s_J and $e_I - s_I, e_J - s_J$ (respectively). Moreover from each spin-network we obtain a sum of four linearly independent terms, which are obtained by assigning the following spins to the new edges $j(e_{IJ}) = \frac{1}{2}$, $j(s_{I,J}) = j(e_{I,J}) \pm \frac{1}{2}$ and keeping other spins unchanged. Note that if $j(e_I) = \frac{1}{2}$ or $j(e_J) = \frac{1}{2}$ we obtain new edges with spins equal to zero. This means that in some cases s_I or s_J just disappear from the original graph.

4.4 Simple examples of solutions

In the classical 2+1 dimensional general relativity, the reduced configuration space is the space of flat connections modulo gauge transformations ($SU(2)$ gauge for Euclidean theory and $SU(1,1)$ gauge for Lorentzian theory). Also, it has been found in other approaches to quantization of 2+1 gravity that the physical Hilbert space consists of functions with support only on flat connections [30]. Are such functions annihilated by the Euclidean Hamiltonian operator found above? The answer is yes. The distributions of the form

$$\Psi(A) = \psi(A_0)\delta(F(A)) \quad (4.27)$$

where A_0 is an $SU(2)$ flat connection, belong to the physical Hilbert space, provided ψ is gauge and diffeomorphism invariant. It is a straightforward consequence of the fact that $h_\alpha(A_0)$ and $h_\alpha(A_0)^{-1}$ are both equal to identity. Therefore, our Hilbert space contains also the semi-classical solutions. These are given by ψ 's which are sharply peaked around some particular value of A_0 .

Let us recall at this point that the physical Hilbert space is a subset of the *dual* space to the space of cylindrical functions. Therefore the action of the Hamiltonian operator should be defined in the dual space. In the previous sections we have constructed operators acting on cylindrical functions. How do we define their action on distributions? We define $\langle \tilde{\Phi} | := \langle \Phi | \hat{H}$ to be the element of the dual space such that $\langle \tilde{\Phi} | \Psi \rangle = \langle \Phi | \hat{H} | \Psi \rangle$ for all cylindrical functions $| \Psi \rangle$ and the last expression is evaluated by acting with \hat{H} on Ψ as defined before. Notice that the operator defined in the previous section acts on a cylindrical function by adding edges in the fundamental representation to the

underlying graph γ (and multiplying it by certain spins dependent number). Therefore it acts on distributions by *subtracting* edges from the underlying graph.² This means that we have just found another family of solutions. Those are (elements dual to) cylindrical functions which do not have edges to remove. More precisely, those for which none of the edges is of the form that is created by the action of \hat{H} on a cylindrical function Ψ_γ . Similarly as for flat connection solutions, these statements are true for any of the Hamiltonian operators constructed in previous sections.

²This description refers only to very specific type of elements of \mathcal{Cyl}^* . Namely the elements $\langle \Phi |$ such that $\langle \Phi | \Psi \rangle = 0$ for any cylindrical state $|\Psi \rangle$ which is linearly independent from $|\Phi \rangle$.

Chapter 5

Solving quantum Hamiltonian constraint

In this chapter we present two strategies for an explicit construction of the physical Hilbert space. In Section 5.1 we discuss the quantum implementation of the idea of the Wick transform. It provides the Lorentzian space given the Euclidean one is already known (Euclidean theory is often easier to solve). We concentrate here on the states which have support only on flat connections as those are not only simple but also relevant in the semi-classical regime. In Section 5.2 we discuss a completely different way of obtaining physical states. Here, we use (and prove) the fact that any solution to the scalar constraint is a sum of "elementary" solutions which can be found by considering certain disjoint finite dimensional subspaces in the kinematical Hilbert space. For simplicity, we only describe the method for the Euclidean theory and demonstrate its application in an explicit example.

It should be stressed here that the motivation for the two sections is very different from the point of view of the techniques used. In Section 5.1 we will use certain simplifications which are specific to 2+1 dimensions. Because the moduli space of flat connections is finite dimensional, we can reduce the quantum gravity, both Euclidean and Lorentzian, to a simple quantum mechanics problem. Therefore the calculations made in this section have no higher dimensional analog. However, since the idea of Wick

transform remained only at the formal level in higher dimensions, we will use this simplified model to provide the first rigorous implementation of quantum Wick transform. Furthermore, in this simplified setting, we already know the Lorentzian physical Hilbert space and we can check if Wick transform provides the correct answers. In Section 5.2, on the other hand, we try to preserve as much analogy as possible with the (Euclidean) 3+1 dimensional theory. Therefore we hope that analogous methods will be useful in higher dimensions.

5.1 Quantum Wick transform

In this section, we are going to illustrate certain strategy for obtaining solutions to the quantum constraints. This strategy has been proposed for 3+1 dimensional theory in [60] and [2]. The idea is to make use of the fact that the Euclidean Hamiltonian constraint takes a much more simple form than the Lorentzian one. Therefore one could start by solving the Euclidean theory and then hope to find a simple map on those states which would provide us with Lorentzian physical states. This was the original motivation to introduce the Wick transform.

Below we will follow the strategy proposed in [2]. Namely, recall from Section 3.4 that the (classical) Wick transform W is a Poisson bracket preserving automorphism on the algebra of phase-space functions. Therefore, its quantum analog \hat{W} will be an automorphism on the algebra of quantum operators. Let us therefore define the Lorentzian Hamiltonian operator as

$$\hat{H} := \hat{W} \circ \hat{H}^E \circ \hat{W}^{-1} := \exp\left(-\frac{1}{i\hbar}\hat{T}\right) \circ \hat{H}^E \circ \exp\left(\frac{1}{i\hbar}\hat{T}\right) \quad (5.1)$$

Using Equations (3.29)-(3.32), one can show that \hat{H} so defined will have the correct classical limit. The Lorentzian physical states can now be obtained by Wick transforming the kernel of \hat{H}^E :

$$\hat{H}^E|\Psi_E\rangle = 0 \Leftrightarrow \hat{H} \circ \hat{W}|\Psi_E\rangle \equiv \hat{H}|\Psi\rangle = 0 \quad (5.2)$$

Since T is a multiple of K of the Section 4.2, the ideas provided there to construct the operator \hat{K} can equally well be used here to construct \hat{T} . In the next subsection we will choose, however, to follow a different route.

5.1.1 Wick operator

In Section 4.4 we have showed that functions which have support only on flat connections are annihilated by the Hamiltonian operators. Below, we will construct the Wick transform operator restricted to the subspace of such functions. We have to fix the topology of the spatial slice M because the space of flat connections modulo gauge transformations $\mathcal{A}_0/\mathcal{G}$ is different for different topologies. A non-trivial and often studied example is the 2-torus \mathbb{T}^2 . Let us recall a simple description of $\mathcal{A}_0/\mathcal{G}$ in terms of representatives in each gauge equivalence class (for a detailed discussion see [44] or [41]).

In the case of $SU(2)$ connection the moduli space of flat connections is particularly simple. It can be described in terms of the following representatives:

$$A_a^i = (a_1(d\phi)_a + a_2(d\psi)_a) t^i \quad (5.3)$$

where $\phi, \psi \in [0, 2\pi)$ are angular coordinates on the torus, $a_1, a_2 \in [0, 1)$ are constants which provide a convenient parameterization of this moduli space and finally t^i is any constant unit vector in the Lie algebra $\mathfrak{su}(2)$. In the case of $SU(1, 1)$ connection, $\mathcal{A}_0/\mathcal{G}$ has three sectors - space-like, null and time-like. We will only need the description of the space-like sector for our purposes here. It turns out that it is again given by (5.3), but with parameters a_1, a_2 being now any positive real numbers.

Finally, in order to obtain a description of the reduced phase-space, we also provide the canonically conjugate momenta. For both gauge groups, the momenta are given by

$$E^{ai} = \left[k_1 (\partial_\phi)^a + k_2 (\partial_\psi)^a \right] t^i + C^a u^i \quad (5.4)$$

where constants $k_1, k_2 \in \mathbb{R}$ provide a convenient parameterization, C^a is any fixed vector field and u^i is constant and perpendicular to t^i . The purpose of adding the second fixed term in the above expression is to make the dyad non-degenerate. The Poisson bracket reduces to

$$\{a_I, k_J\} = \frac{1}{(2\pi)^2} \delta_{IJ} \quad (5.5)$$

with $I, J = 1, 2$ and other brackets vanishing ($(2\pi)^2$ comes from integration over the angles). We are now ready to restrict the classical function T to the reduced phase-space. Using the fact that all the derivatives of the dyad E vanish, we obtain¹:

$$\begin{aligned} T &:= \frac{i\pi}{2} \int_M d^2x K_a^i E_i^a = \frac{i\pi}{2} \int_M d^2x A_a^i E_i^a = \\ &\frac{i\pi}{2} \int_M d^2x (a_1(d\phi)_a + a_2(d\psi)_a) \left(k_1(\partial_\phi)^a + k_2(\partial_\psi)^a \right) t^i t_i = (2\pi)^2 \frac{i\pi}{2} (a_1 k_1 + a_2 k_2) \end{aligned} \quad (5.6)$$

This expression is sufficiently simple for an easy construction of its quantum analog. Upon quantization a_I becomes a multiplication operator and $(2\pi)^2 k_I$ is replaced by $\frac{\hbar}{i} \frac{\partial}{\partial a_I}$. Therefore we obtain the following quantum Wick transform

$$\hat{W} = \exp \left[\frac{i\pi}{4} \sum_{I=1}^2 \left(a_I \frac{\partial}{\partial a_I} + \frac{\partial}{\partial a_I} a_I \right) \right] \quad (5.7)$$

where we chose the symmetric ordering of the operators (this choice makes \hat{W} self-adjoint).

5.1.2 Action on the physical states

In order to evaluate the action of the Wick transform operator on a physical state $\Psi^E(A_0) \equiv \Psi^E(a_1, a_2)$, let us recall that in the Euclidean theory $a_1, a_2 \in [0, 1]$ parameterize a product of two circles $S^1 \times S^1$. In the Lorentzian theory they have different role, these are parameters on $\mathbb{R}_+ \times \mathbb{R}_+$, where \mathbb{R}_+ is the set of positive real

¹Here we are using $A = \Gamma + K$ for convenience. The main results would not change if we used the connection from Section 3.1

numbers. We can, however, equivalently regard the Euclidean functions as periodic functions on \mathbb{R}_+^2 (with the period equal to one in each of the variables). This is what we will do from now on in order to be able to map Euclidean functions to the Lorentzian ones. Therefore the Euclidean Hilbert space is now defined as the space of square integrable functions on a 2-torus: $\mathcal{H}^E = L^2(\mathbb{T}^2, d\mu_0)$, where the measure is given by $d\mu_0 := da_1 \wedge da_2$. It is well known that such space is spanned by the basis which consists of functions $\sin(n_1 a_1 + n_2 a_2)$ and $\cos(m_1 a_1 + m_2 a_2)$, where n_I, m_I ($I = 1, 2$) are arbitrary integers. Each of the basis elements is well defined not only for $(a_1, a_2) \in \mathbb{R}^2$, but also for arbitrary complex values of the two parameters. We will use this property to evaluate the action of the Wick transform operator on certain Euclidean physical states. We will use a dense subset of Hilbert space \mathcal{H}^E for explicit evaluation. The elements of the subset are finite linear combinations of sin and cos functions described above. For the rest of this section Ψ^E denotes a function which belongs to this dense subset. Even though Wick transform operator \hat{W} is Hermitian, it is not bounded. Therefore the properties of its extension to the whole Hilbert space (which can be defined through its action on the basis elements) need to be studied separately. We will leave this for future investigations.

We can now make the following series expansion:

$$\Psi^E(a_1, a_2) = \sum_{n=0}^{\infty} \frac{1}{n!} a_2^n \partial_2^n \Psi^E(a_1, 0) \quad (5.8)$$

Using this series expansion (and the analogous one in the other variable), it is a straightforward calculation to show that:

$$\hat{W}\Psi^E(a_1, a_2) = e^{i\frac{\pi}{2}}\Psi^E(ia_1, ia_2) \quad (5.9)$$

It is important to notice here that multiplication of the flat connection by i is precisely the action of the internal Wick rotation! Indeed, by construction we have $n_i A_0^i = 0$. This also implies that $W_R(A_0)$ belongs to the space-like sector of $SU(1, 1)$ moduli space. The physical state in the $SU(2)$ formulation of the Lorentzian theory is now defined as $\Psi(a_1, a_2)$ such that

$$\hat{W}\Psi^E(a_1, a_2) = \Psi(a_1, a_2) \quad (5.10)$$

Note however that (5.9) provides us with the following prescription for a construction of Ψ . Take a Euclidean solution Ψ^E , continue it analytically to complexified phase-space and evaluate it on the Wick rotated connection, i.e.

$$\Psi(A_0) = \Psi^E(W_R(A_0)) \quad (5.11)$$

Note that *quantum* Wick transform is described in terms of *classical* Wick rotation.

5.1.3 Mapping the physical observables and scalar product

In this section we would like to discuss the applicability of the Wick transform beyond construction of the physical states. We can also use it to map physical observables

via analog of the Equation (5.1), provided the classical Wick transform acts appropriately. Finally, as proposed in [60] it can also be used to construct the scalar product of Lorentzian theory.

As we know (see for example [1]) we can introduce the following, so called, loop variables which provide for us a complete set of observables:

$$T^0[\alpha](A) := \text{tr} h_\alpha(A) \quad (5.12)$$

$$T^1[\alpha](A, E) := \oint_\alpha ds^a \epsilon_{ab} \text{tr} E^b h_\alpha(A) \quad (5.13)$$

When restricted to our reduced (Euclidean) phase-space

$$T^0[n_1, n_2](a_1, a_2) = 2 \cos(a_1 n_1 + a_2 n_2)$$

where n_1, n_2 are winding numbers of the loop α . From this, it is easy to show that

$$\hat{W} \circ \hat{T}^0[n_1, n_2](a_1, a_2) \circ \hat{W}^{-1} = 2 \cosh(a_1 n_1 + a_2 n_2) \quad (5.14)$$

which is indeed equal to the operator T^0 evaluated on corresponding (through Wick rotation) $SU(1, 1)$ connection. Similarly

$$T^1[n_1, n_2](a_1, a_2, k_1, k_2) = 2(k_1 n_2 - k_2 n_1) \sin(a_1 n_1 + a_2 n_2)$$

Using (5.11) one can then show that

$$\hat{W} \circ \hat{T}^1[n_1, n_2](a_1, a_2, k_1, k_2) \circ \hat{W}^{-1} = 2i\hbar \sinh(a_1 n_1 + a_2 n_2) n_I \epsilon_{IJ} \frac{\partial}{\partial a_J} \quad (5.15)$$

which, again agrees with the operator appearing in the Lorentzian $SU(1, 1)$ theory.

In order to discuss the scalar product, recall that on our reduced quantum theory the appropriate product turns out to be [1] $d\mu_0(A_0) = da_1 \wedge da_2$. Thiemann suggested in [60] a strategy for constructing a scalar product on Wick transformed functions in such a way that Wick transform is an isomorphism. The main condition for constructing such a product was to provide a distribution ν on complexified connections defined as:

$$\nu(A^{\mathbb{C}}, \overline{A^{\mathbb{C}}}) = \left((\overline{\hat{W}^\dagger})' \right)^{-1} \left(\overline{(\hat{W}^\dagger)'} \right)^{-1} \delta(A^{\mathbb{C}}, \overline{A^{\mathbb{C}}}) \quad (5.16)$$

where bar denotes complex conjugation, prime denotes analytic continuation and the distribution δ is defined in [60] (it reduces the integration over complex connections to integration over the real section $A^{\mathbb{C}} = \overline{A^{\mathbb{C}}}$). In our case of moduli space of flat connections we can evaluate ν explicitly. Using (5.9) we obtain:

$$\nu(A^{\mathbb{C}}, \overline{A^{\mathbb{C}}}) \equiv \nu(a_I^{\mathbb{C}}, \overline{a_J^{\mathbb{C}}}) = \delta(-ia_I^{\mathbb{C}}, -\overline{ia_J^{\mathbb{C}}}) \quad (5.17)$$

The result is therefore that the measure provided by Thiemann's construction restricts the integration over complex connections to the real section which is exactly the $SU(1, 1)$ section defined by our Wick rotation. Furthermore, it reproduces the measure $da_1 \wedge da_2$ in which the Lorentzian observables \hat{T}^0 and \hat{T}^1 are self-adjoint.

5.2 Finding solutions by reduction to finite dimensional subspaces

In this section we present a new method of finding solutions to the quantum Hamiltonian constraint. First we will state the general theorem of Ashtekar and Lewandowski [15] and then apply it to one of the solutions discussed in Section 4.4 to obtain a new class of interesting solutions. In this section we describe the method only for Euclidean theory but, with some modifications, it can also be applied to simplify the problem of finding solutions to Lorentzian constraint.

5.2.1 General theorem

Recall that the Euclidean Hamiltonian operator acts on spin-network functions by changing the underlying labeled graph (γ, \vec{j}) . Let us call this action at one of the vertices by g_v . Therefore g_v acting on any labeled graph produces four new different labeled graphs. We will call a graph (γ_0, \vec{j}) simple if it can not be obtained by the action of g from any other graph. Consider now an element of \mathcal{Cyl}_{inv}^* , the diffeomorphism (and gauge) invariant subspace of the dual of the space of cylindrical functions, which is associated with the spin-network function based on a simple graph $T_{[\gamma_0, \vec{j}]}^*$ (square brackets denote diffeo- and gauge equivalence class). As explained in Section 4.4, every such element is a solution to the equation

$$\hat{H}^E T_{[\gamma_0, \vec{j}]}^* = 0 \tag{5.18}$$

Let $V_{[\gamma_0, \vec{j}]}^* \subset \mathcal{Cyl}_{inv}^*$ be a finite dimensional space spanned by such elements. Consider now operator h_v ($v \in V(\gamma_0)$) mapping vector spaces, defined as

$$h_v V_{[\gamma_0, \vec{j}]}^* = \bigoplus_{[\gamma, \vec{j}] \in g_v[\gamma_0, \vec{j}]} V_{[\gamma, \vec{j}]}^* \quad (5.19)$$

Finally, let us define a finite dimensional subspace $\mathcal{S}_{[\gamma_0, \vec{j}]}^{*(1)}$ of \mathcal{Cyl}_{inv}^* in the following way:

$$\mathcal{S}_{[\gamma_0, \vec{j}]}^{*(1)} = \bigoplus_{\gamma', \vec{j}', v'} h_{v'} V_{[\gamma', \vec{j}']}^* \quad (5.20)$$

where the sum extends only over the labeled graphs (γ', \vec{j}') and their vertices v' such that

$$h_{v'} V_{[\gamma', \vec{j}']}^* \cap h_v V_{[\gamma_0, \vec{j}]}^* \neq \emptyset \quad (5.21)$$

Analogously one can define $\mathcal{S}_{[\gamma_0, \vec{j}]}^{*(2)}$ as the subspace obtained by the action of h_v on all the subspaces $V_{[\gamma, \vec{j}]}^* \in \mathcal{S}_{[\gamma_0, \vec{j}]}^{*(1)}$ and adding all the other such subspaces which have a non-trivial intersection with this set. This process can be continued to construct a family of finite dimensional subspaces $\mathcal{S}_{[\gamma_0, \vec{j}]}^{*(n)}$ having an important property. Namely, for each two distinct diffeomorphism equivalence classes $[\gamma_0, \vec{j}]$, the corresponding subspaces $\mathcal{S}_{[\gamma_0, \vec{j}]}^{*(n)}$ either overlap or have the trivial intersection. This important fact allows us to write the following theorem.

For every $\Psi^* \in \mathcal{Cyl}_{inv}^*$ there is a *uniquely* defined decomposition

$$\Psi^* = \sum \Psi_{[\gamma_0, \vec{j}]}^{*(n)} \quad (5.22)$$

where $\Psi_{[\gamma_0, \vec{j}]}^{*(n)} \in \mathcal{S}_{[\gamma_0, \vec{j}]}^{*(n)}$ and we extended the range of index n to include zero in the obvious way. Furthermore, this theorem implies an important property of the solutions to the Euclidean Hamiltonian constraint. Namely we have

$$\left(\hat{H}^E \Psi^* = 0\right) \implies \left(\hat{H}^E \Psi_{[\gamma_0, \vec{j}]}^{*(n)} = 0\right) \quad (5.23)$$

Therefore the problem of finding the kernel of the Euclidean scalar constraint reduces to finding elementary solutions of finite dimensional problems.

5.2.2 New family of elementary solutions

In this subsection we will apply the theorem stated in the previous subsection to the distribution

$$\Psi^*(A) = \delta(F(A)) \quad (5.24)$$

This solution has been studied in more detail in [45], where it has been decomposed in the spin-network basis. The result is

$$\delta(F(A)) = \sum_{[I]} k_{[I]} T_{[I]}^*(A) \quad (5.25)$$

where we simplified notation via $[I] = [\gamma, \vec{j}]$ and

$$k_{[I]} := \int_{\mathcal{A}_0/\mathcal{G}} d\nu_0(A_0) T_{[I]}(A_0) \quad (5.26)$$

with ν_0 being the invariant, normalized measure on the moduli space of flat $SU(2)$ connections.

From now on, let us restrict our attention only to those subspaces $\mathcal{S}_{[\gamma_0, \vec{j}]}^{*(n)}$ for which the simple graph γ_0 is three-valent. Since the operation g_v does not create vertices of valence higher than three, it is equivalent to considering a theory in which only three-valent graphs give rise to states in physical Hilbert space. We will now find explicit expressions for the coefficients $k_{[I]}$ up to normalization factor within each of the finite dimensional subspaces.

First, let us recall that evaluation of certain spin network functions on flat connections has been done before. We will use here standard notation and conventions from [51, 50, 27]. $\theta(a, b, c)$ denotes the evaluation of a graph with three edges labeled with colors a, b and c (color is equal to twice the spin), such that all the edges begin in one vertex and all of them end at another vertex. Evaluation of a graph which can be obtained by projecting edges of a tetrahedron onto a plane is called $\text{Tet} \begin{bmatrix} a & b & e \\ c & d & f \end{bmatrix}$, where a through f denote colors of appropriate edges. Explicit formulae are known for both types of graphs and, since they are somewhat complicated, we refer the reader to the references.

Second, recall the following useful formula:

$$T_{g_v(\gamma, \vec{j})}(A_0) = \frac{\text{Tet} \begin{bmatrix} a & s & c \\ r & b & t \end{bmatrix}}{\theta(a, b, c)} T_{\gamma, \vec{j}}(A_0) \quad (5.27)$$

where A_0 is a flat connection, (a, b, c) are colors of the edges of γ meeting at vertex v and (r, s, t) are colors of the appropriate new edges created by g_v . We abuse notation

here and choose to denote by $g_v(\gamma, \vec{j})$ any, but only one, of the graphs produced by g_v . When one of the new colors (r, s, t) is zero, the tetrahedral graph reduces to the theta graph and we obtain simple identity. Let us denote the ratio in the above expression by $\text{Tet}(g_v(\gamma, \vec{j}))$. The formula (5.27) can be used to write down an explicit expression for calculating k 's. Let $T_{[\gamma, \vec{j}]}^* \in \mathcal{S}_{[\gamma_0, \vec{j}]}^{*(n)}$. We have

$$k_{[\gamma, \vec{j}]} = \prod_{i=0}^{n-1} \text{Tet} \left(g_{v_{n-i}} \circ g_{v_{n-i-1}} \circ \dots \circ g_{v_1}(\gamma_0, \vec{j}) \right) k_{[\gamma_0, \vec{j}]} \quad (5.28)$$

Notice, however, that $k_{[\gamma_0, \vec{j}]}$ is a common factor for all the terms in the sum over $[I]$. Therefore, the coefficients $k_{[I]}$ in the expansion of a physical state $\Psi_{[\gamma_0, \vec{j}]}^{*(n)}$ in terms of spin networks, are given (up to an overall constant) by the product of known labeling dependent factors Tet .

We will finish this section with the most simple non-trivial example to illustrate the applications of our result. Let us consider $T_{[\gamma_0, \vec{j}]}^*$ such that its evaluation on a flat connection is equal to $\theta(2, 2, 2)$. In other words, we choose γ_0 to be the theta graph with all the edges labeled by spin equal to one. Using (5.28) we obtain the following solution to the Hamiltonian constraint

$$\Psi_{[\theta(2,2,2)]}^{*(1)} = \frac{1}{N} \left\{ (-3)T_{\theta_{--}}^* + (-2)T_{\theta_{+-}}^* + (-2)T_{\theta_{-+}}^* + \frac{10}{3}T_{\theta_{++}}^* + \dots \right\} \quad (5.29)$$

where $\theta_{\pm\pm}$ are the graphs obtained from $\theta(2, 2, 2)$ through an action of g_v and superscripts $\pm\pm$ denote in the obvious way the new labels assigned to the two edges originating at v . Also, \dots stand for the analogous terms for other pairs of edges as well as for terms

at the second vertex (N is the normalization constant). Using the methods of [27] we can independently check if this state is really annihilated by the Hamiltonian operator. For this purpose it suffices to compute the action of the operator on the spin-network function whose underlying graph is our γ_0 . We will use the operator described at the end of Section 4.3. After a highly non-trivial calculation one obtains the following result:

$$\hat{H}_v T_{[\theta(2,2,2)]} \propto 0 \cdot T_{\theta_{++}} + \frac{1}{4} T_{\theta_{+-}} + \frac{1}{4} T_{\theta_{-+}} + \left(-\frac{1}{3}\right) T_{\theta_{--}} + \dots \quad (5.30)$$

Thus, one can check that, indeed, we have

$$\left(\langle \Psi_{[\theta(2,2,2)]}^{*(1)} | \hat{H} | T_{[\theta(2,2,2)]} \rangle = 0 \right) \implies \left(\hat{H} \delta(F(A)) |_{\mathcal{S}_{[\theta(2,2,2)]}^{*(1)}} = 0 \right) \quad (5.31)$$

where we decided to remind the reader that our solution $\Psi^{*(1)}$ is just a restriction of the δ -function to a suitable finite dimensional subspace.

Summarizing, we have shown here that the general theorem of Subsection 5.2.1 provides a powerful tool for constructing elementary solutions to the Hamiltonian constraint. Given a semi-classical or any other "non-elementary" solution one can decompose it into elementary ones and there exists a constructive procedure for finding explicitly the components of those solutions in the spin-network basis. We have described this procedure on the example of the δ -function solution in the Euclidean theory (Equation (5.28)). Also, we have checked by explicit calculation with one particular solution that "the method is correct", i.e. Euclidean Hamiltonian operator does annihilate an elementary solution obtained from the δ -function.

Chapter 6

Future directions

Let us summarize the thesis with pointing out possible directions for future investigations based on the results presented here. Isolated horizons proved useful for the quantum calculations of entropy in 3+1 dimensions [3, 4]. We showed in Chapter 2 how to construct isolated horizons phase-space in 2+1 dimensions. Therefore a natural question that arises here is whether one could perform the calculation of entropy based on quantization of that phase-space. Given our new formulation of the Lorentzian theory with compact gauge group, one can use the length (analog of the area) operator constructed in Chapter 4. The remaining issues are the proper quantum treatment of the horizon terms in the symplectic structure and quantum implementation of the boundary conditions. For preliminary results in this direction see [34].

In Chapter 5 we constructed explicitly the action of the Wick transform operator on functions of flat connections. However it still remains to check whether the Lorentzian solutions obtained in this way are normalizable and whether this method provides all the known solutions. Also, in order for the Wick transform to be useful in 3+1 dimensions, one needs to find a suitable regularization of the transform in the case of a more general connection than the flat one. This is not an easy task since one has to deal here with the extrinsic curvature which is a complicated functional of the dyad. In particular, one has to face the problem of choosing the appropriate regularization just as in the case of the

Hamiltonian operator. The work presented here may provide some clues towards solving this problem. Namely, we may try to find a regularization for which the quantum Wick transform continues to be described by the classical Wick rotation.

Finally, it has been suggested in [55] that one might be able to find the precise link between the canonical quantum gravity and the spin foam models by expanding out the projector on physical states $\delta(\hat{H})$. The coefficients in the expansion in powers of \hat{H} would then provide the appropriate amplitudes in the definition of a spin foam model [19, 48]. Notice, however, that the 2+1 dimensional Euclidean theory is usually formulated with the constraint $F(A) = 0$. Therefore, in this case, we would like to expand $\delta(F(A))$. In the last part of Chapter 5 we found out that the coefficients $k_{[T]}$ are roughly given by the evaluations of the tetrahedral graphs, which are closely related to 6-j symbols. It is thus natural to ask how they are related to the spin foam vertex amplitudes which are also given by the 6-j symbols.

Appendix

The 2+1 analog of the Newman-Penrose formalism

In this appendix we construct the 2+1 analog of the Newman-Penrose (NP) formalism [47, 49, 58].

A.1 Triads and spin-coefficients

In place of the Newman-Penrose null tetrad, we will use a triad consisting of two null vectors ℓ^a and n^a and a space-like vector m^a , subject to:

$$\ell \cdot \ell = n \cdot n = 0, \quad m \cdot m = 1 \tag{A.1}$$

$$\ell \cdot m = n \cdot m = 0 \tag{A.2}$$

$$\ell \cdot n = -1. \tag{A.3}$$

Note that, unlike in 3+1 dimensions, the vector m^a is real. Therefore, there will be no complex quantities appearing in our 2+1 analog of the NP formalism.

In terms of this triad, the space-time metric g_{ab} can be expressed as

$$g_{ab} = -2\ell_{(a}n_{b)} + m_a m_b, \tag{A.4}$$

and its inverse is given by

$$g^{ab} = -2\ell^{(a}n^{b)} + m^a m^b. \tag{A.5}$$

We will now investigate spin-coefficients, i.e., the derivatives of the triad vectors. The normalization and orthogonality conditions on the triad vectors immediately lead to the following relations:

$$\ell^b \nabla_a \ell_b = n^b \nabla_a n_b = m^b \nabla_a m_b = 0 \quad (\text{A.6})$$

$$\ell^b \nabla_a m_b = -m^b \nabla_a \ell_b \quad (\text{A.7})$$

$$\ell^b \nabla_a n_b = -n^b \nabla_a \ell_b \quad (\text{A.8})$$

$$n^b \nabla_a m_b = -m^b \nabla_a n_b \quad (\text{A.9})$$

If we did not have any relations between ℓ , n and m , we would have had $3 \times 3 \times 3 = 27$ independent spin coefficients. However, the above equations impose $3 \times 6 = 18$ relations between them whence the number of independent parameters is reduced to just 9. To keep as close a contact with the standard NP framework, our notation will closely follow that in [47, 49, 58]. However, since we have only a real spatial triad vector m^a rather than the pair m^a, \bar{m}^a of the standard NP framework, there are some inevitable discrepancies in factors of 2. The notation is summarized in tables A.1 – A.3.

		ℓ^b	n^b	m^b
D	ℓ^a	0	$-\epsilon$	$-\kappa_{NP}$
Δ	n^a	0	$-\gamma$	$-\tau$
δ	m^a	0	$-\alpha$	$-\rho$

Table A.1. The components of $\nabla_a \ell_b$.

		ℓ^b	n^b	m^b
D	ℓ^a	ϵ	0	π
Δ	n^a	γ	0	ν
δ	m^a	α	0	μ

Table A.2. The components of $\nabla_a n_b$.

In terms of these spin coefficients, the covariant derivatives of the triad vectors are given by:

$$\begin{aligned} \nabla_a \ell_b &= -\epsilon n_a \ell_b + \kappa_{NP} n_a m_b - \gamma \ell_a \ell_b \\ &\quad + \tau \ell_a m_b + \alpha m_a \ell_b - \rho m_a m_b \end{aligned} \tag{A.10}$$

$$\begin{aligned} \nabla_a n_b &= \epsilon n_a n_b - \pi n_a m_b + \gamma \ell_a n_b \\ &\quad - \nu \ell_a m_b - \alpha m_a n_b + \mu m_a m_b \end{aligned} \tag{A.11}$$

$$\begin{aligned} \nabla_a m_b &= \kappa_{NP} n_a n_b - \pi n_a \ell_b + \tau \ell_a n_b \\ &\quad - \nu \ell_a \ell_b - \rho m_a n_b + \mu m_a \ell_b \end{aligned} \tag{A.12}$$

Hence the divergences of the triad vectors, used in the main text, are given by:

$$\nabla_a \ell^a = \epsilon - \rho \tag{A.13}$$

$$\nabla_a n^a = \mu - \gamma \tag{A.14}$$

$$\nabla_a m^a = \pi - \tau \tag{A.15}$$

		ℓ^b	n^b	m^b
D	ℓ^a	κ_{NP}	$-\pi$	0
Δ	n^a	τ	$-\nu$	0
δ	m^a	ρ	$-\mu$	0

Table A.3. The components of $\nabla_a m_b$.

We conclude this section with examples 2 (the generalized BTZ black hole) and 3 (the Clément solution) discussed in section 2.1. It is easy to verify that a desired triad in the generalized BTZ space-time is given by:

$$\ell^a = \partial_v + \frac{1}{2}(N)^2 \partial_r - N^\phi \partial_\phi \tag{A.16}$$

$$n^a = -\partial_r \tag{A.17}$$

$$m^a = \frac{1}{r} \partial_\phi \tag{A.18}$$

The corresponding co-triads are

$$\ell_a = -\frac{1}{2}(N)^2 dv + dr \tag{A.19}$$

$$n_a = -dv \tag{A.20}$$

$$m_a = rN^\phi dv + rd\phi. \tag{A.21}$$

For this triad, the spin-coefficients are:

$$\begin{aligned} \epsilon &= \frac{f'(r)}{2} - r(N^\phi)^2, & \gamma &= 0, & \alpha &= N^\phi, \\ \kappa_{NP} &= 0, & \tau &= N^\phi, & \rho &= -\frac{1}{2r}(N)^2 \end{aligned} \quad (\text{A.22})$$

$$\pi = N^\phi, \quad \nu = 0, \quad \mu = -\frac{1}{r}$$

For the BTZ black hole, the function $f(r)$ is given by

$$f(r) = -\frac{M}{\pi} + \frac{r^2}{l^2}. \quad (\text{A.23})$$

For the Clément solution, a convenient triad is

$$\begin{aligned} \ell &= -\frac{1}{2}N^2 dv + \frac{r}{K} dr, \\ m &= K d\phi + KN^\phi dv, \\ n &= -dv, \end{aligned} \quad (\text{A.24})$$

and the corresponding spin-coefficients are given by:

$$\epsilon = -K \left(\Lambda + (N^\phi)^2 \right) - \frac{Q^2 K}{4\pi r^2} \left(1 + \omega N^\phi \right)^2, \quad (\text{A.25})$$

$$\alpha = \pi = \tau = \frac{\omega Q^2}{4\pi K^2} \left(1 - 2 \ln \frac{r}{\bar{r}_0} \right), \quad (\text{A.26})$$

$$\rho = -\frac{N^2}{2K} \left(1 + \frac{Q^2 \omega^2}{4\pi r^2} \right), \quad (\text{A.27})$$

$$\mu = -\frac{1}{K} \left(1 + \frac{Q^2 \omega^2}{4\pi r^2} \right), \quad (\text{A.28})$$

$$\gamma = 0, \quad (\text{A.29})$$

$$\nu = \frac{Q^2 \omega}{4\pi r^2}, \quad (\text{A.30})$$

$$\kappa_{NP} = 0. \quad (\text{A.31})$$

A.2 Curvature

Since we are in $2 + 1$ dimensions, all the information of the curvature tensor is contained in the Ricci tensor R_{ab} . We will thus calculate the different components of R_{ab} in our preferred triads. Our conventions for the Riemann tensor are:

$$\nabla_a \nabla_b t^c - \nabla_b \nabla_a t^c = -R_{abd}^c t^d. \quad (\text{A.32})$$

Using the tables of the previous section we can express components of the Ricci tensor in terms of the spin coefficients as follows:

$$\begin{aligned}
R_{ab}\ell^a\ell^b &= -\pi\kappa_{NP} + 2\alpha\kappa_{NP} - \epsilon\rho - \rho^2 + \kappa_{NP}\tau + \\
&\quad + \mathcal{L}_\ell\rho - \mathcal{L}_m\kappa_{NP}
\end{aligned} \tag{A.33}$$

$$\begin{aligned}
R_{ab}\ell^an^b &= \pi^2 - \pi\alpha + 2\gamma\epsilon - \epsilon\mu + \mu\rho - \pi\tau - \alpha\tau + \\
&\quad + \mathcal{L}_\ell\gamma - \mathcal{L}_\ell\mu - \mathcal{L}_n\epsilon + \mathcal{L}_m\pi
\end{aligned} \tag{A.34}$$

$$R_{ab}\ell^am^b = 2\gamma\kappa_{NP} - \pi\rho - \rho\tau + \mathcal{L}_\ell\tau - \mathcal{L}_n\kappa_{NP} \tag{A.35}$$

$$\begin{aligned}
R_{ab}n^a\ell^b &= -\pi\alpha + 2\gamma\epsilon - \gamma\rho + \mu\rho - \pi\tau - \alpha\tau + \tau^2 + \\
&\quad + \mathcal{L}_\ell\gamma - \mathcal{L}_n\epsilon + \mathcal{L}_n\rho - \mathcal{L}_m\tau
\end{aligned} \tag{A.36}$$

$$R_{ab}n^an^b = -\gamma\mu - \mu^2 + \pi\nu + 2\alpha\nu - \nu\tau - \mathcal{L}_n\mu + \mathcal{L}_m\nu \tag{A.37}$$

$$R_{ab}n^am^b = -\pi\mu + 2\epsilon\nu - \mu\tau + \mathcal{L}_\ell\nu - \mathcal{L}_n\pi \tag{A.38}$$

$$\begin{aligned}
R_{ab}m^a\ell^b &= -\pi\epsilon + \alpha\epsilon + \gamma\kappa_{NP} + \kappa_{NP}\mu - \pi\rho - \alpha\rho + \\
&\quad + \mathcal{L}_\ell\alpha - \mathcal{L}_m\epsilon
\end{aligned} \tag{A.39}$$

$$\begin{aligned}
R_{ab}m^an^b &= \alpha\gamma - \alpha\mu + \epsilon\nu + \nu\rho - \gamma\tau - \mu\tau - \\
&\quad - \mathcal{L}_n\alpha + \mathcal{L}_m\gamma
\end{aligned} \tag{A.40}$$

$$\begin{aligned}
R_{ab}m^am^b &= -\pi^2 + \epsilon\mu + 2\kappa_{NP}\nu + \gamma\rho - 2\mu\rho - \tau^2 + \\
&\quad + \mathcal{L}_\ell\mu - \mathcal{L}_n\rho - \mathcal{L}_m\pi + \mathcal{L}_m\tau
\end{aligned} \tag{A.41}$$

Finally, since the Ricci tensor is symmetric, we obtain the following restrictions on the spin coefficients:

$$0 = \pi^2 - \epsilon\mu + \gamma\rho - \tau^2 - \mathcal{L}_\ell\mu - \mathcal{L}_n\rho + \mathcal{L}_m\pi + \mathcal{L}_m\tau \quad (\text{A.42})$$

$$0 = \pi\epsilon - \alpha\epsilon + \gamma\kappa_{NP} - \kappa_{NP}\mu + \alpha\rho - \rho\tau - \mathcal{L}_\ell\alpha + \mathcal{L}_\ell\tau - \mathcal{L}_n\kappa_{NP} + \mathcal{L}_m\epsilon \quad (\text{A.43})$$

$$0 = -\alpha\gamma - \pi\mu + \alpha\mu + \epsilon\nu - \nu\rho + \gamma\tau + \mathcal{L}_\ell\nu - \mathcal{L}_n\pi + \mathcal{L}_n\alpha - \mathcal{L}_m\gamma \quad (\text{A.44})$$

A.3 Triad rotations

In this section we investigate how our spin-coefficients change under Lorentz transformations. We begin with a boost in the plane spanned by ℓ^a and n^a :

$$\ell^a \longrightarrow c \ell^a \quad (\text{A.45})$$

$$n^a \longrightarrow \frac{1}{c} n^a \quad (\text{A.46})$$

$$m^a \longrightarrow m^a \quad (\text{A.47})$$

Under the action of this boost, we have:

$$\begin{aligned} \kappa'_{NP} &= c^2 \kappa_{NP} & \pi' &= \pi & \epsilon' &= c\epsilon + \ell^a \nabla_a c \\ \tau' &= \tau & \nu' &= \frac{1}{c^2} \nu & \gamma' &= \frac{1}{c} \left(\gamma + \frac{1}{c} n^a \nabla_a c \right) \\ \rho' &= c\rho & \mu' &= \frac{1}{c} \mu & \alpha' &= \alpha + \frac{1}{c} m^a \nabla_a c \end{aligned} \tag{A.48}$$

Next, let us consider a null rotation:

$$\ell^a \longrightarrow \ell^a \tag{A.49}$$

$$n^a \longrightarrow \frac{1}{2} c^2 \ell^a + n^a + c m^a \tag{A.50}$$

$$m^a \longrightarrow c \ell^a + m^a \tag{A.51}$$

The coefficients now transform as follows:

$$\kappa'_{NP} = \kappa_{NP} \tag{A.52}$$

$$\tau' = \tau + \frac{1}{2} c^2 \kappa_{NP} + c\rho \tag{A.53}$$

$$\rho' = \rho + c\kappa_{NP} \tag{A.54}$$

$$\pi' = \pi + \frac{1}{2}c^2\kappa_{NP} + c\epsilon + \ell^a\nabla_a c \quad (\text{A.55})$$

$$\begin{aligned} \nu' = \nu + \frac{1}{2}c^3\epsilon + \frac{1}{4}c^4\kappa_{NP} + c\gamma + \frac{1}{2}c^2\tau + c^2\alpha + c^3\rho + \frac{1}{2}c^2\pi \\ + \frac{1}{2}c^2\ell^a\nabla_a c + n^a\nabla_a c + cm^a\nabla_a c \end{aligned} \quad (\text{A.56})$$

$$\begin{aligned} \mu' = \mu + c^2\epsilon + \frac{1}{2}c^3\kappa_{NP} + c\alpha + c\pi + \frac{1}{2}c^2\rho + \\ + c\ell^a\nabla_a c + m^a\nabla_a c \end{aligned} \quad (\text{A.57})$$

$$\epsilon' = \epsilon + c\kappa_{NP} \quad (\text{A.58})$$

$$\gamma' = \gamma + \frac{1}{2}c^2\epsilon + \frac{1}{2}c^3\kappa_{NP} + c\tau + c\alpha + c^2\rho \quad (\text{A.59})$$

$$\alpha' = \alpha + c\epsilon + c^2\kappa_{NP} + c\rho \quad (\text{A.60})$$

A.4 Components of the gravitational connection A

We can express the covariant derivative operator ∇_a in terms of the connection 1-form A_a^I . Using the relation

$$\nabla_a v_b = A_{aJ}^I v^J e_{Ib}, \quad (\text{A.61})$$

where e_{Ib} is the triad, and using

$$A_a^I e_{Ib} = \epsilon_{KI} A_a^K e_{Ib} \quad (\text{A.62})$$

we arrive at the desired expression:

$$\begin{aligned}
A_a^K &= (\pi n_a + \nu \ell_a - \mu m_a) \ell^K \\
&\quad + (\kappa_{NP} n_a + \tau \ell_a - \rho m_a) n^K \\
&\quad + (-\epsilon n_a - \gamma \ell_a + \alpha m_a) m^K
\end{aligned} \tag{A.63}$$

The analogous expression for the triad is just

$$e_a^I = -\ell_a n^I - n_a \ell^I + m_a m^I. \tag{A.64}$$

A.5 The Maxwell field and equations

To conclude, let us consider the Maxwell field. The components of the field strength \mathbf{F} in our triad define the analogs of the NP Φ_i :

$$\mathbf{F} = \Phi_0 n \wedge m + \Phi_1 \ell \wedge n + \Phi_2 m \wedge \ell. \tag{A.65}$$

Finally, the Maxwell equations are then given by

$$D\Phi_1 - \delta\Phi_0 = (\pi - \alpha)\Phi_0 + \rho\Phi_1 - \kappa_{NP}\Phi_2, \tag{A.66}$$

$$2D\Phi_2 - \delta\Phi_1 = -\mu\Phi_0 + 2\pi\Phi_1 + (\rho - 2\epsilon)\Phi_2, \tag{A.67}$$

$$2\Delta\Phi_0 - \delta\Phi_1 = (2\gamma - \mu)\Phi_0 - 2\tau\Phi_1 + \rho\Phi_2, \tag{A.68}$$

$$\Delta\Phi_1 - \delta\Phi_2 = \nu\Phi_0 - \mu\Phi_1 + (\alpha - \tau)\Phi_2. \tag{A.69}$$

A.6 Horizons

Because of the various boundary conditions, a number of simplifications arise at the horizon Δ . First, it is convenient to assume that the null vector n is exact, $dn \hat{=} 0$. Then, $\alpha \hat{=} \pi$. If Δ is a non-expanding horizon, two of the spin coefficients vanish; $\rho \hat{=} 0$ and $\kappa_{NP} \hat{=} 0$. Furthermore, $\ell^a D_a \pi \hat{=} m^a D_a \epsilon$ and $\ell^a D_a \tau \hat{=} 0$. The Ricci tensor is constrained: $R_{ab} \ell^a \ell^b \hat{=} 0$, $R_{ab} \ell^a m^b \hat{=} 0$. Finally, for the Maxwell field, $\Phi_0 \hat{=} 0$ and $\ell^a D_a \Phi_1 \hat{=} 0$.

On a weakly isolated horizon, spin coefficients are further restricted: $\epsilon \hat{=} const$. In the non-extremal case, $\epsilon \not\hat{=} 0$, the preferred foliation is characterized by $\pi \hat{=} const$. On an isolated horizon, two further conditions hold: $\ell^a D_a \mu \hat{=} 0$ and $\ell^a D_a (R_{cd} m^c m^d) \hat{=} 0$.

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Vita

Jacek Wiśniewski was born in Bydgoszcz, Poland on January 11, 1973. He attended primary school Szkoła Podstawowa Nr 4 and high school Technikum Elektroniczne, both in Bydgoszcz. He graduated from high school with an overall grade very good and with technical diploma in electronics. During the years 1992-97, he studied at Wydział Fizyki (Physics Department) of Uniwersytet Warszawski (Warsaw University). During this time he also spent one semester at the Vrije Universiteit (Free University) in Amsterdam, Netherlands as well as one semester at the University of Oxford, U.K. He graduated with an overall grade very good and earned his M.S. degree specializing in theoretical physics.

Mr. Wiśniewski has joined Physics Department of The Pennsylvania State University in the Fall semester of 1997. He passed the Candidacy Examination in the Spring of 1998. In the Fall of 1999, he passed the Comprehensive Examination and became a member of the Center for Gravitational Physics and Geometry. He completed all the course requirements with the GPA of 3.93 and served as a recitation instructor for introductory physics courses for nine semesters. In his research, supported through the Center and performed under supervision of Prof. Abhay Ashtekar, he specialized in theoretical general relativity and quantum gravity. For the work at the department, he was awarded Roberts Fellowship and two times Duncan Fellowship from the funds of Eberly College of Science.

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