Matrix methods in loop quantum cosmology

Daniel Cartin[1] Gaurav Khanna[2]

[1]Naval Academy Preparatory School
cartin@naps.edu

[2]University of Massachusetts, Dartmouth
gkhanna@umassd.edu

Quantum Gravity in the Americas III
August 2006

Outline

1. Introduction
2. Examples in LQC
   - Schwarzschild interior model
   - Isotropic model with scalar field
3. Conclusions

Outline

1. Introduction
2. Examples in LQC
   - Schwarzschild interior model
   - Isotropic model with scalar field
3. Conclusions

Loop quantum cosmology

- Loop quantum cosmology (LQC) is a simplifed model used to study the full theory of loop quantum gravity
- LQC uses the machinery of the full theory as much as possible, i.e spin networks, etc.
- We look at a symmetry reduced version of the Hamiltonian constraint, adapted to the model under consideration
- Because of the discrete nature of spin networks, this constraint will be a *recursion relation* or difference equation acting on the wave function

Solution methods in LQC

For these recursion relations arising in LQC, there are several methods to solve for and characterize the wave functions

- A one-parameter sequence can simply be iterated from initial data to find all other values
- This may not be easy for multi-parameter relations; this is why generating function techniques were employed (see gr-qc/0602025 and articles cited therein)
- Generating functions can quickly become onerous to solve for – e.g. the Bianchi I model with cosmological constant has a third-order differential equation for its generating function

Matrix methods

Here we discuss another possible method to characterize the wave function solutions, that of matrix techniques.

- The evolution of the wave function sequence is written in terms of a matrix equation
- In the asymptotic limit (i.e. large values of the appropriate parameter), the behavior of the sequence is dominated by those eigenvectors of the matrix with eigenvalues $|\lambda| > 1$
- The eigenvectors themselves show whether this behavior is simple growth without bound, or oscillatory (e.g. the sequence flips sign with every step increment in the parameter)

Outline

1. Introduction
2. Examples in LQC
    - Schwarzschild interior model
    - Isotropic model with scalar field
3. Conclusions

The Schwarzschild interior in LQC

Ashtekar and Bojowald (gr-qc/0509075)

- The interior of the spherically symmetric black hole is equivalent to the Kantowski-Sachs cosmological model, so it is amenable to LQC analysis
- The wave function Ψ is parametrized by the two triad eigenvalues, μ and τ; the minimum length δ and Immirzi parameter γ also appear in the model as quantum ambiguities
- $\mu = 0$ corresponds to the event horizon; $\tau = 0$ is the event horizon

Using separation of variables

Cartin and Khanna (gr-qc/0602025)

- Using a separation of variables method (for $\mu \geq 3\delta$), the Hamiltonian constraint can be written in terms of two one-parameter recursion relations for sequences α_μ and β_τ (i.e. $\Psi_{\mu,\tau} = \alpha_\mu \beta_\tau$)
- Here we focus exclusively on the α_μ sequence, since its recursion relation and its properties depend on a combination the Immirzi parameter γ and the minimum length δ
- The recursion relation is

$$(\mu+2\delta)\alpha_{\mu+4\delta}-(1+2\gamma^2\delta^2)\mu\alpha_\mu+(\mu-2\delta)\alpha_{\mu-4\delta} = \Lambda(\alpha_{\mu+2\delta}-\alpha_{\mu-2\delta})$$

Here, Λ is a separation constant

Defining the matrix relation

This relation for α_μ is equivalent to the following matrix equation

$$\begin{bmatrix} \alpha_{\mu+4\delta} \\ \alpha_{\mu+2\delta} \\ \alpha_\mu \\ \alpha_{\mu-2\delta} \end{bmatrix} = \begin{bmatrix} \frac{\Lambda}{\mu+2\delta} & \frac{\kappa\mu}{\mu+2\delta} & -\frac{\Lambda}{\mu+2\delta} & -\frac{\mu-2\delta}{\mu+2\delta} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \alpha_{\mu+2\delta} \\ \alpha_\mu \\ \alpha_{\mu-2\delta} \\ \alpha_{\mu-4\delta} \end{bmatrix}$$

or, to simplify notation,

$$\vec{\Psi}_{\mu+4\delta} = Q_{\mu+2\delta}\vec{\Psi}_{\mu+2\delta}$$

We have defined $\kappa = 1 + 2\gamma^2\delta^2$; the value of κ will determine the large μ behavior of the sequence

Large μ behavior of the sequence

In the large μ limit, i.e. $\mu \gg \delta$ and $\mu \gg \Lambda$, then $Q_{\mu+2\delta}$ becomes

$$Q_\mu \sim Q \equiv \begin{bmatrix} 0 & \kappa & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

The eigenvalues and eigenvectors of Q characterize the asymptotic evolution of α_μ and its associated vector $\vec{\Psi}_\mu$, since

$$\vec{\Psi}_{\mu+2M\delta} \sim Q^M \vec{\Psi}_\mu$$

Eigenvalues of Q

The eigenvectors $\vec{v}_\mu$ corresponding to eigenvalues $|\lambda| > 1$ grow without bound, since

$$\vec{v}_{\mu+2M\delta} \sim \lambda^M \vec{v}_\mu$$

The characteristic equation of Q is

$$\det(Q - I\lambda) = \lambda^4 - \kappa\lambda^2 + 1 = 0$$

Thus, the eigenvalues λ split into two categories:

- For $\kappa \leq 2$, λ is of the form $e^{i\theta}$, so $|\lambda| = 1$
- For $\kappa > 2$, two eigenvalues λ have $|\lambda| > 1$

Boundary conditions on α_μ

- In the classical phase space, solutions do not extend beyond a certain value in p_b (the triad component whose quantum eigenvalue is μ)
- We carry this over to the boundary condition $\alpha_\mu \to 0$ as $\mu \to \infty$ to ensure the wave function has the proper semi-classical behavior
- As we just saw, this can only happen when $\kappa \equiv 1 + 2\gamma^2\delta^2 \leq 2$

Limit on the Immirzi parameter

- In order for the minimum area in the reduced model to match the smallest eigenvalue of the area operator in the full theory, we set $\delta = 2\sqrt{3}$
- This implies $\kappa \leq (2\sqrt{6})^{-1} = 0.204124\ldots$
- Calculations by Meissner (gr-qc/0407054) using black hole entropy give $\gamma_M = 0.237533\ldots$
- Other possible versions which *do* satisfy the bound include changing the gauge group to SO(3) and/or neglecting higher order spin contributions

Outline

1. Introduction

2. Examples in LQC
   - Schwarzschild interior model
   - Isotropic model with scalar field

3. Conclusions

Isotropic model with scalar field

Ashtekar, Pawlowski and Singh (gr-qc/0607039)

The Hamiltonian constraint for the wave function $\Psi(v, \phi)$ for volume eigenvalue v and field ϕ can be written as

$$\frac{\partial^2 \Psi(v, \phi)}{\partial \phi^2} = -\hat{\Theta} \Psi(v, \phi)$$

where the operator $\hat{\Theta}$ is related to the action of the gravitational constraint, and is of the form

$$\hat{\Theta} \Psi(v, \phi) = F^+(v) \Psi(v+4, \phi) + F^0(v) \Psi(v, \phi) + F^-(v) \Psi(v-4, \phi)$$

Simplifying the constraint

- In order to use matrix methods for this constraint, we first discretize the field derivative:

$$\frac{\partial^2\Psi(v,\phi)}{\partial\phi^2} \to \frac{\Psi(v,\phi+2h) - 2\Psi(v,\phi) + \Psi(v,\phi-2h)}{4h^2}$$

- We also can use eigensequences of the operator $\hat{\Theta}$ (although this is not essential):

$$\hat{\Theta}\Psi_\omega(v,\phi) = \omega^2\Psi_\omega(v,\phi)$$

The evolution matrix and its eigenvalues

Putting all of this together, we get the equivalent matrix equation as

$$\begin{aligned} \left[\begin{array}{c} \Psi_\omega(v,\phi+2h) \\ \Psi_\omega \end{array}\right] &= \left[\begin{array}{cc} 2-4h^2\omega^2 & -1 \\ 1 & 0 \end{array}\right] \left[\begin{array}{c} \Psi_\omega(v,\phi) \\ \Psi_\omega(v,\phi-2h) \end{array}\right] \\ &\equiv Q_\phi(\omega) \left[\begin{array}{c} \Psi_\omega(v,\phi) \\ \Psi_\omega(v,\phi-2h) \end{array}\right] \end{aligned}$$

The eigenvalues of Q_ϕ, in the limit $h \longrightarrow 0$, are given by

$$\lambda = 1 \pm 2ih\omega + O(h^2\omega^2)$$

As a practical matter, instability can be kept in check by the appropriate choice of finite h in a numerical evolution

Mode analysis

Rosen, Jung and Khanna (gr-qc/0607044)

- The framework just covered works only for single parameter relations
- In order to deal with multiple parameters, we choose one as a "time", and the rest as "space"
- The wave function is then decomposed into its "spatial" Fourier modes, and we are back to a single evolution parameter (using eigensequences as we did here is the same idea)
- Cases analyzed by Rosen et al. include the Schwarzschild interior and (non-self-adjoint) Bianchi I

Outline

Summary

- Matrix methods offer a nice way to characterize the behavior of wave functions arising in LQC
- In particular, they show that...
  - physical considerations in the Schwarzschild interior model give an upper limit on the Immirzi parameter γ, and
  - models with scalar fields are inherently stable