

Gluon propagator from Yang-Mills spin foams

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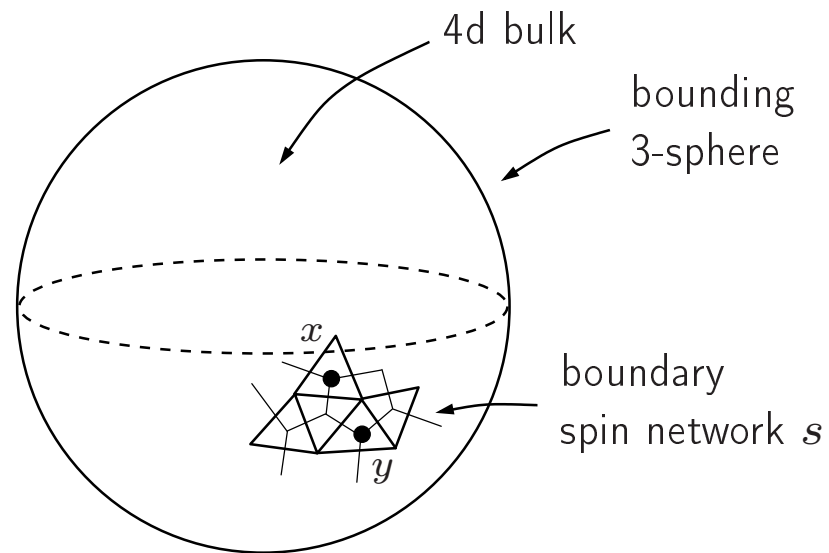
Plan of talk

1. Brief review: gravitons from spin foams
2. Why consider Yang-Mills spin foams?
3. $SU(2)$ YM theory and its representations
4. Gluons from Yang-Mills spin foams
5. Comparison to Rovelli's approach
6. Conclusion

1. Brief review: Gravitons from spin foams

Work by: Rovelli, Modesto, Bianchi, Speziale,
Willis, Livine . . .

Definition of the propagator



$$\mathbf{G}_{\mathbf{q}}^{abcd}(\mathbf{x}, \mathbf{y}) = \frac{\sum_s W[s] \hat{h}^{ab}(\mathbf{x}) \hat{h}^{cd}(\mathbf{y}) \Psi_{\mathbf{q}}[s]}{\sum_s W[s] \Psi_{\mathbf{q}}[s]},$$

Ingredients

Sum over spin foam amplitude for a given boundary spin network s :

$$W[s] = \frac{1}{Z} \int \mathbf{D}\phi \ f_s(\phi) \ e^{-\int \phi^2 - \frac{\lambda}{5!} \int \phi^5}$$

Field insertions:

$$\tilde{h}^{ab}(\vec{x}) = (\det g) g^{ab}(\vec{x}) - \delta^{ab} = E^{ai}(\vec{x}) E^{bi}(\vec{x}) - \delta^{ab}$$

Ingredients

The boundary state is a Gaussian that is peaked at a background geometry:

$$\Psi_{\mathbf{q}}[s] = C_{\Gamma} \exp \left\{ -\frac{1}{2} \sum_{l'} \alpha_{l'} \frac{j_l - j_l^{(0)}}{(j_l^{(0)})^{\frac{k}{2}}} \frac{j_{l'} - j_{l'}^{(0)}}{(j_{l'}^{(0)})^{\frac{k}{2}}} + i \sum_l \Phi_l^{(0)} j_l \right\}$$

The phase term determines the average momentum of the state.

Asymptotic formula for $10j$ symbol:

$$B(j_{ij}) = \sum_{\sigma} P(\sigma) \frac{1}{2} \left[e^{i S_{\text{Regge}}(\sigma) + i k \pi/4} + e^{-i S_{\text{Regge}}(\sigma) - i k \pi/4} \right] + D(j_{ik})$$

Recipe

- perturbative expansion of group field theory
- sum over complexes + sum over spin foams on each complex
- lowest order: complex = 4-simplex
 - asymptotic formula for $10j$ symbols
 - expansion of Regge action
 - boundary state “picks out” one of the three terms
- graviton propagator

2. Why consider Yang-Mills spin foams?

- difference to gravity: fixed background
- similarity: have to extract physics from spin foam amplitudes
- advantage: we already know what the result should be

↪ test if our methods make sense

- also inherent interest: confinement problem

3. $SU(2)$ YM theory and its representations

SU(2) lattice gauge theory in $d = 3$

Partition function:

$$Z = \int \left(\prod_{e \in \mathcal{C}_\kappa} dU_e \right) \exp \left(-\beta \mathcal{S}(U) \right)$$

Coupling:

$$\beta = \frac{4}{ag^2} + \frac{1}{3}$$

The action is a sum of face/plaquette actions:

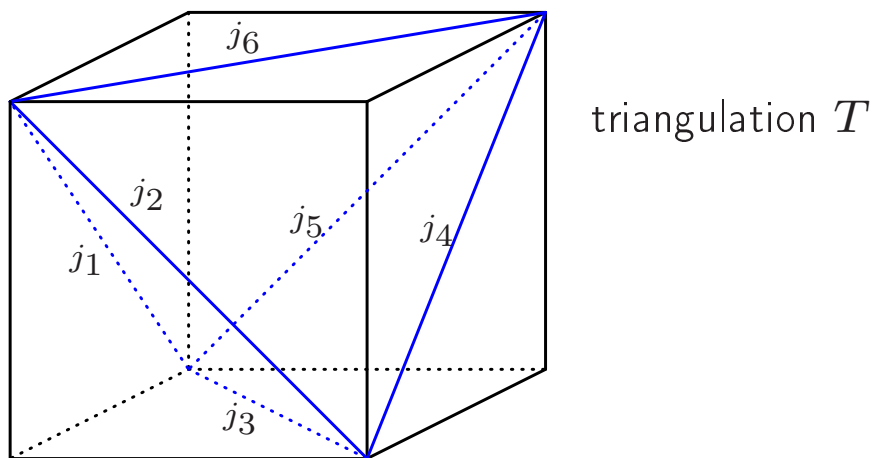
$$\mathcal{S} = \sum_{f \in \mathcal{C}_\kappa} \mathcal{S}_f$$

Spin foam representation

The spin foam representation results from the lattice path integral by 1. expanding the plaquette actions in loops, and 2. integrating out the connection variables U_e . There are several different ways to do this.

In $d = 3$ we can do it in such a way that the amplitudes become *6j-symbols*. [Anishetty, Cheluvvaraja, Sharatchandra]

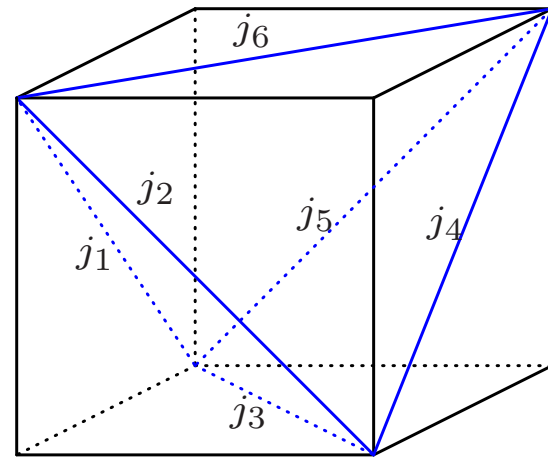
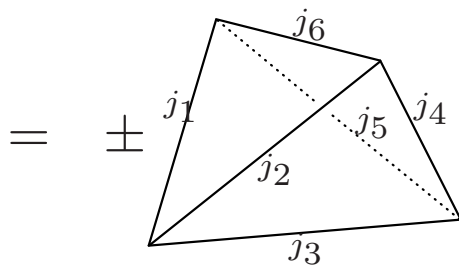
The cubes of the dual lattice κ^* are split into tetrahedra, giving a triangulation T :



Spin foam representation

$$Z = \sum_{\vec{j}} \left(\prod_{e \in CT} (2j_e + 1) \right) \left(\prod_{t \in CT} A_t \right) \left(\prod_{e \in \kappa^*} e^{-\frac{2}{\beta} j_e(j_e+1)} \right)$$

$$A_t = (-1)^{\sum_i j_i} \begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix}$$



3d SU(2) YM theory as a deformation of 3d gravity

3d SU(2) Yang-Mills theory can be seen as a deformation of 3d gravity:

$$Z_{\text{YM}} = \sum_{\vec{j}} \left(\prod_{e \in T} (2j_e + 1) \right) \left(\prod_{t \in T} A_t \right) \left(\prod_{e \in \kappa^*} e^{-\frac{2}{\beta} j_e(j_e+1)} \right)$$

$$Z_{\text{BF}} = \sum_{\vec{j}} \left(\prod_{e \in T} (2j_e + 1) \right) \left(\prod_{t \in T} A_t \right)$$

4. Gluons from Yang-Mills spin foams

Diakonov & Petrov

Diakonov & Petrov suggested an argument for extracting gluons from spin foams: [hep-th/9912268]

- replace sums over spins by integrals
- use asymptotic formula for $6j$ symbols
- stationary phase approximation

→ flatness constraint on spin foams

- solve constraint

→ gluons

Present work

We **extend** this argument by including static quarks as sources and effects due to discreteness of spin.

- Poisson summation formula

→ trade discreteness of spins for topological excitations

- use asymptotic formula for $6j$ symbols
- stationary phase approximation

→ flatness constraint on spin foams
+ defect created by source

- solve constraint

→ gluons + topological excitations

→ static potential

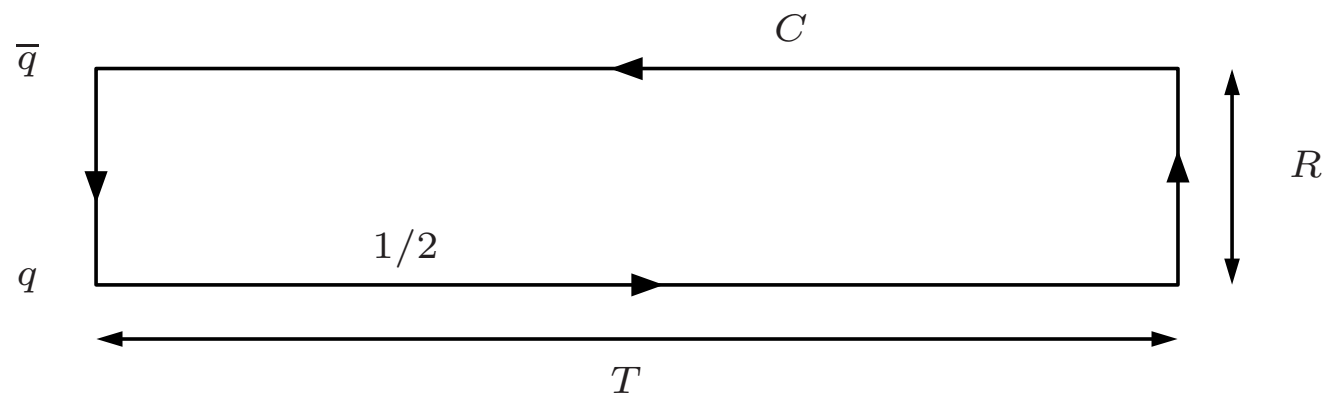
Topological excitations

In this talk we ignore discreteness effects and replace the sums over spins simply by integrals.

The proper treatment of discreteness leads to additional degrees of freedom that are similar to monopoles in $U(1)$, and may be of interest for the confinement problem.

Potential between test quarks

Static “test” quarks can be represented by a rectangular Wilson loop



The potential between the quarks is extracted from the expectation value of the Wilson loop:

$$V(R) = \lim_{T \rightarrow \infty} -\frac{1}{T} \ln \langle W_C \rangle$$

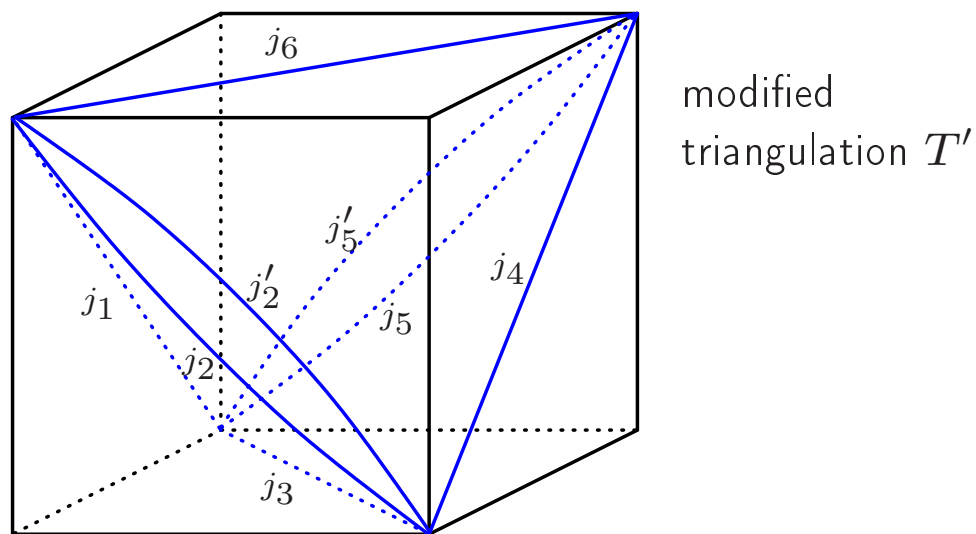
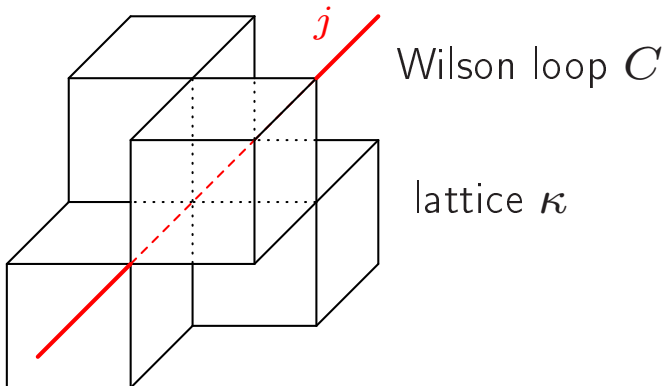
Wilson loop in the spin foam representation

Qualitatively:

$\langle W_{jC} \rangle =$ sum over all spin foams that are bounded by the Wilson loop

More precisely:

$$\langle W_{jC} \rangle = \frac{1}{Z} \sum_{\vec{j}} \left(\prod_{e \subset T} (2j_e + 1) \right) \left(\prod_{t \subset T'} A_t \right) \left(\prod_{v \subset C} A_v^{9j} \right) \left(\prod_{e \subset \kappa^*} e^{-\frac{2}{\beta} j_e(j_e+1)} \right)$$



$$A_v^{9j} = \pm \left\{ \begin{array}{ccc} j_1 & j_2 & j_3 \\ j'_5 & j & j_5 \\ j_6 & j'_2 & j_4 \end{array} \right\} = \pm \left\{ \begin{array}{ccc} j_1 & j'_5 & j_6 \\ j_2 & j & j'_2 \\ j_3 & j_4 & j_5 \end{array} \right\}$$

Starting point

$$\langle W_{jC} \rangle = \frac{1}{Z} \sum_{\vec{j}} \left(\prod_{e \in T} (2j_e + 1) \right) \left(\prod_{t \in T'} A_t \right) \left(\prod_{v \in C} A_v^{9j} \right) \left(\prod_{e \in \kappa^*} e^{-\frac{2}{\beta} j_e(j_e+1)} \right)$$

a small, β large \leadsto typical spins large. We therefore approximate

$$2j_e + 1 \approx 2j_e, \quad j_e(j_e + 1) \approx j_e^2 - 1/4$$

We replace the sums over spins by integrals:

$$\sum_{j_e} \longrightarrow \int_0^\infty dj_e$$

Asymptotic formula for 6j symbols

Amplitude of a tetrahedron:

$$A_t = (-1)^{\sum_i j_i} \left\{ \begin{array}{ccc} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{array} \right\}$$

Interpret the spin assignments j_i as edge lengths $j_i + 1/2$ of the tetrahedron.

Then, for large j_i and a non-degenerate tetrahedron

$$A_t \approx \frac{1}{\sqrt{6\pi V}} \left(e^{iI_R} + e^{-iI_R} \right) .$$

Here, V is the volume of the tetrahedron, and I_R is the contribution of the tetrahedron to the Regge action—a discretized version of $\int d^3x \sqrt{g} R$ on triangulations.

Stationary phase approximation

Given the asymptotic formula, we expect an oscillatory behaviour of the gravity + source amplitude, compared to a slowly changing factor

$$e^{-\frac{2}{\beta} j_e^2}.$$

\rightsquigarrow replace path integral by a path integral over stationary points of the gravity + source part, i.e.

$$\int D j \ A_{\text{grav+source}}(j) \ A_{\text{rest}}(j) \ \approx \ \int D j_{\text{cl}} \ A_{\text{grav+source}}(j_{\text{cl}}) \ A_{\text{rest}}(j_{\text{cl}})$$

where the spin foams j_{cl} are stationary points of the gravity + source part.

Outside the loop

How are these stationary points characterized?

For tetrahedra away from the loop C , the amplitude corresponds to pure 3d gravity.

\leadsto stationary points correspond to the flat geometry.

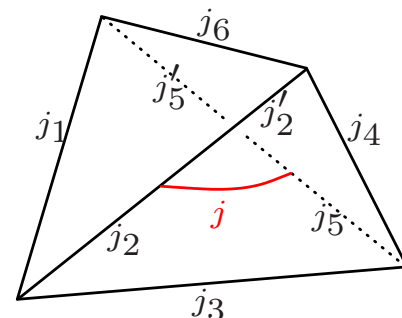
\leadsto Regge action $I_R = 0$ in asymptotic formula.

We ignore the volume factors, and set the amplitude $A_t = 1$.

We assume the same for degenerate tetrahedra.

Along the loop

We have no asymptotic formula for the $9j$ -symbol



We know, however, that the j_i are typically much larger than the j of the source, and that the $9j$ -symbol is zero unless j'_i , j_i and j satisfy the triangle inequality.

The simplest possibility: $9j$ -symbol has just the effect of ensuring the triangle inequality and can be otherwise treated like the $6j$ -symbols, i.e.

$$A_v^{9j} \approx \begin{cases} 1, & |j_i - j| \leq j'_i \leq j_i + j, \\ 0, & \text{otherwise.} \end{cases}$$

Flatness constraint

The stationary points are given by spin foams that are flat outside of C and satisfy the additional triangle inequality

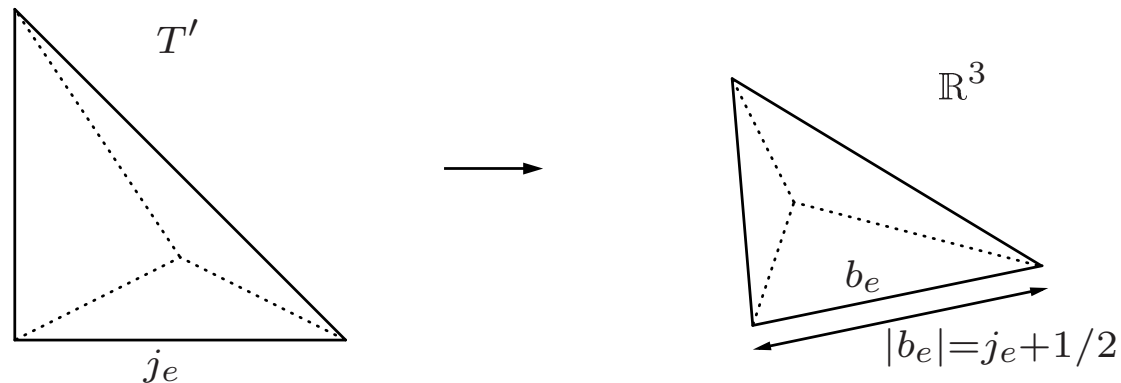
$$|j_i - j| \leq j'_i \leq j_i + j,$$

along C .

What do we mean by a “flat” spin foam?

The part of the spin foam outside C can be embedded in flat Euclidean \mathbb{R}^3 such that each edge has length $j_i + 1/2$.

Embedding in \mathbb{R}^3 color space



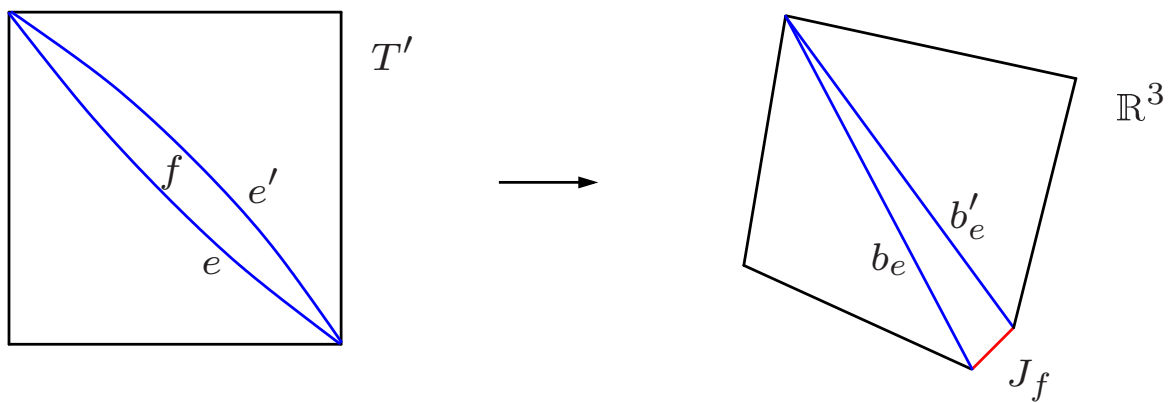
Each edge e of the triangulation T' is mapped into a difference vector b_e in \mathbb{R}^3 of length $j_e + 1/2$.

Along closed curves the vectors b_e close, i.e. $db = 0$.

Let us call the target space \mathbb{R}^3 **color** space.

Defect along C

Along C we do not have flatness. We only have the additional triangle inequalities.



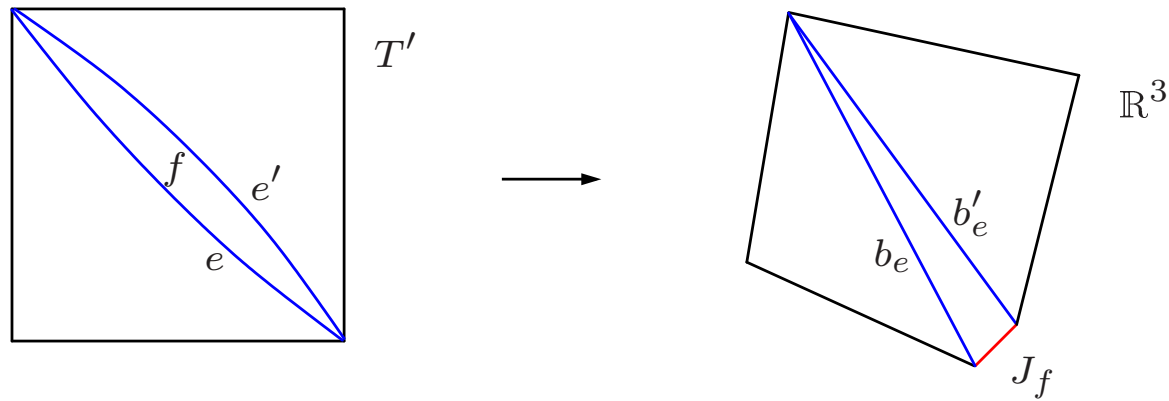
Around faces $f \subset T'$ dual to C the vectors b_e do not close, in general. We only have the triangle inequality

$$\left| |b_e| - j \right| \lesssim |b'_e| \leq |b_e| + j.$$

We represent this by a deficit vector

$$J_f = \begin{cases} j n_f, & f \text{ dual to } C, \\ 0, & \text{otherwise.} \end{cases}$$

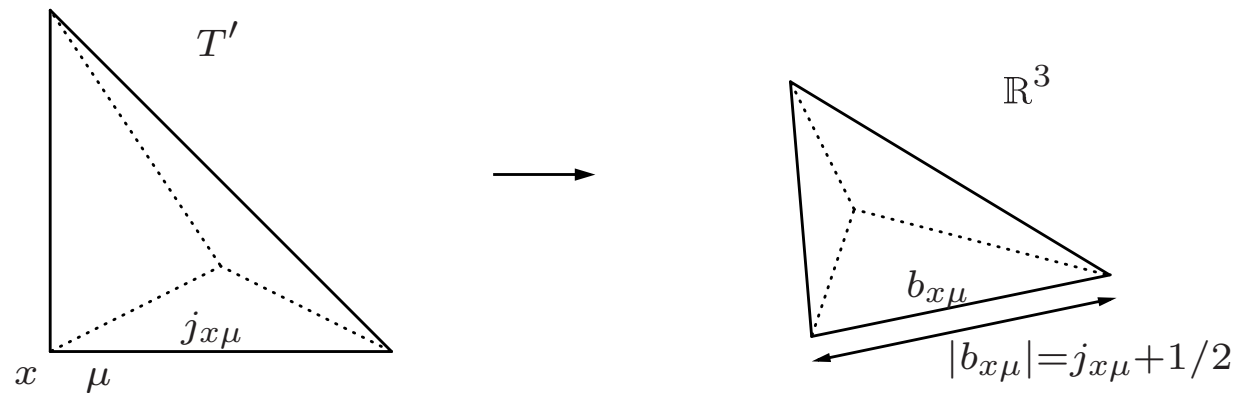
Defect along C



Thus, we can express flatness + triangle inequalities along C by the single condition

$$db = J.$$

From integrals over spins to integral over embeddings



We replace the integrals over flat spin foams by an integral over embeddings of T' in the color space \mathbb{R}^3 .

$$\langle W_{jC} \rangle = \frac{1}{Z} \int_{\mathbb{R}^3} \left(\prod_{x\mu} d^3 b_{x\mu} \right) \int_{S^2} dn \delta(db - J) \exp \left[-\frac{1}{2\beta} \sum_x b_{x\mu}^2 \right]$$

Solving the constraint

General solution to constraint:

$$b_{x\mu} = \nabla_{\mu}\varphi_x + \bar{b}_{x\mu},$$

where φ is an \mathbb{R}^3 -valued scalar on T' , and \bar{b} is a particular solution to the inhomogeneous equation.

The particular solution can be chosen as

$$\bar{b}_{x\rho} = \epsilon_{\rho\mu\nu}u_{\mu}(u \cdot \nabla)^{-1}J_{x\nu}$$

where u is a unit vector in one of the lattice directions (say $u = \hat{1}$).

Solving the constraint

φ is determined (up to a constant) by b and $\bar{b} \rightsquigarrow$ can replace **constrained** integral over b by an **unconstrained** integral over φ :

$$\langle W_{jC} \rangle = \frac{1}{Z} \int \left(\prod_x d\varphi_x \right) \int_{S^2} dn \exp \left[-\frac{1}{4\beta} \sum_x (\nabla_\rho \varphi_x + \epsilon_{\rho\mu\nu} u_\mu (u \cdot \nabla)^{-1} J_{x\nu})^2 \right]$$

φ plays the role of the gluons: it has 3 degrees of freedom corresponding to 1 physical degree of freedom per gluon in $d = 3$.

Potential

Gaussian integration over φ yields

$$\langle W_{jC} \rangle = \frac{1}{Z} \int_{S^2} dn \exp \left[-\frac{2}{\beta} \sum_{xy} J_{x\mu}^a G_{x,y} J_{y\nu}^a \right],$$

where $G_{x,y}$ is the lattice propagator.

Recalling that

$$J = j n C \quad \text{and} \quad \beta = \frac{4}{ag^2},$$

we obtain

$$\langle W_{jC} \rangle = \exp \left[-\frac{j^2 a g^2}{2} \sum_{xy} C_{x\mu} G_{x,y} C_{y\mu} \right].$$

Potential

Up to lattice corrections this gives

$$V(R) = j^2 g^2 \int_{-\pi/a}^{\pi/a} \frac{d^2 k}{(2\pi)^2} \left(\frac{1}{k^2} - \frac{e^{ik_1 R}}{k^2} \right),$$

which is roughly in agreement with the tree-level perturbative calculation:

$$V(R) = j(j+1) g^2 \int \frac{d^2 k}{(2\pi)^2} \left(\frac{1}{k^2} - \frac{e^{ik_1 R}}{k^2} \right)$$

Physical picture

In the absence of a source, the gluon field has the action

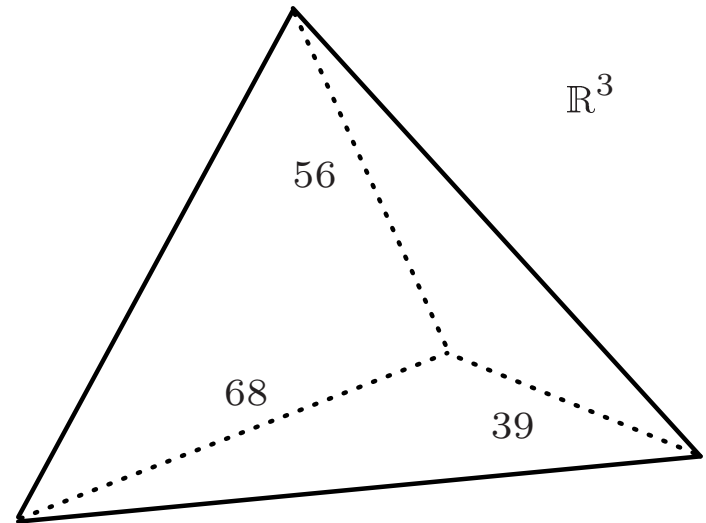
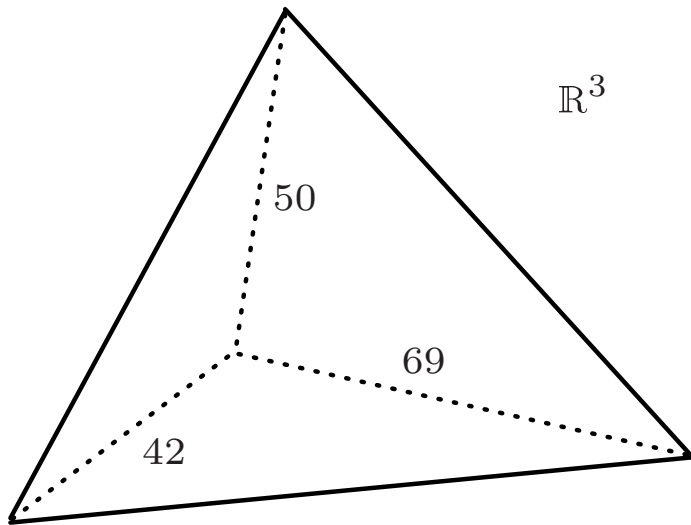
$$S = \frac{1}{4\beta} \sum_x (\nabla_\mu \varphi_x)^2 .$$

If we Wick-rotate back to $+ - - -$, the classical solutions are waves.

Recall that the vectors $b_{x\mu} = \nabla_\mu \varphi_x$ describe the difference vectors of the embedded triangulation.

\leadsto φ defines the embedding $T' \rightarrow \mathbb{R}^3!$

Gluons as waves of spin



A wave in φ corresponds to a wave of spins on the triangulation!

Comparison to Rovelli's approach

	Rovelli et al.	3d SU(2) YM
source	field insertions	Wilson loop
approximation	asymptotics of $10j$ expansion around background	asymptotics of $6j$ expansion around background
triangulation indep.	GFT expansion	refinement limit
coupling	fixed G	tuning of g

6. Conclusion

We have presented a heuristic derivation of gluons from spin foams in 3d SU(2) Yang-Mills theory, extending earlier work by Diakonov & Petrov.

Questions:

- how could one develop a systematic perturbation theory?
- generalization to $d > 3$ and SU(N)?

The comparison gravity/YM suggests an alternative to the group field theory approach: a refinement limit with a tuned Planck length.

References

C.Rovelli, Graviton propagator from background-independent quantum gravity
gr-qc/0508124

E.Bianchi, L.Modesto, C.Rovelli, S.Speziale, Graviton propagator in loop
quantum gravity **gr-qc/0604044**

D.Diakonov, V.Petrov, Yang-Mills theory in three dimensions as quantum
gravity theory **hep-th/9912268**

F.Conrady, Gluons and topological excitations from spin foams
in preparation