Entanglement as a Source of Black Hole Entropy

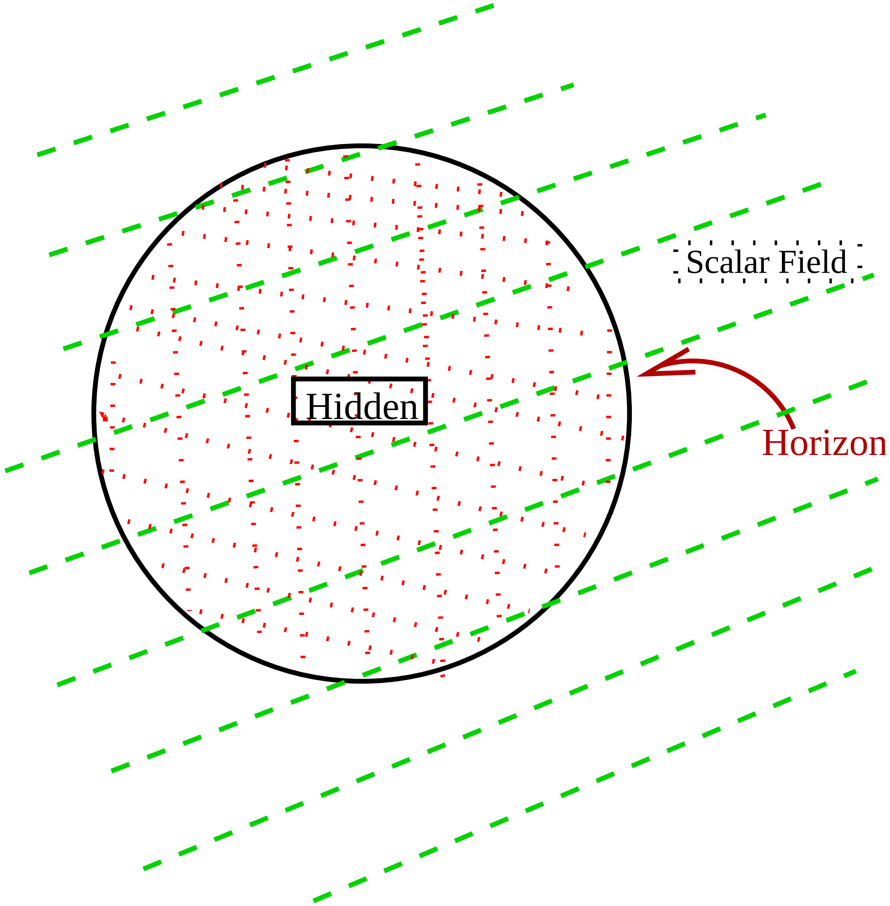
Saurya Das
University of Lethbridge
CANADA

Plan:

- Entanglement entropy of fields in presence of horizon: role of states. Is $S \propto A$?

- Where are the degrees of freedom?

- Entanglement entropy of fields in presence of black hole horizon



Incomplete Information = Entanglement Entropy= k(Area)

Can S_{BH} result from entanglement of fields outside/inside the horizon?
Consider a Free Scalar Field in Flat Space.
e.g. Gravitational perturbations

$$H = \frac{1}{2} \int d^3x \left[\pi^2(x) + |\vec{\nabla}\varphi(\vec{x})|^2 \right]$$

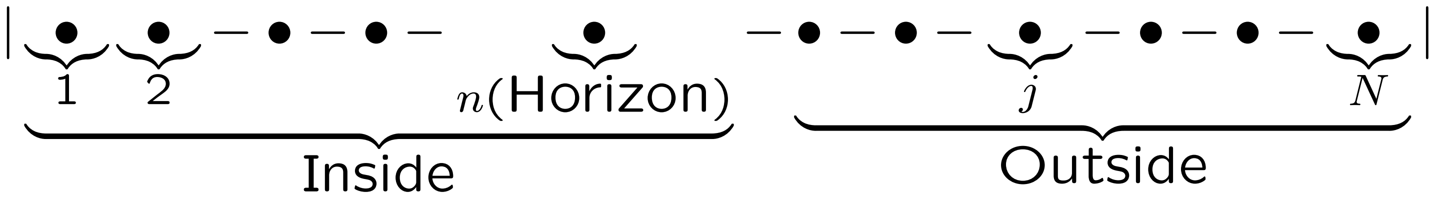
Partial wave decomposition

$$\varphi(\vec{r}) = \sum_{lm} \frac{\varphi_{lm}(r)}{r} \; Y_{lm}(\theta, \phi)$$

$$\pi(\vec{r}) = \sum_{lm} \frac{\pi_{lm}(r)}{r} \; Y_{lm}(\theta, \phi)$$

$$H = \sum_{lm} H_{lm} = \sum_{lm} \frac{1}{2} \int_0^\infty dr \left\{ \pi_{lm}^2(r) + x^2 \left[\frac{\partial}{\partial r} \left(\frac{\varphi_{lm}(r)}{r} \right) \right]^2 + \frac{l(l+1)}{r^2} \; \varphi_{lm}^2(r) \right\}$$

Discretise



$$r \to r_i$$

$$r_{i+1} - r_i = a$$

$L = (N+1)a =$ Box size. $\qquad na =$ 'Horizon'

$$H_{lm} = \frac{1}{2a} \sum_{j=1}^{N} \left[\pi^2_{lm,j} + \left(j + \frac{1}{2}\right)^2 \left(\frac{\varphi_{lm,j}}{j} - \frac{\varphi_{lm,j+1}}{j+1} \right)^2 + \frac{l(l+1)}{j^2} \varphi^2_{lm,j} \right]$$

Trace over first $n < N$ sites and find entanglement entropy

N coupled oscillators

$$H = \frac{1}{2}\sum_{i=1}^{N} p_i^2 + \frac{1}{2}\sum_{i,j=1}^{N} x_i K_{ij} x_j \quad \leftarrow \text{Hamiltonian}$$

Density Matrix, tracing over the first n of N oscillators ($x \equiv x_{n+1}, \ldots, x_N$)

$$\rho\left(x; x'\right) \quad = \int \prod_{i=1}^{n} \; dx_i \; \psi(x_1, \ldots, x_n; x_{n+1}, \ldots, x_N) \; \psi^{\star}(x_1, \ldots, x_n; x_{n+1}, \ldots, x'_N)$$

where ($\underline{x} = Ux$, $UKU^T =$ Diagonal):

$$\psi(x_1, \ldots, x_N) = \prod_{i=1}^{N} N_i \; H_{\nu_i}\left(k_{D\;i}^{\frac{1}{4}} \; \underline{x}_i\right) \exp\left(-\frac{1}{2} k_{D\;i}^{\frac{1}{2}} \; \underline{x}_i^2\right)$$

Note: $\rho_{out}^2 \neq \rho_{out}$

$\Rightarrow \;\; \rho_{out} =$ Mixed, although full state is pure

$\rightarrow$ *Entanglement Entropy* $= -Tr\left(\rho \ln \rho\right) > 0$

For the scalar field

$$H_{lm} = \frac{1}{2a}\sum_{j=1}^{N}\left[\pi_{lm,j}^2 + \underbrace{\left(j+\frac{1}{2}\right)^2\left(\frac{\varphi_{lm,j}}{j} - \frac{\varphi_{lm,j+1}}{j+1}\right)^2 + \frac{l(l+1)}{j^2}\varphi_{lm,j}^2}_{\varphi_i K_{ij}\varphi_j}\right]$$

$$K_{ij} =$$

$$\frac{1}{i^2}\left[l(l+1) + \frac{9}{4}\,\delta_{i1}\delta_{j1} + \left(N-\frac{1}{2}\right)^2\delta_{iN}\delta_{jN} + \left(\left(i+\frac{1}{2}\right)^2 + \left(i-\frac{1}{2}\right)^2\right)\delta_{i,j(i\neq 1,\right.$$

$$-\left[\frac{(j+\frac{1}{2})^2}{j(j+1)}\right]\delta_{i,j+1} - \left[\frac{(i+\frac{1}{2})^2}{i(i+1)}\right]\delta_{i,j-1} \leftarrow$$ **nearest neighbour interaction**

$$K = \begin{pmatrix} \times & \times & & & & & & \\ \times & \times & \times & & & & & \\ & \times & \times & \times & & & & \\ & & \times & \times & \times & & & \\ & & & \times & \times & \times & & \\ & & & & \times & \times & \times & \\ & & & & & \times & \times & \times \\ & & & & & & \times & \times \end{pmatrix}$$

Off-diagonal = Interactions

State of the scalar field ($\varphi_i \to x_i$):

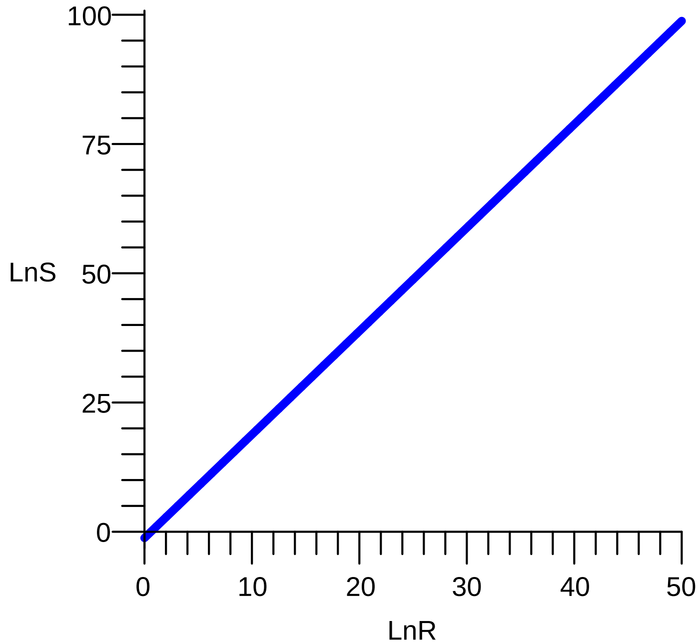
Ground state:

$$\psi(x_1,\ldots,x_N) = \prod_{i=1}^{N} N_i H_{\nu_i}\left(k_{Di}^{\frac{1}{4}}\,\underline{\mathrm{x}}_i\right) \exp\left(-\frac{1}{2}k_{Di}^{\frac{1}{2}}\,\underline{\mathrm{x}}_i^2\right)$$

$\nu_i = 0$ $, \forall i, \to$ $\quad \psi(x_1,\ldots,x_N) = \prod_{i=1}^{N} N_i \exp(-\frac{1}{2}k_{Di}^{\frac{1}{2}}\underline{\mathrm{x}}_i^2)$

Entropy:

$S = \sum_{l=0}^{l_{max}}(2l+1)S_l = 0.3\left(\frac{R}{a}\right)^2 \sim$ Area, $\qquad [R = a(n+1/2)]$



ln(R) vs ln(S)

Note: Ground State Entropy $\propto$ Area

L. Bombelli, R. K. Koul, J. Lee, R. Sorkin PRD **34** (1986) 373; M. Srednicki, PRL **71** (1993) 666

We ask: What happens for *Excited States?*

- Coherent States

- Squeezed States

- First Excited State

M. Ahmadi, SD, S. Shankaranarayanan, hep-th/0507228 (Can. J. Phys)
SD, S. Shankaranarayanan, gr-qc/0511066 (Phys. Rev. D., Rapid Comm.)
SD, S. Shankaranarayanan, in preparation

Coherent state ('Shifted' ground state) $\Delta p \Delta x = \hbar/2$:

$$\psi(x_1, \ldots, x_N)_{CS} = \prod_{i=1}^{N} N_i \exp(-\frac{1}{2} k_{Di}^{\frac{1}{2}} (\underline{x}_i - \alpha_i)^2)$$

$$= \prod_{i=1}^{N} \exp(-i \underline{p}_i \alpha_i) N_i \exp(-\frac{1}{2} k_{Di}^{\frac{1}{2}} \underline{x}_i^2)$$

Defining:

$$\tilde{x} \equiv x - U^{-1}\alpha \ , \quad d\tilde{x} = dx$$

Density Matrix:

$$\rho_{CS}\left(x; x'\right) = \int \prod_{i=1}^{n} dx_i \ \psi_{CS}(x_1, \ldots; x_{n+1}, \ldots) \psi^{\star}_{CS}(x_1, \ldots; x_{n+1}, \ldots) = \rho\left(\tilde{x}; \tilde{x}'\right)$$

S (CS) = S (GS) $\sim$ Area

Squeezed state. $\Delta p \gg 1$, $\Delta x \ll 1$ (or vice-versa) $\Delta p \Delta x = \hbar/2$:

$$\psi_{SS}(x_1, \ldots, x_N) \;=\; r^{N/2} \prod_{i=1}^{N} N_i \exp\left[-\sum_i r\kappa_{Di}^{1/2} \underline{x}_i^2\right]$$

$$\tilde{x} \;\equiv\; \sqrt{r}\; \underline{x} \;, \quad d\tilde{x} = \sqrt{r}\; d\underline{x}$$

<u>Density Matrix:</u>

$$\rho_{SS}\left(x; x'\right) = \int \prod_{i=1}^{n} \; dx_i \; \psi_{SS}(x_1, \ldots; x_{n+1}, \ldots)\psi^{\star}_{SS}(x_1, \ldots; x_{n+1}, \ldots) = \rho\left(\tilde{x}; \tilde{x}'\right)$$

S (SS) = S (GS) $\sim$ Area

Superposition of First Excited States

$$\psi(x_1,\ldots,x_N) = \prod_{i=1}^{N} N_i H_{\nu_i}\left(k_{Di}^{\frac{1}{4}}\underline{\mathsf{x}}_i\right)\exp(-\tfrac{1}{2}k_{Di}^{\frac{1}{2}}\underline{\mathsf{x}}_i^2)$$

Superposition:

$$\psi_1(x_1\ldots x_N) \quad = \quad \sum_{i=1}^{N} a_i N_i H_1\left(k_{Di}^{\frac{1}{4}}\underline{\mathsf{x}}_i\right)\exp\left[-\frac{1}{2}\sum_j k_{Dj}^{\frac{1}{2}}\ \underline{\mathsf{x}}_j^2\right]$$

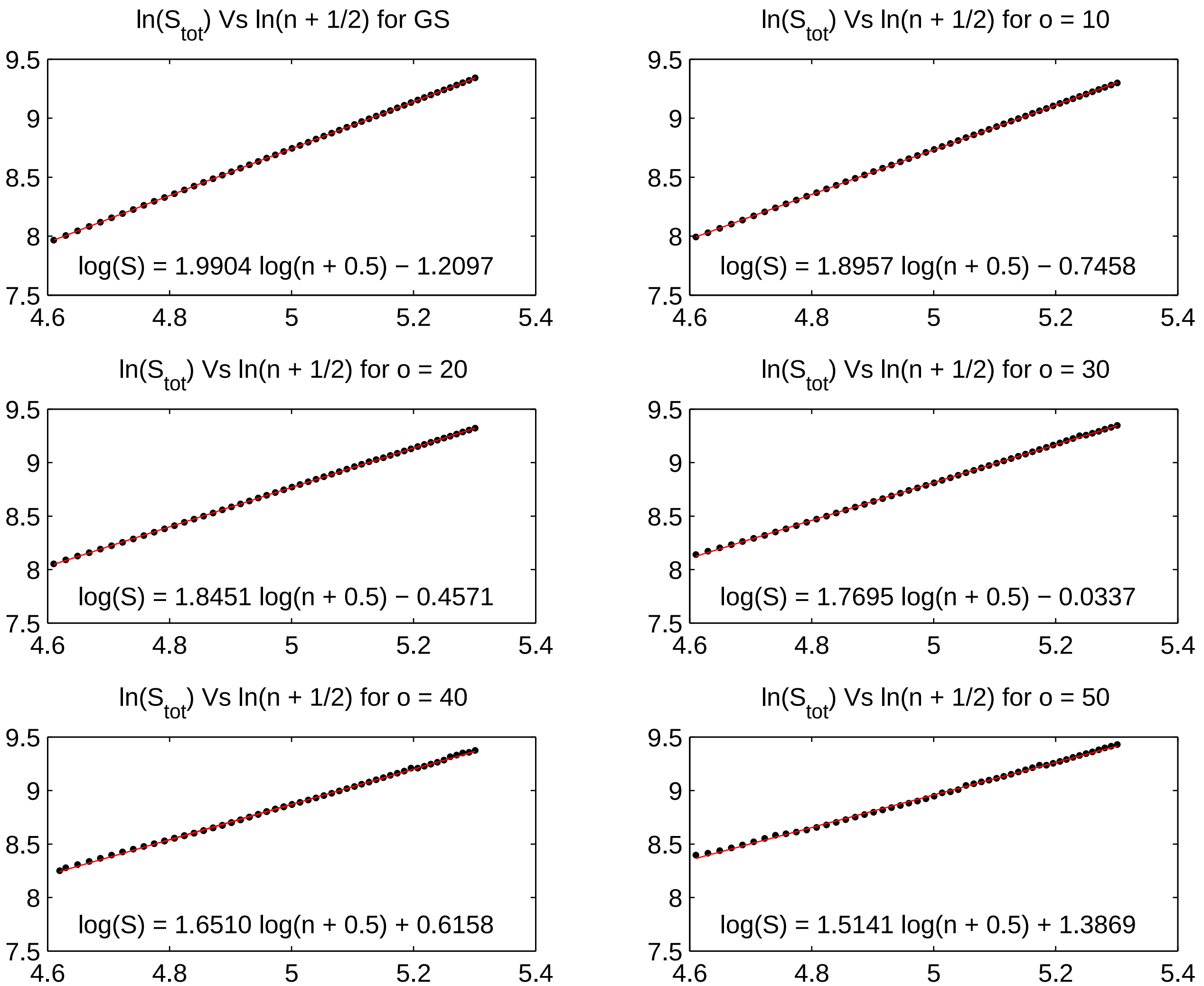
$$= \quad \sqrt{2}\ \left(a^T K_D^{\frac{1}{2}}\underline{\mathsf{x}}\right)\ \psi_0\left(x_1,\ldots,x_N,\right)\ [a^T = (a_1,\ldots,a_N),(a^T a = 1)]$$

Density Matrix:

$$\rho(x;x') = 2\int\prod_{i=1}^{n} dx_i\left[x'^T \wedge x^T\right]\psi_0\left(x_i;x\right)\psi_0^\star\left(x_i;x'\right)$$

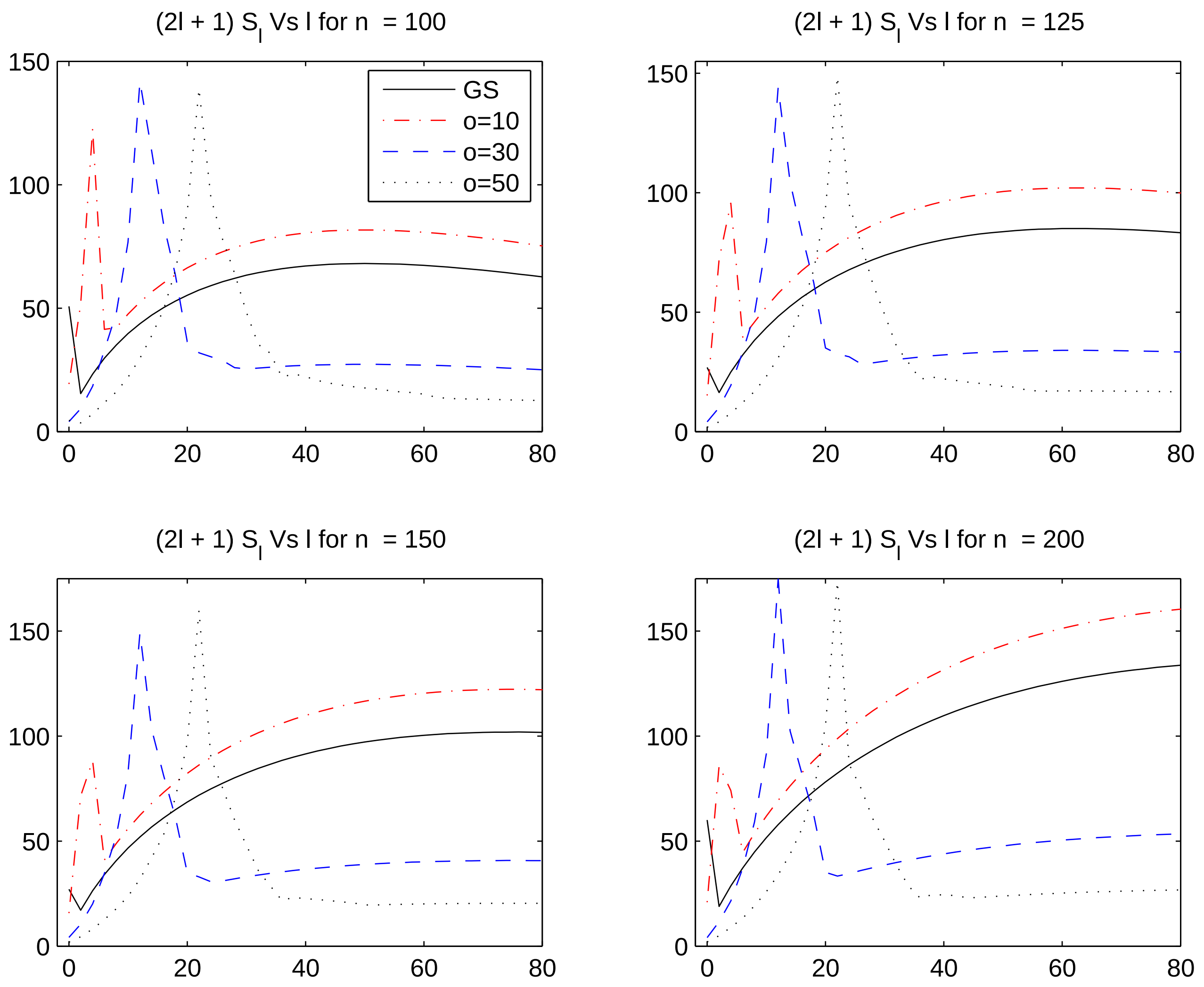
$$a^T = \frac{(0,\cdots,0,\overbrace{1,\cdots,1}^{o})}{\sqrt{o}} \quad N = 300. \quad n = 100 - 200. \quad o = 10 - 50.$$

$$\epsilon_{1,2} \leq 10^{-3}$$



$$S \propto A^{\alpha} \; , \quad \alpha < 1$$

$(2l+1)S_l$ vs l



Higher ES → Higher partial waves excited

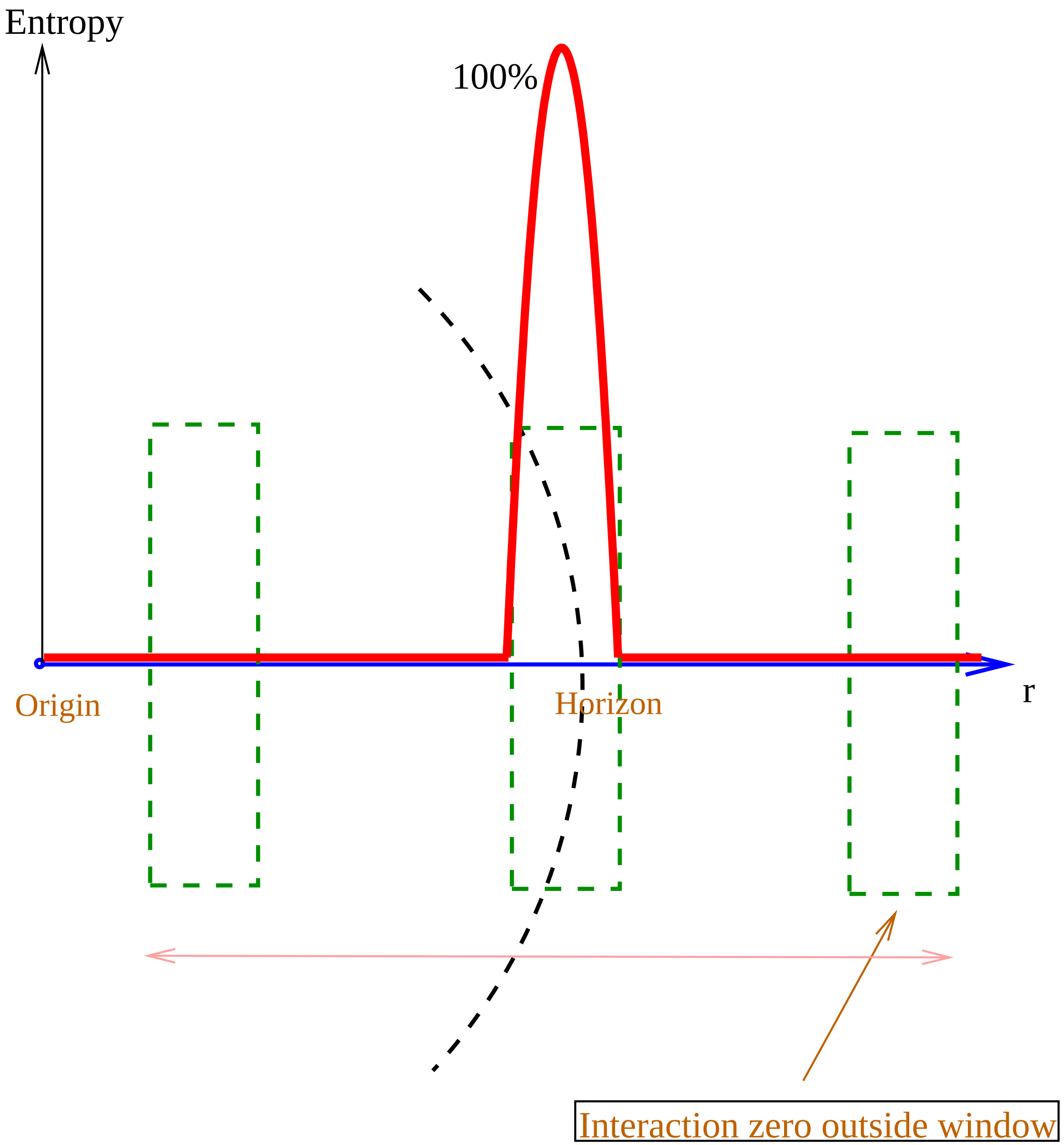
Where are the Degrees of Freedom?

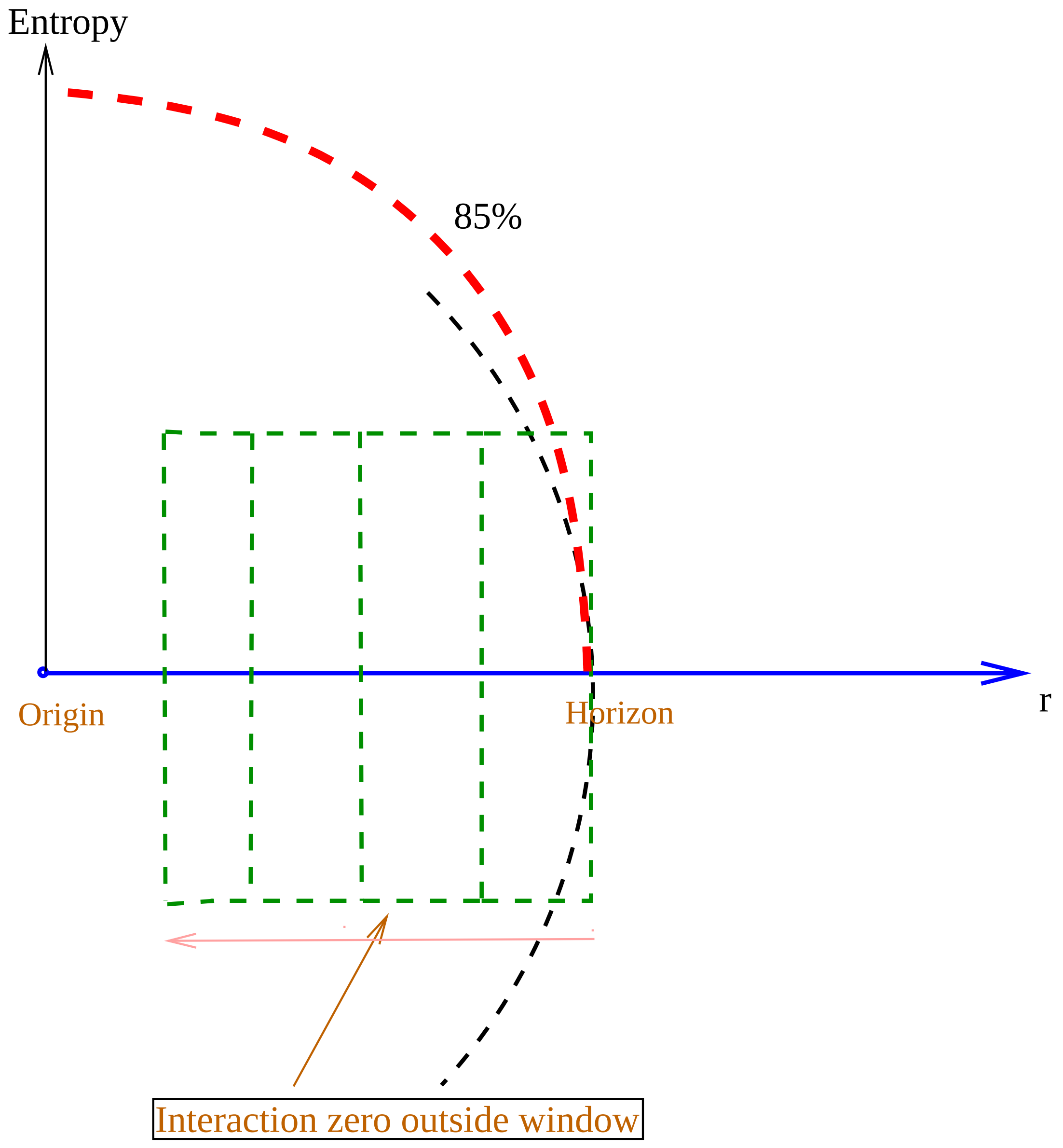
$$K = \begin{pmatrix} \times & \times & & & & & & \\ \times & \times & \times & & & & & \\ & \times & \times & \times & & & & \\ & & \times & \times & \times & & & \\ & & & \times & \times & \times & & \\ & & & & \times & \times & \times & \\ & & & & & \times & \times & \times \\ & & & & & & \times & \times \end{pmatrix}$$

$$\downarrow$$

$$K = \begin{pmatrix} \times & & & & & \\ & \times & & & & \\ & & \times & & & \\ & & & \times & & \\ & & & & \left| \begin{matrix} \overline{\times} & \overline{\times} & \\ \times & \times & \times \\ & \underline{\times} & \underline{\times} \end{matrix} \right| & \\ & & & & & \times \end{pmatrix}$$

Where are the Degrees of Freedom?





Entropy
85%
Origin
Horizon
r
Interaction zero outside window

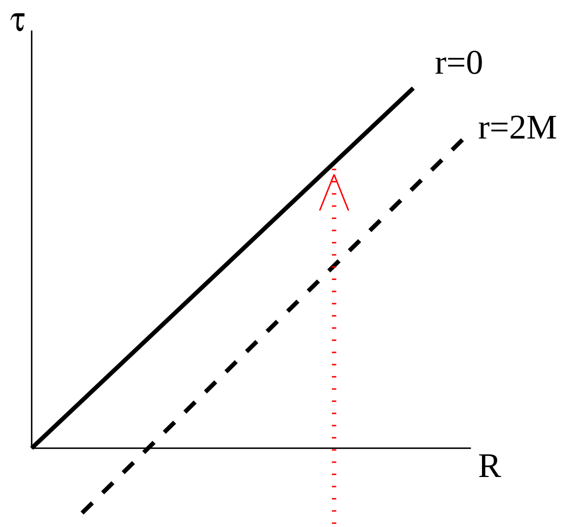
Black Hole Spacetimes: Results go through

$$ds^2 = -f(r)dt^2 + \frac{dr}{f(r)} + r^2 d\Omega^2 \ , \quad f(r) = 1 - \frac{r_0}{r}$$

Define:

$$\tau = t + r_0 \left[\ln \left(\frac{1-\sqrt{r/r_0}}{1+\sqrt{r/r_0}} \right) + 2\sqrt{\frac{r}{r_0}} \right] \ , \ R = \tau + \frac{2}{3} \frac{r^{\frac{3}{2}}}{\sqrt{r_0}} \rightarrow r = \left[\frac{3}{2} (R-\tau) \right]^{\frac{2}{3}} r_0^{\frac{1}{3}}$$

$$ds^2 = -d\tau^2 + \frac{dR^2}{\left[\frac{3}{2r_0}(R-\tau)\right]^{\frac{2}{3}}} + \left[\frac{3}{2r_0}(R-\tau)\right]^{\frac{4}{3}} r_0^{\frac{2}{3}} \ d\Omega^2$$



Hamiltonian:

$$H(\tau) = \frac{1}{2}\int_{\tau}^{\infty} dR \left[\frac{2\pi(\tau,R)^2}{3(R-\tau)} + \frac{3}{2}\; r\; (R-\tau)\,(\partial_R\phi(\tau,R))^2\right]$$

Transform @ $\tau = 0$, say:

$$\pi(r) = \sqrt{r}\pi_1(r)\ , \quad \phi(r) = \frac{\phi_1(r)}{r}$$

Hamiltonian:

$$H(0) = \frac{1}{2}\int_{0}^{\infty} dr \left[\pi_1(r)^2 + r^2\left(\partial_r \frac{\phi_1}{r}\right)^2\right]$$ ← Hamiltonian in flat spacetime

$$r = r_0 \Rightarrow R = \frac{2}{3}r_0$$

Trace: $R = 0 \to \frac{2}{3}r_0$ or $\mathrm{R} = \frac{2}{3}\mathrm{r}_0 \to \infty$

All results go through for a fixed τ

Summary

- Black Hole Entropy = Entanglement Entropy ??
- GS, CS, SS $\Rightarrow S \propto A$
- ES $\Rightarrow S \propto A^{\alpha}$ $, \alpha < 1.$ Higher partial waves excited
- DOF near horizon contribute to most (but not all!) of the entropy
- Results hold for BH spacetimes

Work in progress