The Master Constraint Programme

Bianca Dittrich

Perimeter Institute for Theoretical Physics, Waterloo, Canada

Quantum Gravity in the Americas III, PennState, August 2006

Plan

- Motivation from LQG
- What is a Master Constraint?
- The Master Constraints in LQG
- Algebraic Quantum Gravity and Semiclassical Limit
- Approximations

Motivation

- 1996: anomaly free quantization of Hamiltonian Constraint Operator (T. Thiemann)
- difficulties:
  - infinit. generator of diffeomorphism does not exist
  - Can the constraint algebra realized on the quantum level?
  - to avoid anomalies Hamiltonian has to be graph changing
  - semi–classical limit: semi–classical states bad wrt graph changing operators
- how to define physical inner product on the space of solutions?

The Master Constraint Programme

T. Thiemann: The Phoenix Project: Master Constraint Programme for Loop Quantum Gravity

- constraints $\hat{C}_i$
- want to impose $\hat{C}_i\psi = 0$ on physical states ψ for all i
- classically equivalent: $C_i = 0\ \forall i \ \Leftrightarrow M := \sum_{ij} C_i K_{ij} C_j = 0$
- replace above quantum conditions by $\hat{M}\psi = 0$

$\Rightarrow$ have to deal with only one condition
$\Rightarrow$ no anomalies (question is postponed) $\Rightarrow$ more freedom
$\Rightarrow$ Master Constraint can be defined directly either on H_{diff} or on H_{kin}
$\Rightarrow$ construction of physical Hilbert space via direct integral decomposition

Direct Integral Decomposition

- mathematical method to construct the “generalized eigenspaces” of $\hat{M}$ together with an inner product:

  $$H_{kin} = \int_{\mathrm{spec}(\hat{M})} d\mu(\lambda) H(\lambda)$$

- $\mu(\lambda)$ spectral measure
- set H_{phys} equal to $H(\lambda = 0)$:
  physical Hilbert space inherits inner product from H_{kin}
- H_{phys} carries representation of observable algebra
- method ensures that unphysical solutions do not enter into physical Hilbert space
- method succesfully tested for several examples: second class constraints, structure functions, compact/ non–compact Lie groups, free field theories, Gauss constraints in LQG (B. Dittrich, T. Thiemann: Testing the MCP for LQG: I - V)

The Master Constraints for Loop Quantum Gravity

T. Thiemann: The Phoenix Project: Master Constraint Programme for LQG
T. Thiemann: QSD VIII

Graph Changing Version:

- $M = \int d^3x \frac{[C(x)]^2}{\sqrt{q(x)}}$
- diff.–invariant, density weight one, could be defined on H_{diff}
- formally: $\hat{M} = \lim_{\epsilon \to 0} \sum_{\Delta} \hat{C}_{\epsilon}(\Delta)^{\dagger} \hat{C}_{\epsilon}(\Delta)$
- where $C_{Eucl}(\Delta) = \int_{\Delta} \mathrm{Tr}(F \wedge \{A, \sqrt{V(\Delta)}\})$
- similar to original constraint: $V(\Delta)$ replaced by $\sqrt{V(\Delta)}$

- $\hat{M}$ can be defined via a quadratic form on H_{diff}, however one has to take care of the fact, that $C_{\epsilon}(\Delta)$ is not diff.–invariant (QSD VIII; M. Han, Y. Ma: Master Constraint Operator in LQG)
- can make $\hat{M}$ more "non–local"

The Master Constraints for Loop Quantum Gravity

Non–Graph–Changing Version / Extended Master Constraint

- can define a non–graph–changing version: loops are added along already existing edges (minimal loop description)
- this operator can be defined on H_{kin}
- can add diffeo. constraints (and Gauss constraints): $M_E = \int d^3x \, \frac{C(x)^2 + q^{ab}(x) C_a(x) C_b(x)}{\sqrt{(}q)}$
- leads to Algebraic Quantum Gravity: semi–classical limit available

Algebraic Quantum Gravity and Semiclassical Limit

K. Giesel, T. Thiemann, Algebraic Quantum Gravity I,II,III;

- use non graph changing Master Constraint Operator
- $\Rightarrow$ use a fixed algebraic (non embedded) graph
- LQG kinematical Hilbert space is replaced by infinite tensor product Hilbert space (consider graph with infinitely many edges)
- operators are “lifted” from LQG to AQG Hilbert space
- do not have action of diffeomorphism in AQG, need however to get rid of corresponding degrees of freedom (problem in lattice gravity)
- use extended Master Constraint

Semiclassical Limit of the Master Constraint Operator

- semiclassical limit can be calculated and is the correct one!
    - fixed cubic graph
    - Euclidean part of the constraint
    - use complexifier coherent states
    - replace $SU(2)$ by $U(1)^3$
- last item can be justified:
    - "perturbation" for (non–integral) powers of the Volume Operator is developed with error–control
    - to zeroth order in $\hbar$ one can replace $SU(2)$ with $U(1)^3$ for calculations of expectation values of coherent states

$\Rightarrow$ correct classical limit
$\Rightarrow$ explicit calculations possible by expanding (non–integral) powers of the Volume Operator

Approximating the Inner Product

B. Bahr, T. Thiemann: Approximating the physical inner product of Loop Quantum Cosmology;

- problem: direct integral decomposition very complicated: approximations?
- physical inner product $< \phi|\phi' >_{phys} := < \phi|\, \delta(\hat{M})\, |\phi' >$
- $\delta(\hat{M}) = \lim_{t\to 0} t^{-\alpha} \exp(-\frac{\hat{M}^2}{t})$
  where α depends on the (unknown) spectral measure of $\hat{M}$ near zero
- $t^{-\alpha}$ drops out if we normalize the above expression with the help of a reference vector:
  $< \phi|\phi' >_{phys} := \frac{<\phi|\, \delta(\hat{M})\, |\phi'>}{<\Psi_0|\, \delta(\hat{M})\, |\Psi_0>}$
- this could be calculated with the help of coherent states, which provide resolution of unity

Approximating the Inner Product

- Loop Quantum Cosmology ($\Lambda = 0$):
- $\hat{M} = \hat{C}^{\dagger}\hat{C}$ with $\hat{C} = \left(\frac{\sin \mu_0 \hat{c}}{\mu_0}\right)^2 \hat{p}\widehat{\sqrt{|p|}}$
- there are “unphysical” solutions to $\hat{M}$
- calculation can be done approximately
- leads to the same physical Hilbert space as an exact calculation using (a rescaled) $\hat{C}$
- in particular “unphysical” solutions have zero norm

$\Rightarrow$ methods available to calculate physical inner product

Conclusions and Discussion

- semiclassical limit for the non–graph changing version
- approximation methods are being developed
- explicit calculations possible

- connection to path integrals?
- non–graph–changing Master Constraint: effective theory?