Quantum field theory
and
symmetry reduced models

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Outline

- Introduction & motivation
- Review of definitions and notation for free KG theory
- Look at different approaches to imposing symmetry
- Analysis of b-symmetry; comparison with c-symmetry
- Explicit expression for ‘b’ embedding; corresponding embedding of LQC into LQG
- Summary and questions

Introduction: basic motivating questions

1. What is the meaning of symmetry in QFT? That is, what is the *ideal* that LQC is trying to approximate? There is more than one answer to this in the literature. Yet this question is crucial.
2. What is the meaning of specific *states* in LQC? Can we identify them with homogeneous and isotropic states in LQG? (If not, then what is their meaning?)
3. Does LQC accurately model the (exactly) homogeneous and isotropic sector of LQG?

Note: I am not talking about inhomogeneities here. That is a different question which can be asked meaningfully only after the above questions are answered.

Elementary motivation for different approach to symmetry

Quantum analogues of classical conditions on states:

The analogue of $\mathcal{O} = \lambda$ is $\hat{\mathcal{O}}\Psi = \lambda\Psi$.

So we ask, given a symmetry group G with action on classical phase space:

> What is the quantum analogue of classical G-symmetry, in the above sense?

Suppose $\{C_i\}$ are classical constraints sufficient to isolate the classical symmetric sector. We propose to define the quantum analogue to be roughly

$$\{\Psi \mid \hat{C}_i\Psi = 0\}.$$

That is the idea.

This is different from invariance under the (induced) action of G on quantum states.

Free KG theory and symmetry reduced model: definitions and notation

Full theory

Γ	phase space
$\langle \cdot, \cdot \rangle$	1 particle inner product on Γ
$\mathcal{H}$	field theory Hilbert space
$\hat{\varphi}[f], \hat{\pi}[g]$	smeared field operators
$a(\xi)$	annih. op. associated with $\xi \in \Gamma$

Symmetry reduced theory (symmetry group: rotations about z-axis)

Γ_{inv}	reduced phase space
$\langle \cdot, \cdot \rangle$	1 particle inner product on Γ_{inv}
$\mathcal{H}_{red}$	reduced field theory Hilbert space

Different methods of imposing symmetry

1. Invariance symmetry: $\hat{\mathbb{L}}_z \Psi = 0$
   Let space of solutions to this be denoted $\mathcal{H}_{inv}$

2. Imposing field-operator symmetries à la Dirac

   - Classical symmetry constraints: $\mathcal{L}_\phi \varphi(x) = 0$ and $\mathcal{L}_\phi \pi(x) = 0$
     smeared: $\varphi[\mathcal{L}_\phi f] = 0$ and $\pi[\mathcal{L}_\phi g] = 0$
   - Problem: is 2^{nd} class system — cannot impose in quantum theory.
     Reformulate as 1^{st} class system. Two natural ways:

     (a) $\{\varphi[\mathcal{L}_\phi f] = 0\}_{f \in \mathcal{S}(\Sigma)}$ ⇒ wavefns w/ support on symm. φ
     (b) $\{a[\mathcal{L}_\phi f, \mathcal{L}_\phi g] = 0\}_{f,g \in \mathcal{S}(\Sigma)}$

     Call these reformulations "c" and "b", and call the associated notions of symmetry "c-symmetry" and "b-symmetry".

In the case $\Gamma = \{(q^i, p_i)\} = \mathbb{R}^{2n}$ with constraints $q^1 = 0$ & $p_1 = 0$, these reformulations are analogous to

$$\{q^1 = 0\} \quad \text{and} \quad \{q^1 + i p_1 = 0\}$$

Results

Both $\mathcal{H}_b$ and $\mathcal{H}_c$ are $\overset{\text{nat.}}{\cong}$ $\mathcal{H}_{red}$.

$\mathcal{H}_b$ can be completely characterized in 3 diff. ways:

1. $\mathcal{H}_b =$ solution space to a set of constraints whose class. analogues uniquely isolate the symmetric sector of the classical phase space
2. $\mathcal{H}_b =$ span of coherent states assoc. with symm. sector of classical theory
3. $\mathcal{H}_b =$ space of states in which all non-symm. modes are unexcited

In addition,

- Fluctuations from axisymmetry are under complete contol in $\mathcal{H}_b$ and are in a certain sense minimized.
- $\hat{\mathbb{H}}$ preserves $\mathcal{H}_b$.

→ c-symmetry only shares 1 of the above 5 desireable properties with b-symmetry

→ Invariance symmetry ($\mathcal{H}_{inv}$) doesn't achieve commutation of reduction & quantization even kinematically, does not have same control on fluctuations from axisymmetry, & becomes physically vacuous in the case of quantum gravity any way.

Conclusion: At least in KG case, b-symmetry is the best notion of symmetry when relating the full theory to the reduced theory. c-symmetry is not ideal but also works.

Explicit 'b' embedding of $\mathcal{H}_{red}$ into $\mathcal{H}$:

$$\iota \overset{r}{\Psi} := \int_{\xi \in \Gamma_{inv}} \mathrm{d}\nu^{o}_{red} \langle \overset{r}{\Psi}{}^{coh}_{\xi}, \overset{r}{\Psi} \rangle \Psi^{coh}_{\xi}$$

where $\mathrm{d}\nu^{o}_{red}$ is a rigorization of the "Lesbesgue measure" (or, $e^{\langle \xi, \xi \rangle}$ times a Gaussian measure). ι intertwines $\hat{\mathbb{H}}$ and $\hat{\mathbb{H}}_{red}$ ($\hat{\mathbb{H}} \circ i = i \circ \hat{\mathbb{H}}_{red}$).

Can do same in LQG:

$$\iota \overset{r}{\Phi} := \int_{\xi \in \Gamma_{inv}} \mathrm{d}\nu \, \overline{\overset{r}{\eta}{}^{coh}_{\xi} \left(\overset{r}{\Phi} \right)} \eta^{coh}_{\xi}$$

Gives embedding of $\mathrm{Cyl}_{LQC} \hookrightarrow \underline{\mathrm{Cyl}}^{*}_{LQG}$, where $\underline{\mathrm{Cyl}}_{LQG}$ denotes space of cyl. functions based on graphs with only straight edges. Thus, its at least as well-defined as Bojowald's 'c' embedding.

Summary

- We have clarified the notion of symmetry in QFT
- New proposal for embedding LQC states into LQG states: ‘b’ embedding

Questions:

1. In constructing the ‘b’ embedding of LQC into LQG, there is an ambiguity: the choice of a complexifier. Is there a way to rid of this ambiguity without appealing to dynamics?
2. Bojowald used ‘c’ embedding to justify quantization of the elementary variables in spherically symmetric reduced model. Can one also use an embedding to justify quantization of the elementary variables in LQC? Will the ‘b’ embedding lead to a different quantization?
3. What about an embedding at the level of the physical Hilbert space(s)? Formally it can be done, but details need to be worked out.

Extra Slides

Quantization of free KG theory: Review

Classical : $\mathcal{C} =$ smooth fns of $\varphi : \Sigma \to \mathbb{R}$ with appropriate fall-off.

$$\Gamma = T^*\mathcal{C} = \{(\varphi, \pi) \mid \varphi, \pi : \Sigma \to \mathbb{R}\}$$

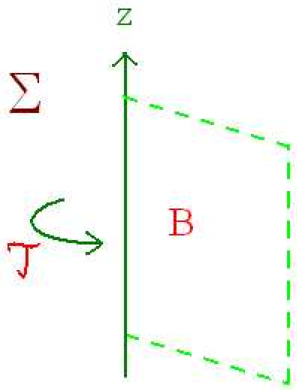
$$\Rightarrow \Omega([\varphi, \pi], [\varphi', \pi']) = \int_\Sigma (\pi\varphi' - \varphi\pi')\mathrm{d}^3x$$

$$\mathbb{H} = \frac{1}{2}\int_\Sigma (\pi^2 + (\vec{\nabla}\varphi)^2 + m^2\varphi^2)\mathrm{d}^3x$$

Fock quantization

- $J[\varphi, \pi] := [-\Theta^{-\frac{1}{2}}\pi, \Theta^{\frac{1}{2}}\varphi]$, complex structure, where $\Theta := -\Delta + m^2$
- J determines $\langle \cdot, \cdot \rangle := \frac{1}{2}\Omega(J\cdot, \cdot) - \frac{i}{2}\Omega(\cdot, \cdot)$
- $h :=$ completion of Γ w.r.t. $\langle \cdot, \cdot \rangle$
- $\mathcal{H} := \oplus_{n=0}^{\infty} \overset{n}{\otimes}_s \, h$
- For each $\xi = [\varphi, \pi] \in h$, we have the usual creation & annihilation operators $a^\dagger(\xi)$,$a(\xi)$.
- Representation of (smeared) field operators:
  $\hat{\varphi}[f] := i\{a([0, f]) - a^\dagger([0, f])\}$
  $\hat{\pi}[g] := -i\{a([g, 0]) - a^\dagger([g, 0])\}$

Reduced theory & its quantization



- Γ_{inv} = axisymmetric subspace of Γ
- $\mathcal{C}_{inv}$ = axisymmetric subspace of $\mathcal{C}$
  $\equiv$ smooth fns on $B := \Sigma/\mathfrak{T}(\equiv \mathbb{R}^+ \times \mathbb{R})$
  with sufficient fall off.
- $\Gamma_{inv} = T^*\mathcal{C}_{inv}$

Since $\mathcal{C}_{inv}$ is a vector space & dynamics are linear, Fock quantization can again be done, with complex structure on Γ_{inv} determined by the usual pos.-neg. freq. decomposition rule.

Everything is done in the standard way, as if it were a theory just living on B.

This is what we mean by the "reduced theory". Associated structures will be subscripted with "red".

Expressions for the field operators can be inverted:

$$a([f,g]) = \tfrac{1}{2}\hat{\varphi}[\Theta^{\frac{1}{2}}f - ig] + \tfrac{1}{2}\hat{\pi}[\Theta^{-\frac{1}{2}}g + if]$$
$$a^\dagger([f,g]) = \tfrac{1}{2}\hat{\varphi}[\Theta^{\frac{1}{2}}f + ig] + \tfrac{1}{2}\hat{\pi}[\Theta^{-\frac{1}{2}}g - if]$$

taking these expressions over to classical theory we obtain classical analogues of the creation & annihilation operators:

$$a([f,g]) = \tfrac{1}{2}\varphi[\Theta^{\frac{1}{2}}f - ig] + \tfrac{1}{2}\pi[\Theta^{-\frac{1}{2}}g + if] = \langle [f,g],[\varphi,\pi]\rangle$$
$$a([f,g])^\dagger = \cdots = \langle [\varphi,\pi],[f,g]\rangle \qquad \text{(functions on } \Gamma \ni (\varphi,\pi))$$

Schrödinger quantization (is unitarily equivalent to Fock)

- $\overline{\mathcal{C}} = \mathcal{S}'(\Sigma)$, tempered distributions
- "$\mathrm{d}\mu = \exp\{-\frac{1}{2}(\varphi, \Theta^{\frac{1}{2}}\varphi)\}\mathcal{D}\varphi$", $\mathcal{D}\varphi =$ "Lesbesgue measure on $\mathcal{S}'(\Sigma)$"
- $\mathcal{H} = L^2(\overline{\mathcal{C}}, \mathrm{d}\mu)$
- A "cylindrical function" on $\mathcal{S}'(\Sigma)$ is a function Ψ s.t. $\Psi[\varphi] = F(\varphi(e_1), \ldots, \varphi(e_n))$ for some smooth $F : \mathbb{R}^n \to \mathbb{C}$ & some $e_1, \ldots, e_n \in \mathcal{S}(\Sigma)$.
- Representation of (smeared) field operators:

$$\begin{aligned}
(\hat{\varphi}[f]\Psi)[\varphi] &:= \varphi[f]\Psi[\varphi] \\
(\hat{\pi}[g]\Psi)[\varphi] &:= \{\text{self-adj. part of} - i\textstyle\int_\Sigma \mathrm{d}^3x g \frac{\delta}{\delta\varphi}\}\Psi[\varphi] \\
&= -i\textstyle\int_\Sigma \mathrm{d}^3x (g\frac{\delta}{\delta\varphi} - \varphi\Theta^{\frac{1}{2}}g)\Psi[\varphi]
\end{aligned}$$

- Motivated by inverted eq'ns on slide 10, we define

$$\begin{aligned}
a([f,g]) &:= \tfrac{1}{2}\hat{\varphi}[\Theta^{\frac{1}{2}}f - ig] + \tfrac{1}{2}\hat{\pi}[\Theta^{-\frac{1}{2}}g + if] \\
a^\dagger([f,g]) &:= \tfrac{1}{2}\hat{\varphi}[\Theta^{\frac{1}{2}}f + ig] + \tfrac{1}{2}\hat{\pi}[\Theta^{-\frac{1}{2}}g - if]
\end{aligned}$$

- Unique, normalized vacuum annihilated by all $a(\xi)$, $\xi \in h$: $\Psi_0[\varphi] \equiv 1$.

Some definitions

$$
\begin{aligned}
\mathcal{H}_{inv} &:= \{\Psi \in \mathcal{H} \mid \hat{\mathbb{L}}_z \Psi = 0\} \\
\mathrm{Cyl}^*_{inv} &:= \{\eta \in \mathrm{Cyl}^* \mid \hat{\mathbb{L}}^*_z \eta = 0\} \\
\mathrm{Cyl}^*_c &:= \{\eta \in \mathrm{Cyl}^* \mid \hat{\varphi}[\mathcal{L}_\phi f]^* \eta = 0 \quad \forall f \in \mathcal{S}(\Sigma)\} \\
\mathcal{H}_b &:= \{\Psi \in \mathcal{H} \mid a([\mathcal{L}_\phi f, \mathcal{L}_p hig])\Psi = 0 \quad \forall f, g \in \mathcal{S}(\Sigma)\}
\end{aligned}
$$

$\mathcal{H}_{inv}$ and Cyl_{inv} are implementations of “invariance symmetry”.

Cyl^*_c is solution space to constraint set ‘c’ at quantum level. Constraint set ‘c’ forces its sol’ns to have support on $\mathcal{S}'(\Sigma)_{inv}$. But $\mu(\mathcal{S}'(\Sigma)_{inv}) = 0$, whence any solution of set ‘c’ in $\mathcal{H}$ has zero norm and so is zero — only admits non-normalizable solutions. Must go to Cyl^*. ($\mathrm{Cyl} \hookrightarrow \mathcal{H} \hookrightarrow \mathrm{Cyl}^*$)

c-symmetry is analogous to Bojowald and Kastrup’s notion of “symmetric state” in LQG (distributional states w/ supp. on symm. config.).

The decomposition $\mathcal{H} = \mathcal{H}_{red} \otimes \mathcal{H}_{\perp}$

$$\overline{\mathcal{C}} = \mathcal{S}'(\Sigma), \quad \overline{\mathcal{C}}_{red} = \mathcal{S}'(B) \overset{\text{nat.}}{\cong} [\mathcal{S}'(\Sigma)]_{inv}$$

Let $\overline{\mathcal{C}}_{\perp}$:= kernel of group averaging map $\Pi : \mathcal{S}'(\Sigma) \to [\mathcal{S}'(\Sigma)]_{inv}$.

Then one can show

$$\overline{\mathcal{C}} = \overline{\mathcal{C}}_{red} \oplus \overline{\mathcal{C}}_{\perp}$$

topologically, $\overline{\mathcal{C}} = \overline{\mathcal{C}}_{red} \times \overline{\mathcal{C}}_{\perp}$.

Lemma: $\exists$ measure $\mu_{\perp}$ on $\overline{\mathcal{C}}_{\perp}$ s.t.

$$\mu = \mu_{red} \times \mu_{\perp}$$

where μ is quantum measure in full theory & μ_{red} is quantum measure in reduced theory.

$\Rightarrow$ $$L^2(\overline{\mathcal{C}}, \mathrm{d}\mu) = L^2(\overline{\mathcal{C}}_{red}, \mathrm{d}\mu_{red}) \otimes L^2(\overline{\mathcal{C}}_{\perp}, \mathrm{d}\mu_{\perp})$$

Let $\mathcal{H}_{\perp} := L^2(\overline{\mathcal{C}}_{\perp}, \mathrm{d}\mu_{\perp})$.

$\Rightarrow$ $$\boxed{\mathcal{H} = \mathcal{H}_{red} \otimes \mathcal{H}_{\perp}}$$

Analysis of c-symmetry

Embedding of $\mathcal{H}_{red}$ in Cyl^*

$$\begin{aligned} \mathfrak{E} : \mathcal{H}_{red} &\quad\rightarrow\quad \mathrm{Cyl}^*_{red} \otimes \mathrm{Cyl}^*_{\perp} \subset \mathrm{Cyl}^* \\ \Psi &\quad\mapsto\quad \Psi \otimes \delta \end{aligned}$$

Image of $\mathfrak{E}$ is in Cyl^*_c. Define $\mathcal{H}_c := \mathrm{Im}\mathfrak{E}$.

We have

$$\hat{\mathbb{H}}(\Psi \otimes \delta) = (\hat{\mathbb{H}}_{red}\Psi) \otimes \delta + \Psi \otimes (\hat{\mathbb{H}}_{\perp}\delta)$$

but $\hat{\mathbb{H}}_{\perp}\delta$ is *not* prop. to δ.

Therefore: $\hat{\mathbb{H}}(\Psi \otimes \delta)$ *is never again in* $\mathcal{H}_c$*.* Maximal non-preservation of $\mathcal{H}_c$ by $\hat{\mathbb{H}}$.

Additionally, one can show $\mathcal{H}_c \subset \mathrm{Cyl}^*_{inv}$.

Analysis of b-symmetry

$\mathcal{H}_b := \{\Psi \in \mathcal{H} \mid a(\xi)\Psi = 0 \quad \forall \xi \in h_\perp\}$

Rewrite using coherent states

For each $\eta \in h$, $\Psi^{coh}_\eta \in \mathcal{H}$. Satisfies

$$a(\xi)\Psi^{coh}_\eta = \langle \xi, \eta \rangle \Psi^{coh}_\eta .$$

The coherent states which satisfy $a(\xi)\Psi^{coh}_\eta = 0 \quad \forall \xi \in h_\perp$ are precisely those for which $\eta \in h_{inv}$. $\therefore \mathcal{H}_b = \mathrm{span}\{\Psi^{coh}_\xi \mid \xi \in h_{inv}\}$.

$\mathcal{H}_b$ as space in which "all non-symmetric modes are unexcited"

Can show

$$\mathcal{H}_b = \{\Psi \in \mathcal{H} \mid a^\dagger(\xi_1) \cdots a^\dagger(\xi_n)\Psi_0 \mid \xi_1, \ldots, \xi_n \in h_{inv}\}.$$

Equivalently, in terms of $\mathcal{H} = \mathcal{H}_{red} \otimes \mathcal{H}_\perp$, one can show

$$\mathcal{H}_b = \mathcal{H}_{red} \otimes 1$$

where $1 \in \mathcal{H}_\perp = L^2(\mathcal{S}'(\Sigma)_\perp, \mathrm{d}\mu_\perp)$ is constant function 1. 1 is unique vacuum of $\hat{\mathbb{H}}_\perp$, whence again this expresses $\mathcal{H}_b$ as sector in which "all non-symmetric modes are unexcited".

Three further important facts from the form $\mathcal{H}_b = \mathcal{H}_{red} \otimes 1$.

1. $\mathcal{H}_b$ is naturally isomorphic to $\mathcal{H}_{red}$.
2. Since $\hat{\mathbb{H}}_\perp 1 = 0$, $mat\hat{h}bbH$ now preserves $\mathcal{H}_b$. Therefore $\hat{\mathbb{H}}$ induces a Hamiltonian on $\mathcal{H}_{red}$ via the isomorphism b/w $\mathcal{H}_b$ and $\mathcal{H}_{red}$. It is easy to see this Hamiltonian is just $\hat{\mathbb{H}}_{red}$, so that everything is consistent. $\mathcal{H}_b$ achieves full dynamical commutation of reduction and quantization.
3. Since the action of the symmetry group on the first factor of $\mathcal{H} = \mathcal{H}_{red} \otimes \mathcal{H}_\perp$, it is easy to see $\mathcal{H}_b \subseteq \mathcal{H}_{inv}$.

Minimization of fluctuations

For all $\Psi \in \mathcal{H}_{inv}$ (and hence all $\Psi \in \mathcal{H}_b$),

$$\langle \Psi, \hat{\varphi}(x)\Psi \rangle \text{ is axisymmetric.}$$

Proof: use fact that $\varphi(g \cdot x) = U_g \hat{\varphi}(x) U_g^{-1}$.

However, $\mathcal{H}_b$ has the further property that *fluctuations* from axisymmetry are completely controlled: For $\Psi = \Upsilon \otimes 1 \in \mathcal{H}_b$,

$$\begin{aligned} \Delta_\Psi \hat{\varphi}[\mathcal{L}_\phi f] &= \sqrt{\tfrac{1}{2} \int \mathrm{d}^3 x (\mathcal{L}_\phi f) \Theta^{-\frac{1}{2}} \mathcal{L}_\phi f} \\ \Delta_\Psi \hat{\pi}[\mathcal{L}_\phi f] &= \sqrt{\tfrac{1}{2} \int \mathrm{d}^3 x (\mathcal{L}_\phi f) \Theta^{\frac{1}{2}} \mathcal{L}_\phi f} \end{aligned}$$

In particular for f an eigenfunction of Θ,

$$\Delta_\Psi \hat{\varphi}[\mathcal{L}_\phi f] \Delta_\Psi \hat{\pi}[\mathcal{L}_\phi f] = \tfrac{1}{2}$$

saturating the Heisenburg bound.

Misconception to be cleared up:

Sometimes people have a false intuition that there is no difference between “invariance symmetry” and imposing symmetry as a system of constraints à la Dirac.

For example: consider quantum theory of point particle, $\mathcal{H} = L^2(\mathbb{R}^3)$, with symm. group: rotations about z-axis. The two notions of symmetry are then:

1. Ψ is invariant under z-rotations
2. $\hat{L}_z \Psi = 0$

But these are the same!

Answer: $L_z = 0$ is not a symmetry constraint at the classical level.

The 2nd notion of symmetry only makes sense in the field theory case.

Definition of $a^\dagger(\xi)$ and $a(\xi)$:

For $\Psi = (\psi, \psi^A, \psi^{A_1A_2}, \psi^{A_1A_2A_3}, \dots) \in \mathcal{H}(= \oplus_{n=0}^\infty \overset{n}{\otimes}_s h)$,

$$\begin{aligned} a^\dagger(\xi)\Psi &:= (0, \psi\xi^A, \sqrt{2}\,\xi^{(A_1}\psi^{A_2)}, \sqrt{3}\,\xi^{(A_1}\psi^{A_2A_3)}, \dots) \\ a(\xi)\Psi &:= (\overline{\xi}_A\psi^A, \sqrt{2}\;\overline{\xi}_A\psi^{AA_1}, \sqrt{3}\;\overline{\xi}_A\psi^{AA_1A_2}, \dots) \end{aligned}$$