The power spectrum from superinflation in LQC

Nelson Nunes
University of Minnesota

- **Power spectrum in standard inflation**
- **Superinflation in LQC**
- **Power spectrum in LQC**
- **Comparison with other works**

D. J. Mulryne and NJN (2006)
G. Hossain (2004)
G. Calcagni and M. Cortês (2006)

1. Recipe to compute the power spectrum

1. Perturb scalar field $\phi = \phi_b + \delta\phi$

2. Write action to 2nd order in perturbations

3. Find field $u = u(\delta\phi, ...)$ s.t. $u'' + (k^2 + m_{\rm eff}^2)u = 0$ and define conjugate momentum $\pi = \frac{\partial \mathcal{L}}{\partial u'}$

4. Promote u to operator $\hat{u}$ and expand $\hat{u} = \sum (w_k a_k + w_k^* a_k^\dagger) e^{ik.x}$

5. Normalize modes s.t. $[a, a] = [a^\dagger, a^\dagger] = 0$; and $[a, a^\dagger] = \delta$ (Wronskian condition).

6. Find asymptotic value for large modes ($k \ll aH$),
$\mathcal{P}_\phi \propto k^3 \langle |\delta\phi|^2 \rangle \propto H^2$

7. Relate field perturbation to curvature perturbation $\mathcal{P}_\mathcal{R} = \frac{H^2}{\dot{\phi}^2} \mathcal{P}_\phi$

2. Loop Quantum Cosmology (LQC)

LQG - Infinite degrees of freedom

LQC - Simplified situations obtained by implementing symmetries.

Evolution of the Universe divided into three phases:

1. Quantum phase: $a < a_i$ and $a_i^2 = \gamma \ell_{\rm pl}^2$ described by a difference equation. A key consequence is the removal of initial singularity.

2. Semi-classical phase: $a_i < a < a_*$

$$a_*^2 = \frac{j}{3} a_i^2$$

Equations modified due to non-perturbative quantization effects.

3. Classical phase: $a > a_*$. Continuous cosmological equations recovered.

3. Inverse volume operator in LQC

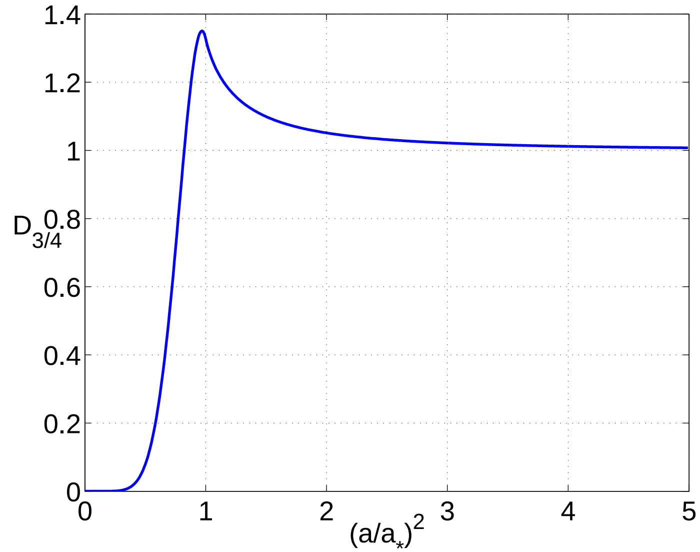
Classically: $d(a) = a^{-3}$

LQC: $d_{l,j}(a) = D_l(q)a^{-3}$ where $q = \left(\frac{a}{a_*}\right)^2$

$$D_l(q) = \left\{\frac{3}{2l}q^{1-l}\left[(l+2)^{-1}\left((q+1)^{l+2} - |q-1|^{l+2}\right) - \frac{1}{1+l}q\left((q+1)^{l+1} - \mathrm{sgn}(q-1)|q-1|^{l+1}\right)\right]\right\}^{3/(2-2l)}$$

for $a \ll a_*$

$$D_{3/4}(q) \approx \left(\frac{12}{7}\right)^6 \left(\frac{a}{a_*}\right)^{15}$$



4. Modified semi-classical equations

1. Modified Friedmann equation

Hamiltonian density is

$$\mathcal{H}_\phi = \frac{1}{2} d_{l,j}(a) p_\phi^2 + a^3 V(\phi)$$

and from the Hamiltonian constraint $\mathcal{H} = 0$, obtain

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{1}{a^2} = \frac{8\pi \ell_{\rm pl}^2}{3} \left(\frac{1}{2} \frac{\dot{\phi}^2}{D} + V(\phi) \right)$$

2. Modified Klein-Gordon equation

From the Hamilton's equations

$$\dot{\phi} = \{\phi, \mathcal{H}_\phi\} = d_{l,j} p_\phi, \qquad \dot{p}_\phi = \{p_\phi, \mathcal{H}_\phi\} = -a^3 \frac{dV}{d\phi}$$

and get

$$\ddot{\phi} + 3\frac{\dot{a}}{a} \left(1 - \frac{1}{3} \frac{d \ln D}{d \ln a}\right) \dot{\phi} + D \frac{dV}{d\phi} = 0$$

Antifrictional term when $d \ln D / d \ln a > 3$ in expanding Universe and frictional term in a contracting Universe.

3. Modified Raychaudhuri equation

$$\frac{\ddot{a}}{a} = -\frac{8\pi\ell_{\rm pl}^2}{3}\frac{\dot{\phi}^2}{D}\left(1 - \frac{1}{4}\frac{d\ln D}{d\ln a}\right) + \frac{8\pi\ell_{\rm pl}^2}{3}V(\phi)$$

Even when $V \equiv 0$, the Universe accelerates provided $d\ln D/d\ln a > 4$.

Can also write

$$\dot{H} = -4\pi\ell_{\rm pl}^2\frac{\dot{\phi}^2}{D}\left(1 - \frac{1}{6}\frac{d\ln D}{d\ln a}\right) + \frac{1}{a^2}$$

4. Modified equation of state

$$w = -1 + \frac{2\dot{\phi}^2}{\dot{\phi}^2 + 2DV}\left(1 - \frac{1}{6}\frac{d\ln D}{d\ln a}\right)$$

Superacceleration ($w < -1$) when $d\ln D/d\ln a > 6$.

5. Exact background solutions

1. Massless scalar field

Using $\ddot{\phi} + 3H(1 - n/3)\dot{\phi} = 0$ and $H^2 = 8\pi\ell_{\rm pl}^2\dot{\phi}^2/6D \quad \Longrightarrow$

$$a = A(-\tau)^p\,, \qquad p = \frac{2}{4-n} < 0$$

2. Scaling solution

$\frac{\dot{\phi}^2}{DV}$ = constant $\quad \Longrightarrow$

$$a = A(-\tau)^p\,, \qquad p = -\frac{4}{n\beta + 4} < 0$$

$$V(\phi) = V_0\,|\phi|^\beta\,, \qquad \phi = F(-\tau)^{np/2}$$

(Lidsey, 2004)

N.B.

$$' = \frac{d}{d\tau} = a\,\frac{d}{dt} \qquad \tau = \frac{p}{aH} \qquad n \equiv \frac{d\ln D}{d\ln a}$$

6. Perturbation equations

Define effective action that gives background equations of motion

$$S = \int d\tau\, d^3x\, a^4 \left(\frac{\phi'^2}{2Da^2} - \frac{\delta^{ij}}{a^2}\partial_i\phi\partial_j\phi - V \right)$$

Perturb field $\phi = \phi_b + \delta\phi$

Define $u = a\delta\phi/\sqrt{D}$ and get

$$\delta S = \int d\tau\, d^3x\, \left(u'^2 - D\delta^{ij}\partial_i u\partial_j u - m_{\rm eff}^2 u^2 \right)$$

$$m_{\rm eff}^2 = -\frac{(a/\sqrt{D})''}{a/\sqrt{D}} + a^2\, D\, \frac{\partial^2 V}{\partial\phi^2}$$

Action is $similar$ ⚠ to the action of a scalar field u in flat spacetime with time dependent effective mass. Can construct quantum theory in an analogous way to that of scalar field u propagating on Minkowski spacetime in the presence of external field ϕ .

7. Quantization

Momentum canonically conjugate to u

$$\pi(\tau, \mathrm{x}) = \frac{\partial \mathcal{L}}{\partial u'} = u'$$

Promote u and π to operators $\hat{u}$ and $\hat{\pi}$ s.t.

$$[\hat{u}(\tau,\mathrm{x}), \hat{u}(\tau,\mathrm{y})] = [\hat{\pi}(\tau,\mathrm{x}), \hat{\pi}(\tau,\mathrm{y})] = 0 \ , \qquad [\hat{u}(\tau,\mathrm{x}), \hat{\pi}(\tau,\mathrm{y})] = i\delta^{(3)}(\mathrm{x}-\mathrm{y})$$

Expand $\hat{u}$ in terms of plane waves:

$$\hat{u}(\tau,\mathrm{x}) = \int \frac{d^3\mathrm{k}}{(2\pi)^{3/2}} \left[\omega_k(\tau)\hat{a}_{\mathrm{k}} + \omega_k^*(\tau)\hat{a}^\dagger_{-\mathrm{k}}\right] e^{-i\mathrm{k}.\mathrm{x}}$$

Obtain equation of motion

$$\omega_k'' + \left(Dk^2 + m_{\mathrm{eff}}^2\right)\omega_k = 0$$

$$\omega_k'' + \left(D_* A^n (-\tau)^{np} k^2 + \frac{m_{\mathrm{eff}}^2 \tau^2}{\tau^2}\right)\omega_k = 0$$

8. General solution

$$\omega_k'' + \left(D_* A^n (-\tau)^{np} k^2 + \frac{m_{\text{eff}}^2 \tau^2}{\tau^2} \right) \omega_k = 0$$

Massless scalar field $m_{\text{eff}}^2 \, \tau^2 = -p(p-1)$

Scaling solution $m_{\text{eff}}^2 \, \tau^2 = -2 + (3-2n)p + \frac{1}{2}(6 - 2n - n^2)p^2$

General solution is:

$$\omega_k(\tau) = c_1 \sqrt{-\tau} J_{|\nu|}(x) + c_2 \sqrt{-\tau} Y_{|\nu|}(x)$$

$$x \propto k(-\tau)^{(2+np)/2} \propto \frac{\sqrt{D}k}{aH}, \qquad \nu = -\frac{\sqrt{1 - 4m_{\text{eff}}^2}}{2+np}$$

We are interested in the asymptotic behavior when $x \gg 1$ and $x \ll 1$. Define, by analogy with standard inflation, effective horizon $d_H = \frac{\sqrt{D}}{aH}$ or effective wavenumber $k_* = \sqrt{D}k$.

9. Normalization and asymptotic limits

Ensure that creation and annihilation operators $\hat{a}_{\rm k}^\dagger$ and $\hat{a}_{\rm k}$ satisfy usual commutation relations $[\hat{a}_{\rm k}, \hat{a}_{\rm l}] = [\hat{a}_{\rm k}^\dagger, \hat{a}_{\rm l}^\dagger] = 0$, $[\hat{a}_{\rm k}, \hat{a}_{\rm l}^\dagger] = \delta^{(3)}({\rm k} - {\rm l})$

Using the commutation relations for u and π get

$$\omega_k^* \, \omega_k' - \omega_k \, \omega_k^{*\prime} = 0 \qquad \text{(Wronskian condition)}$$

General normalized solution is

$$\omega_k(\tau) = \sqrt{\frac{\pi}{2|2+np|}} \sqrt{-\tau} \, H_{|\nu|}^{(1)}(x)$$

1. Small wavelength limit ($\sqrt{D} k/aH \gg 1$)

$$H_\nu^{(1)}(x) \to \sqrt{\frac{2}{\pi x}} \, e^{i(x - \pi\nu/2 - \pi/4)}$$

$$\omega_k(\tau) = \frac{(-\tau)^{-np/4}}{\sqrt{|2+np|\alpha k}} \, e^{i\alpha k (-\tau)^{(2+np)/2}} , \qquad \alpha = 2 \frac{\sqrt{D_* A^n}}{|2+np|}$$

We *do not* recover flat spacetime solution ($\omega_k = e^{-ik\tau}/\sqrt{2k}$) unless $n = 0$ (which only happens at the end of the superinflationary phase).

9. Normalization and asymptotic limits (*cont.*)

2. Long wavelength limit ($\sqrt{D}k/aH \ll 1$)

$$J_{|\nu|}(x) \to \frac{1}{\Gamma(|\nu|+1)} \left(\frac{x}{2}\right)^{|\nu|}, \qquad Y_{|\nu|}(x) \to -\frac{\Gamma(|\nu|)}{\pi} \left(\frac{x}{2}\right)^{-|\nu|}$$

$$H^{(1)}_{|\nu|}(x) = J_{|\nu|}(x) + iY_{|\nu|}(x)$$

(a) Massless field:

$x \propto k(-\tau)^{2p}$ increases $\Leftrightarrow$ modes enter effective horizon.

$Y_{|\nu|}$ can be dominant solution initially but $J_{|\nu|}$ is the dominant asymptotic solution at late time as $\tau \to 0$.

(b) Scaling solution:

$x \propto k(-\tau)^{(2n\beta+8-4n)/(n\beta+4)}$. For $\beta > 2 - 4/n$, x decreases $\Leftrightarrow$ modes exit the effective horizon.

$Y_{|\nu|}$ is the late time dominant solution.

10. Power spectrum of scalar field perturbations

Using $\omega_k(\tau) \propto \sqrt{-\tau}\, Y_{|\nu|}$, $\mathcal{P}_u = \frac{k^3}{2\pi^2}|\omega_k|^2$ and $\mathcal{P}_\phi = D\mathcal{P}_u/a^2$

$$\mathcal{P}_\phi \propto \frac{H^2}{\sqrt{D}} \left(\frac{\sqrt{D}\, k}{aH} \right)^{3-2|\nu|} \propto k^{3-2|\nu|} \, (-\tau)^{1+p(n-2)-|\nu|(np+2)}$$

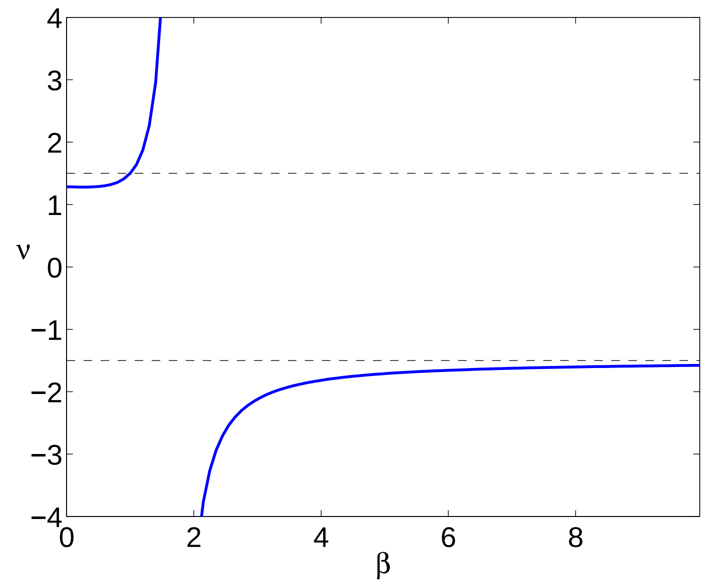
1. Massless field: $\nu = -\frac{n}{8}$

   Scale invariance for $3 - 2|\nu| = 0 \Rightarrow n = 12$ ONLY!

2. Scaling solution:

$$\nu = - \frac{\sqrt{9-12p+8np-12p^2-4p^2n+2n^2p^2}}{2+np}$$

Scale invariance for large β (small p).



11. Other works (massless scalar field)

Mulryne and NJN: Near scale invariance only for $n \approx 12$;

Hossain: Near scale invariance with blue tilted spectrum ($n_s - 1 > 0$).

Using as time variable η , s.t. $\frac{d}{dt} = \frac{1}{a^m}\frac{d}{d\eta}$, and with $m = 1 - n/2$, the equation of motion for the perturbations $f = a^m\, \delta\phi$ is

$$f'' + \frac{a'}{a}(1-m)f' + \left(k^2 - m\frac{a''}{a}\right) f = 0$$

1. Using $H = \text{const.}$, have $a \propto \eta^{1/m}$

$$f'' + \frac{1}{\eta}\left(1 - \frac{1}{m}\right) f' + \left[k^2 - \frac{1}{\eta^2}\left(\frac{1}{m} + 1\right)\right] f = 0$$

(Hossain, 2004)

2. Using $\left(\frac{\dot a}{a}\right)^2 = \frac{\dot\phi^2}{2D}$ and $\dot\phi \propto D a^{-3}$ we have $a \propto \eta^{1/2}$

$$f'' + \frac{1-m}{2\eta} f' + \left(k^2 + \frac{m}{\eta^2}\right) f = 0$$

12. Other works (scaling solution)

Mulryne and NJN: Scale invariance with red tilted spectrum for large β.

Calcagni and Cortês: ”Most of the parameter space generates a strongly scale-dependent spectrum”.

Also use $H = \mathrm{const.}$, have $a \propto \eta^{1/m}$ with $m = 1 - n/2$

$$f'' + \frac{1}{\eta}\left(1 - \frac{1}{m}\right) f' + \left[k^2 - \frac{1}{\eta^2}\left(\frac{1}{m} + 1\right)\right] f = 0$$

(same as Hossain, 2004) where $D\,V''(\phi)$ term was droped out. Massless case applied to scaling solution !!??

”*The solution was derived under the assumption that both* ϵ *and* η *are small enough* $\ll \mathcal{O}(|m|)$ *(massless field approximation). The last examples, however, show that* $|\epsilon|$ *and* $|\eta|$ *can be* $\mathcal{O}(|m|)$*. There may be doubts about the self consistency of the spectral index thus found* ”.

Should use fast roll parameters as in cyclic scenario?

13. Summary, discussion and questions

- Massless scalar field: Scale invariant spectrum of scalar field perturbations is possible only under fine tuning.
- Scaling solution: Scale invariant power spectrum is possible for *steep negative* potentials of the form $V = V_0 \, |\phi|^\beta$.
- What to expect with more general potentials? Must use fast roll parameters?
- Numerical simulations needed to take account of full $D(a)$.
- Correction function in the gradient term, say $\mathcal{H} \propto a^r (\partial\phi)^2$. Obtain

$$\nu \to -\frac{\sqrt{1-4m_{\rm eff}^2\tau^2}}{2+np+rp}$$

- What is the spectrum of curvature perturbations? Can we write $\mathcal{P}_\mathcal{R} = (H/\dot\phi)^2 \mathcal{P}_\phi$.