Asymptotically safe Quantum Gravity

Nonperturbative renormalizability and fractal space-times

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Towards a quantum theory of gravity

- Classical General Relativity:
  - phenomenologically successful at scales $l \gg l_{\mathrm{Pl}}$ (laboratory, solar system, galaxies, $\ldots$)
- Quantized General Relativity:
  - theory is perturbatively non-renormalizable: need infinite number of counter terms
    $\Longleftrightarrow$ theory has no predictive power
  - belief that General Relativity is an effective theory: not fundamental $\Longleftrightarrow$ not valid at arbitrary small distances

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    need infinite number of counter terms
    $\Longleftrightarrow$ theory has no predictive power
  - belief that General Relativity is an effective theory:
    not fundamental $\Longleftrightarrow$ not valid at arbitrary small distances
- there are fundamental theories which are not perturbatively renormalizable
  - so-called non-perturbatively renormalizable theories
  - same predictive power as a perturbative renormalizable theory

Generalized principles of renormalization (K. Wilson)

- theory is fundamental $=$ infinite UV cutoff limit exists
- infinite cutoff limit requires fixed point in RG flow
    - Gaussian fixed point (GFP) ($u_* = 0$) (e.g. pert. renormalization)
    - non-Gaussian fixed point ($u_* \neq 0$)
- fundametal theory is defined through fixed point
- predictivity linked to how coupling constants approach the fixed point

Weinbergs “asymptotic safety” conjecture

(Weinberg (1979) [hep-th/9702027])

conjecture: Euclidean quantum gravity is asymptotically safe

- RG flow of gravity has non-trivial UV fixed point defining a fundamental theory
- evidence: non-Gaussian fixed point (NGFP) in $2+\epsilon$ gravity

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if NGFP also exists in $d = 4$, it defines a fundamental theory of gravity:

Quantum Einstein Gravity (QEG)

- not clear that fundamental theory is of Einstein-Hilbert form
  $\Longrightarrow$ approach differs form quantizing General Relativity

Outline

- introduce tool to study non-perturbative RG flows:
  exact RG equation (ERGE) for effective average action Γ_k
- evidence for NGFP in $d = 4$ quantum gravity
- consequences of asymptotic safety:
  - fractal structure of QEG space-times
  - dynamical dimensional reduction to $d = 2$

The effective average action Γ_k

- Γ_k is Wilson-type (coarse grained) effective action, based on path integral
- IR cutoff at momentum k^2:
  - includes all quantum effects with momenta $p^2 > k^2$
  - quantum fluctuations with $p^2 < k^2$ suppressed by mass$^2 = \mathcal{R}_k(p^2)$
- limits: $k \longrightarrow \infty$ = bare/classical action S
  $k \longrightarrow 0$ = ordinary effective action Γ
- Γ_k satisfies exact evolution equation

$$k\partial_k \Gamma_k = \frac{1}{2}\mathrm{Tr}\left[(\delta^2\Gamma_k + \mathcal{R}_k)^{-1} k\partial_k \mathcal{R}_k\right]$$

- nonperturbative approximation scheme:
  “truncating” the space of action functionals

Constructing Γ_k for gravity

- Starting point: path integral for euclidean metrics, $\int \mathcal{D}\gamma \; \exp[-S_{\text{grav}}[\gamma_{\mu\nu}]]
- background gauge fixing:
  - decompose quantum metric: $\gamma_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$
  - add: background gauge fixing $S_{\text{gf}}[h;\bar{g}] + S_{\text{gh}}[h, C, \bar{C}; \bar{g}]$

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- expand $h_{\mu\nu}$ in $\bar{D}^2$-eigenmodes and introduce IR cutoff $\Delta_k S$
    - modes with $-\bar{D}^2$-eigenvalue $> k^2$ are integrated out
    - modes with $-\bar{D}^2$-eigenvalue $< k^2$ suppressed by mass-term $\mathcal{R}_k$

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- adding sources:
    - scale dependent generating function for connected Green's functions

$$W_k[\text{sources}; \bar{g}]$$

Constructing Γ_k for gravity

- adding sources:
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- Obtaining the effective average action Γ_k:
  - introduce classical fields: $g_{\mu\nu} = \langle \gamma_{\mu\nu} \rangle_k, \ldots$
  - $\Gamma_k\left[g_{\mu\nu}, \bar{g}_{\mu\nu}, \mathrm{ghosts}\right]$ = (modified) Legendre transform of W_k

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- exact RG equation for Γ_k:

$$k\partial_k \Gamma_k = \frac{1}{2}\mathrm{Tr}\left[\left(\delta^2\Gamma_k + \mathcal{R}_k(-\bar{D}^2)\right)^{-1} k\partial_k \mathcal{R}_k(-\bar{D}^2)\right] + \mathrm{ghost\ contribution}$$

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- diffeomorphism invariant effective action:

$$\Gamma_k[g] = \Gamma_k[g, \bar{g} = g, \text{ghost} = 0]$$

- theory: specified by RG trajectory:

$$k \mapsto \Gamma_k[g]$$

The Einstein-Hilbert truncation

- Ansatz for Γ_k

$$\Gamma_k = \frac{1}{16\pi G(k)} \int d^d x \sqrt{g}\, \{-R + 2\Lambda(k)\} + \text{classical gauge fixing \& ghost terms}$$

- running couplings

  Newton constant $G(k)$, dimensionless $g(k) = k^{d-2} G(k)$

  cosmological constant $\Lambda(k)$, dimensionless $\lambda(k) = \Lambda(k)/k^2$

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- substitute ansatz into ERGE and project on subspace:
  $\Longrightarrow$ autonomous β-functions for g, λ

$$k\partial_k g = \beta_g(g,\lambda) \; , \qquad k\partial_k \lambda = \beta_\lambda(g,\lambda).$$

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$$k\partial_k g = \beta_g(g,\lambda)\ , \qquad k\partial_k \lambda = \beta_\lambda(g,\lambda).$$

- β-functions have a non-Gaussian fixed point:
  - exists $g^* > 0, \lambda^* > 0$ where $\beta_g(g,\lambda) = \beta_\lambda(g,\lambda) = 0$
  - IR repulsive in both g, λ

Evidence for asymptotic safety

Evidence within the exact RG approach:

- within the Einstein-Hilbert approximation:
  - test $\mathcal{R}_k$-dependence of physical quantities (critical exponents, $g_* \lambda_*, \dots$)
    $\Longrightarrow$ k-independent in exact theory
- generalized truncation including R^2-term:
  - NGFP stable under extending truncation space
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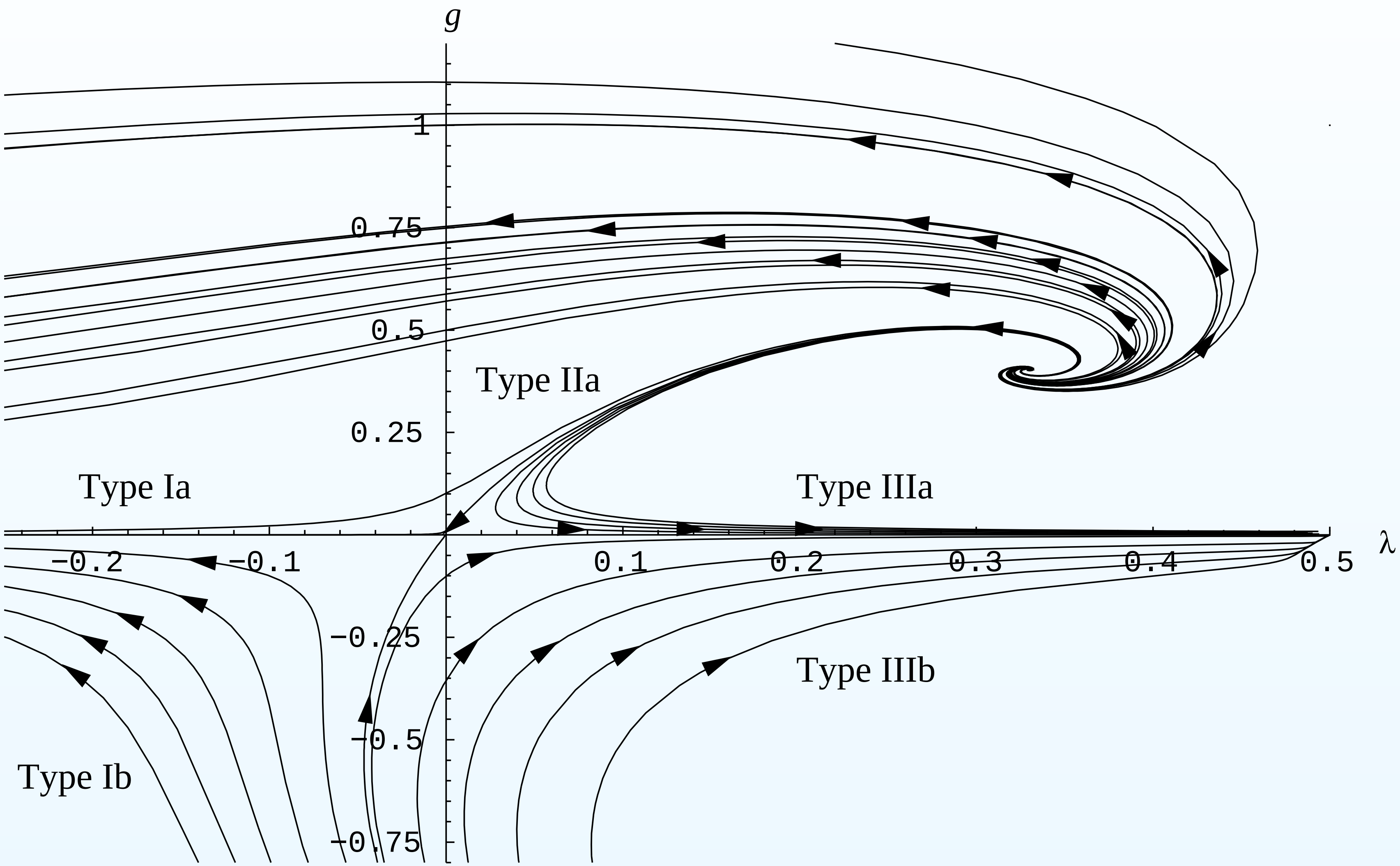
NGFP also found via:

- exact path-integral calculation for metrics with 2 Killing vectors
  (P. Forgács, M. Niedermaier, hep-th/0207028)
- proper-time RG equation
  (A. Bonanno, M. Reuter, hep-th/0410191)

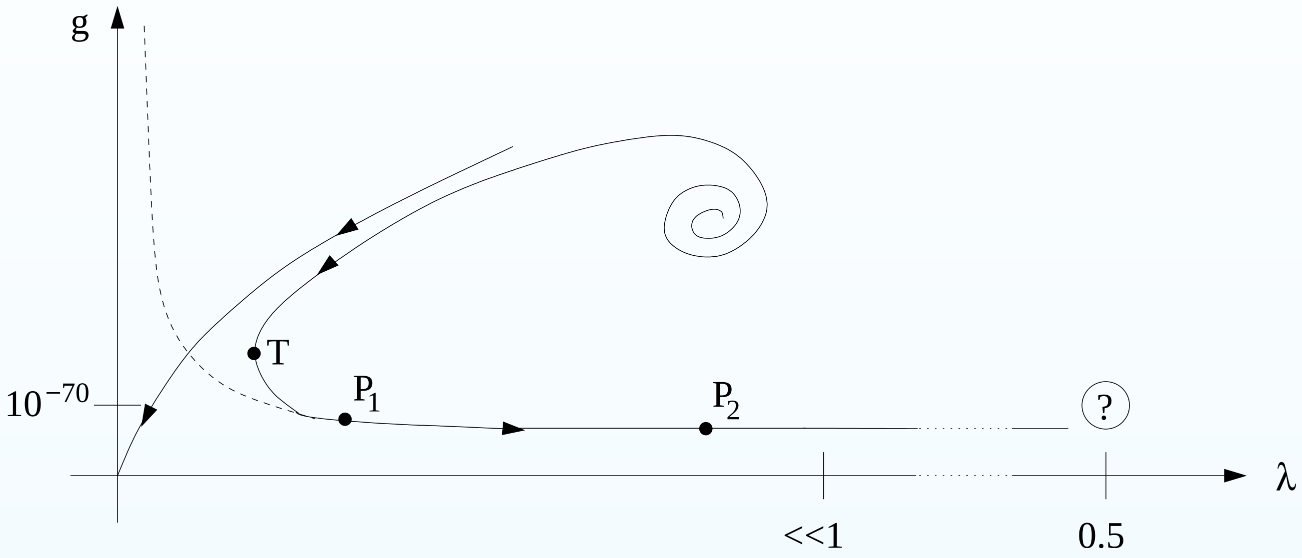
The RG flow of QEG in the Einstein-Hilbert-truncation

(M. Reuter, F. Saueressig, Phys. Rev. D **65** (2002) 065016 [hep-th/0110054])



The RG trajectory realized in Nature

(M. Reuter, H. Weyer, JCAP **0412** (2004) 001 [hep-th/0410119])



- originates at NGFP (quantum regime: $G(k) = k^{-2} g_*, \Lambda(k) = k^2 \lambda_*$)
- linear regime: oscillations around NGFP
- passing *extremely* close to the GFP
- long classical GR regime (classical regime: $G(k) = \mathrm{const}, \Lambda(k) = \mathrm{const}$)
- $\lambda \lesssim 1/2$: strong IR renormalization effects?

Classical versus Quantum space-time

(O. Lauscher and M. Reuter, hep-th/0508202, hep-th/0511260)

RG running of $\Gamma_k[g] \Longrightarrow$ profound consequences for space-time structure

- Classical General Relativity, $S_{\rm EH}$:
  - vacuum Einstein equations:

$$R_{\mu\nu} = \Lambda \, g_{\mu\nu}$$

  - solution $g_{\mu\nu}$ = single metric valid at all length scales

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- effective action $\Gamma_k[g]$:
  - 1-parameter family of equations of motion:

$$\frac{\delta \Gamma_k[\langle g_{\mu\nu}\rangle_k]}{\delta g_{\mu\nu}} = 0$$

  - solution $\langle g_{\mu\nu}\rangle_k$ = metric seen by physical process with momentum k^2
  - proper distances calculated from $\langle g_{\mu\nu}\rangle_k$ depend on k^2
    $\Longrightarrow$ typical fractal behavior (Coastline of England)

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- Classical limit:
  - fractal behavior = linked to k-dependece of $\Gamma_k[g_{\mu\nu}]$
  - trajectory realized in nature $\Longleftrightarrow$ long classical regime :
    $\Gamma_k[g_{\mu\nu}]$ is k-independent $\Longleftrightarrow$ recover classical space-time picture

Relating k to the coarse graining scale $\ell(k)$

- Interpretation: $\langle g_{\mu\nu}\rangle_k$ averaged over proper distances $\ell(k)$

Algorithm to determining $\ell(k)$:

1. Recall: $\Gamma_k[g] = \Gamma_k[g, \bar{g} = g, \mathrm{ghost} = 0]$
   $\Longrightarrow k^2 = -\bar{D}^2 = -D^2$ is last eigenmode integrated out
2. construct “on-shell”-Laplacians $\Delta(k) = D^2(\langle g_{\mu\nu}\rangle_k)$
3. find eigenfunction $\psi(x)$ with eigenvalue k^2
4. determine typical coordinate distance Δx^μ on which $\psi(x)$ varies
5. calculate proper length $\ell(k)$

$$\ell(k) = \sqrt{\langle g_{\mu\nu}\rangle_k \, \Delta x^\mu \, \Delta x^\nu}$$

scaling argument in the quantum regime:

$$\ell \propto k^{-1}$$

Quantum regime: QEG space-times are self-similar fractals

- in the quantum regime ($k^2 \gg m_{\rm Pl}^2$)

$$\Lambda(k) = \lambda_*\, k^2\ , \qquad G(k) = g_*\, k^{2-d}\ , \qquad k \propto 1/\ell$$

- effective field equations in the Einstein-Hilbert approximation:

$$R_{\mu\nu}(\langle g_{\mu\nu}\rangle_k) = \Lambda(k)\langle g_{\mu\nu}\rangle_k$$

- radius of curvature r_c for a typical solution:

$$r_c(k) \propto \Lambda(k)^{-1/2} \propto k^{-1}$$

- Radius of curvature at resolution ℓ:

$$r_c(\ell) \propto \ell$$

$\Longrightarrow$ zooming into the space-time structure does not change the image

Dynamical dimensional reduction in QEG space-times

Investigate effective graviton propagator

$$\Gamma_k = -\frac{1}{16\pi G(k)} \int d^d x \, \sqrt{g} \, R + \ldots$$

expanding around flat space

$$\tilde{\mathcal{G}}_k(p) \propto \frac{G(k)}{p^2}$$

classical regime: $G(k) = \text{const}$

$$\tilde{\mathcal{G}} \propto \frac{1}{p^2} \quad \Longrightarrow \quad \mathcal{G}(x,y) \propto \frac{1}{|x-y|^{d-2}}$$

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fixed point regime: cutoff is $p^2 \Longrightarrow G(k^2 = p^2) \propto p^{2-d}$

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effective graviton propagator reduces dynamically:

macroscopically: $d = 4$ $\Longleftrightarrow$ microscopically: $d = 2$

Spectral dimension for classical manifolds

- diffusion of scalar test particle on Riemannian manifold with metric g

$$\partial_T K_g(x, x'; T) = \Delta_g K_g(x, x'; T)$$

$$\Delta_g \phi \equiv g^{-1/2} \partial_\mu (g^{1/2} g^{\mu\nu} \partial_\nu \phi)$$

- define average return probability

$$\begin{aligned} P_g(T) &\equiv \frac{1}{V} \int d^d x \sqrt{g(x)}\, K_g(x, x; T) \\ &= \frac{1}{V} \mathrm{Tr}\left[\exp(T\Delta_g)\right] \\ &= \left(\frac{1}{4\pi T}\right)^{d/2} \sum_{n=0}^{\infty} A_n T^n \end{aligned}$$

- asymptotic expansion contains information about space-time dimension

$$d = -2 \frac{d \ln P_g(T)}{d \ln T}$$

Spectral dimension of QEG space-times

- in QEG: metric of manifold is k-dependent

  $\Longrightarrow$ diffusion process “with momentum k” sees metric $\langle g_{\mu\nu} \rangle_k$

  $\Longrightarrow$ diffusion equation and return probability will become k-dependent

- Computation of the spectral dimension:

  1. determine k-dependence of $\Delta(k)$
  2. solve the k-dependent heat equation
  3. evaluate “quantum return probability” $P(T)$
  4. obtain spectral dimension

$$\mathcal{D}_s = -2 \frac{d \ln P(T)}{d \ln T}$$

Spectral dimension $\mathcal{D}_s$ of QEG space-times

- Quantum return probability:

$$P(T) = \int \frac{d^d p}{(2\pi)^d} \exp[-p^2 \, F(p^2) \, T] \, , \quad F(p^2) = \Lambda(p)/\Lambda(k_0)$$

Limits:

- ○ $T \to \infty$:

  long random walks $\Longleftrightarrow$ probe space-time at large distance

  $\Longrightarrow$ classical regime: $F(p^2)$ with $p^2 \to 0$

- ○ $T \to 0$:

  short random walks $\Longleftrightarrow$ probe space-time at small distance

  $\Longrightarrow$ fixed point regime: $F(p^2)$ with $p^2 \to 0$

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- classical regime: no running, $F(p^2) = 1$:

$$P(T) \propto T^{-d/2} \Longrightarrow \mathcal{D}_s = d$$

- fixed point regime: $\Lambda(p) \propto p^2 \to F(p^2) \propto p^2$:

$$P(T) \propto T^{-d/4} \Longrightarrow \mathcal{D}_s = d/2$$

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$d = 4$: QEG predicts continuous change of fractal dimension

$\mathcal{D}_s = 4$ macroscopically $\Longrightarrow$ $\mathcal{D}_s = 2$ microscopically

Spectral dimension $\mathcal{D}_s$ from Causal dynamical triangulations

(J. Ambjørn, et. al. Phys. Rev. Lett. 95 (2005) 171301 [hep-th/0505113])

- CDT $\Longleftrightarrow$ Monte Carlo evaluation of Lorentzian Path Integral
- mesured D_s for $d = 4$

$$\begin{aligned} \mathcal{D}_s(T \to \infty) &= 4.02 \pm 0.1 \\ \mathcal{D}_s(T \to 0) &= 1.80 \pm 0.25 \end{aligned}$$

- supports analytic results from exact renormalization group

Summary

- mounting evidence that Euclidean Quantum Gravity is asymtotically safe
- exact renormalization group:

  space-time carries a one-parameter family of metrics $\langle g_{\mu\nu} \rangle_k$

  describe metric structure at different length-scales
- regimes with non-trivial RG running:

  space-time obtains fractal properties
- fractal dimension of space-time continuously changes:

  $\mathcal{D}_s = 4$ at macroscopic distances

  $\mathcal{D}_s = 2$ at microscopic scales