ABHAY ASHTEKAR

ASYMPTOTIC QUANTIZATION
Based on 1984 Naples Lectures
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Based on 1984 Naples Lectures

BIBLIOPOLIS
To
Anne Magnon
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**Preface**

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**Part I. Quantum Gravity: what and why**

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**Part II. Asymptotic Quantization**

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The material presented in the following chapters is based on a series of lectures given at Naples in May 1984. However, this is not a mere transcript of the talks. Although the issues covered in this text were discussed in the lectures, I have included many details and updated the material at several points.

The first lecture was held in the Palazzo Serra di Cassano, the premises of the Istituto Italiano per gli Studi Filosofici, and was addressed to a general audience. The subsequent lectures were hosted by the Istituto di Fisica Teorica of the University of Naples and were addressed to research physicists. Consequently, there is a substantial difference in the technical levels of the two parts into which the text is divided. The first is addressed to laypersons actively interested in science, say, the readership of Scientific American. An article based on this chapter has appeared in French in the semi-popular journal La Recherche (Volume 15, pages 1400-1409, 1984). This part may be used to introduce undergraduates to the world of quantum gravity. The second part is intended primarily for advanced graduate students and other researchers who want to become acquainted with the intricacies of this world.

The technical material is devoted to Asymptotic Quantization. (The lectures also introduced some new ideas on canonical quantization of the gravitational field, based on the choice of certain spinorial variables in terms of which Einstein’s theory can be formulated. These ideas have evolved significantly over the last two years. However, since the current state of the art in this area is likely to change in the near future, these development are not discussed here.) Asymptotic Quantization is a general framework designed to obtain a scattering-matrix description for gravity as well as for field theories in Minkowski space-time. It is especially well-suited to handle the infrared problems. The incoming states are specified at past null infinity and to outgoing ones, at future null infinity. Since the characteristic data at null infinity for non-linear physical fields is the same as that for their linearized versions, one can construct the spaces of asymptotic states in the non-linear case directly, without having to introduce heuristic considerations involving mappings from the space of interacting fields to their non-interacting counterparts.

In quantum gravity, the fundamental scale is set by the Planck length. Since this scale is so tiny compared to the familiar ones from elementary particle physics — if one were to picture the Planck length as the radius of the first Bohr orbit in the hydrogen atom, one is led to picture hadrons as occupying a space larger than the earth-moon system! — the in-out description of the asymptotic quantization scheme should suffice largely in the investigation of quantum gravity effects in particle physics. On a more fundamental level, a key advantage of this scheme is that the topology of the interior of space-time is not fixed a priori. One can therefore investigate the issue of topological
fluctuations. For example, one can construct a semi-classical picture of the quantum vacuum in which virtual black-holes contribute to the vacuum to vacuum amplitude. One can study the physical effects of these contributions. Although the framework is sufficiently flexible to accommodate non-trivial topologies in the interior, it is also sufficiently rigid to enable the introduction of notions familiar from Minkowskian quantum field theory, such as 1-particle (i.e. graviton) states, spin and mass of these particles, operations of parity and time reversal, etc. Therefore, it provides the mathematical machinery required for precise formulations and analysis of issues such as the possibility of CPT violation in quantum gravity. Finally, the use of this framework sheds considerable light on the origin of the Bondi-Metzner-Sachs (BMS) group in the gravitational radiation theory of exact general relativity. The enlargement of the asymptotic symmetry group from the Poincaré to the (infinite dimensional) BMS group is an imprint left on the classical gravitational theory by the infrared behavior of the quantum gravitational field.

Some of the results reported here were obtained in collaboration with Kumar Narain and Michel Streubel. The work on infrared problems owes a great deal to discussions with Iwo Birula, Malcolm Ludvigsen and especially Ted Newman. Conversations with Bob Geroch, Joshua Goldberg, Richard Hansen, Gary Horowitz, Anne Magnon, Ted Newman, Roger Penrose, Bernd Schmidt and Amitabha Sen helped me in understanding the subtle issues related to null infinity. To all these colleagues, I am most grateful. I would also like to take this opportunity to thank Avv. Gerardo Marotta, President of the Istituto Italiano per gli Studi Filosofici, and Professor Giuseppe Marmo, a scientific organizer of this series, for their hospitality in Naples. Additional thanks go to Dr. Marmo for his patience and gentle handling of administrative matters which made this endeavor a real pleasure. Finally, my task was greatly simplified by the efficient typing of Ms. Deborah Donahue.

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PART I

QUANTUM GRAVITY: WHAT AND WHY
The Cosmic Snake

Subtle is the Lord

-Albert Einstein

The domain of known physics is indeed very vast. The typical size of galaxies is ten billion billion kilometers while that of elementary particles such as protons is as small as a tenth of a millionth of a millionth of a centimeter. What is even more astonishing is that only four different types of basic interactions are at the root of all complex phenomena that can occur in this vast range. The first is the gravitational interaction which governs astrophysical objects. Because the gravitational force is long range and because it is always attractive, it reigns supreme at astronomical scales. The second is the electromagnetic interaction which dominates the everyday life. In particular, it governs light, is responsible for radio and television and is at the base of all chemical reactions. It is the primary force responsible for things that happen from the everyday length scales down to the molecular and atomic distances. The last two interactions—the weak and the strong—govern the world of atomic nuclei and elementary particles; they are extremely short range. The weak interactions are responsible for disintegration of elementary particles such as neutrons and muons and hence for much of what was called radioactivity in the beginning of this century. The strong interactions bind protons and neutrons in atomic nuclei and quarks within protons and neutrons. The length scale involved is so tiny that the world of these two interactions belongs entirely to quantum physics. In terms of our everyday perceptions, it is a "fuzzy" world in which Heisenberg's uncertainty dominates, waves and particles lose clear-cut distinction and one can only calculate probabilities of occurrence of events. The domain of electromagnetic
interactions is an intermediate one. In many phenomena they encompass—such as chemical reactions—quantum mechanics is crucial. However, in others, such as propagation of radio waves, quantum effects can be safely ignored. They exist but are unimportant both conceptually and in magnitude. The pattern continues on astronomical scales where the quantum effects associated with the gravitational force are completely negligible.

And yet, the gravitational interaction has another facet. There are several reasons to believe that it would dominate again in another domain: at scales below millionth of a billionth of a billionth of a centimeter. This is the smallest fundamental length that arises in today's physics and it lies deep in to the realm of quantum physics. In fact, its existence was first pointed out in 1890 by Max Planck, the founder of quantum mechanics, and it is now called the planck length. It is clear that the manifestations of gravity at the planck scale would necessarily involve quantum mechanics, if not an even more radical theory. This is the realm of quantum gravity.

It is indeed very puzzling that one and the same force should dictate physics at the smallest and the largest scales that can be envisaged today. Nobel laureate Sheldon Glashow has a very picturesque description of this state of affairs (Figure 1). He pictures the world of physics as a snake with four parts. The long part near the tail represents the world of astronomy and cosmology, governed by gravity; the next part stands for the everyday world, dominated by electromagnetism; and the last part of the body denotes the world of atomic nuclei and elementary particles, ruled by weak and strong interactions. The head, representing the planck length and below, belongs once again to gravity; the snake eats its own tail!
The problem of quantum gravity is thus unique at least in two ways. First, because quantum gravity is concerned with the smallest length scale envisageable today, and because physics has been very successful so far in describing and predicting the behavior of systems in terms of smaller ones — the "constituents" — it is quantum gravity that ultimately carries the burden of accounting for everything in the world of fundamental physics. Second, because "the snake eats its own tail", quantum gravity seems to hold the key in our search of obtaining a complete, truly unified picture of all physics.

**Difficulties**

*Traveler, there are no paths; Paths are made by walking.*

—Antonio Machado

In spite of significance of quantum gravity, at least until recently, relatively little research was done in this area. The foremost reason is simply the lack of concrete experimental facts which bear directly on the structure of quantum gravity. To understand this state of affairs, one has to develop a feeling for how small the planck length really is. The radius of a proton or a neutron, for example, is a hundred billion billion times larger than the planck length. Put differently, the ratio of the proton radius to the planck length is bigger than the ratio of the earth-moon distance to the radius of an atom! Thus, to measure directly the effects which occur at planck length using the technology available, say, ten years ago in high energy physics would have been almost as hopeless as trying to detect directly the orbit of the electron in the hydrogen atom using a telescope! To examine directly a sub-atomic process one has to use probes whose dimension is smaller
than the distance scale associated with the process. The most convenient probes available are elementary particles. As the distance scale associated with processes decreases, one must use elementary particles which have smaller and smaller de Broglie wave lengths and hence, by Heisenberg's uncertainty principle, larger and larger momenta, and therefore, bigger and bigger energy. Thus, in the beginning of this century, to show that atoms have nuclei, for example, Rutherford had to use projectiles whose energy was only a few million electron volts. To show that protons and neutrons themselves have constituents, on the other hand, one now needs to accelerate particles to tens of billions of electron volts. Such energies have become attainable only recently. The energy needed to probe directly the planck length is clearly way above the current technology.

However, recently, different types of experiments have been made in high energy physics to check if the proton can disintegrate\(^1\). The processes believed to be responsible for this decay are mediated by very heavy particles whose range of interaction is only about ten thousand times larger than the planck length. These experiments do not use high energy accelerators; the particles responsible for the interactions are not produced in the laboratory. Rather, one works with an extremely large number of protons and waits for a long time to see if one of them disintegrates. The fact that it is possible to carry out such experiments has provided, at least indirectly, an impetus to work in quantum gravity. The idea is that although it is hopelessly beyond our technology to produce, in the laboratory, excitations of the quantum gravitational field at the planck length, it may not be out of question to look for some indirect effects associated with these excitations. Consequently, theoreticians are now going beyond formal developments and have begun to look for physical effects.
Until very recently, however, the emphasis was on constructing a formal, consistent quantum theory of gravity. Even this task has proven to be formidable and one is still far from the goal. Here, the major difficulty is the following. The best theory of gravity available today is Einstein's general relativity. Einstein was led to this theory in an attempt to reconcile Newton's theory of gravity, based on absolute space and absolute time, and special relativity, in which simultaneity of distant events ceases to have an absolute, observer independent meaning and space and time fuse into a single, four-dimensional continuum, the space-time. To bring about the reconciliation, Einstein had to introduce curvature in the four dimensional space-time continuum. Because the gravitational field is universal—everything is a source of the field and gravity affects everything in the same way—it can be coded in the very geometry of space-time. Thus, in general relativity, space-time is curved and the amount of curvature is a measure of the strength of the gravitational field. The basic equations of the theory, the Einstein's equations, tell us, quantitatively, how matter acts as a source of this curvature. The fundamental dynamical entity in general relativity—the entity which "evolves in time" and whose properties constitute the primary goal of one's study—is the geometry of space-time itself. For comparison, let us consider the electromagnetic field produced by charged particles. If the charges (or, rather, the dipoles) oscillate, as, for example in the emitter of a radio station, the field is disturbed and the disturbance propagates in the form of electromagnetic waves. All these things are supposed to take place in a space-time which itself is inert and inactive and serves only as an arena or a background. Thus, here, space-time is like a stage on which the drama of evolution unfolds, the various charges and the electromagnetic field being the actors. In general relativity, the situation
is quite different. Here, the space-time geometry itself is the dynamical entity. There is no longer a fixed background, immune to change; the stage disappears and joins the troop of actors! This unique property of general relativity introduces a host of conceptual as well as technical difficulties in the problem of constructing a quantum theory of gravity. For, to construct this theory, one has to obtain a quantum description of the very geometry of space-time. Not only has one to regard this geometry as a dynamical entity, but, in place of one, evolving geometry, one must consider a probability distribution of them. One has to investigate the detailed microscopic structure of space-time itself. Clearly, this makes the problem very difficult. However, it also brings out why the solution to the problem is likely to have impact on all of fundamental physics including areas which are not directly concerned with gravitation.

Expectations

*What is now proved was once only imagin’d*

—William Blake

Because of lack of experiments, theoreticians have had very few pointers to direct them, or constraints to weed out unpromising ideas. Therefore, the field of quantum gravity has been more speculative than some other branches of theoretical physics. Nonetheless, overall, detailed work has proceeded with care and caution and the basic ideas are deeply rooted in the well-tested principles of special and general relativity and quantum mechanics. Before going on to discuss concrete and somewhat technical results that have emerged recently, — results which represent a surge of activity in the field during
the last few years - we shall first describe, qualitatively, the type of results that one expects from quantum gravity. This description will also serve to illustrate why one is interested in the subject.

In quantum gravity, there appear three fundamental constants of nature: Planck's constant, ħ, comes from quantum mechanics; Newton's constant, G, from gravity; and, the velocity of light, c, comes from special relativity. The three constants have physical dimensions. As Table 1 shows, there is a unique combination of them with the dimension of length (centimeters): the planck length referred to earlier. No other physical theory has a fundamental length built into it. The length scale - the Bohr radius which is basic to atomic physics, for example, involves the mass and the charge of the electron in addition to the universal constant ħ, and is, therefore, significant only to a special class of physical systems - the atoms. The planck length, on the other hand, refers only to universal constants and not to parameters specific to a sub-class of physical systems. Therefore, quantum effects of gravity are expected to be significant to all of physics around and below the planck-length. The common belief is that our usual picture of space-time as a four dimensional continuum would cease to be a good approximation at the planck length and that the "microscopic" structure of space-time may be very complicated. This would have a profound effect both on general relativity and quantum theory.

To illustrate how this may come about, let us first consider general relativity. Although this theory has been accurately verified on macroscopic scales - e.g. in the solar system, the universe as a whole, etc. - like all so-called "classical theories", it has an internal blemish: It permits, and even predicts, the occurrence of singular configurations in which physical quantities become infinite. Let us first consider Newton's law of gravita-
tion. It predicts that a dust cloud (which is sufficiently cool that the thermal agitations can be neglected) will collapse under the gravitational attractive force, ultimately forming a singular point of infinite density at which all mass would be concentrated. In the sixties, English mathematical physicists Roger Penrose and Steven Hawking proved that the situation is similar in general relativity: In a generic situation, matter evolves to form singularities. Now, one believes that such infinities do not actually occur in Nature and their occurrence in a theory is a signal that one is applying the theory beyond its domain of validity. The general picture is that quantum effects would become extremely important once the density of matter becomes comparable to the planck density - which, incidently, is ten followed by eighty zeros, times the density of nuclear matter! - and would intervene to rescue general relativity. To see how this may come about, it is instructive to consider the example of a hydrogen atom, in which the light, negatively charged electron orbits around a heavy, positively charged proton. If one uses the classical - i.e. "non-quantum" - theory to describe this system, one encounters a severe problem: the energy of the system can attain arbitrarily large negative values if one shrinks the radius of the orbit, so that, under external disturbances, the electron can fall inwards and eventually (actually, rather quickly) fall into the proton, creating a singular configuration with infinite negative energy. That is, the classical theory predicts that the atoms are unstable, and hence, we shouldn't exist! Fortunately, this does not occur in Nature. What happens is that quantum principles intervene and bring the lowest permissible value of energy up from minus infinity to \(-\frac{me^4}{2\hbar^2(1+\frac{m}{M})}\), where m and M are, respectively, the masses of the electron and the proton, and \(-e^2\), the product of their charges. The classically predicted instability
and singularity is thus avoided. (Note that letting $h$ go to zero corresponds to increasingly ignoring quantum effects and, in the limit, one gets the infinity of the classical theory.) Very roughly, the intervention of quantum principles may be picturized as follows. Due to Heisenberg's uncertainty principle, the electron cannot have a sharp, well-defined trajectory; it only has a probability amplitude for various trajectories. These amplitudes interfere in such a way that the electron cannot keep falling towards the proton without limit. Returning to the singularities of general relativity, the hope is that, in quantum gravity, a similar interference would occur between probability amplitudes for various space-time geometries and prevent the occurrence of infinities.

In a similar way, quantum gravity may cure some of the problems faced by the quantum theory today. Unification of the principles of quantum mechanics and special relativity has given rise to the quantum theory of fields. An outstanding example of such theories is quantum electrodynamics, the quantum theory of charged particles and electromagnetic fields. In the development of this theory, the focus was on calculations leading to predictions which can be tested against experiments, rather than on issues of mathematical rigour. And the theory has had brilliant successes with experiments, some of which are, in fact, among the most sensitive ones that have been ever performed. However, in the calculations leading to these predictions, one has to integrate certain expressions over energies of virtual photons which mediate the interaction, and these integrals diverge because the range of integration extends to infinite energies. (The resulting infinities are called ultra-violet divergences because, in the visible spectrum, the violet colour corresponds to the most energetic photons.) A systematic procedure - called renormalization - has been invented to subtract out these infinities and to obtain finite answers.
and it is these answers that have had the experimental success. Thus, the
procedure "works", but seems rather ad-hoc. Now, integration up to infinitely
high energies and momenta corresponds, in the physical-space language, to
integration down to infinitely small time and space intervals. Thus, as was
emphasized by the founders of renormalization theory, infinities arise because
one assumes that the smooth-continuum picture of space-time is valid to
arbitrarily small distances. The true structure of space-time is presumably
very complicated and the renormalization procedure may be only a convenient
trick to get the correct answer without bothering about the details of these
complications. Thus, it is only when we have a reasonably good picture of the
quantum structure of space-time that we can really understand why the renormal-
ization procedure works. In this sense, it is only after we can handle
quantum gravity reasonably well that we can hope to have a complete under-
standing of quantum theories of other interactions.

Indeed, the impact of quantum gravity on the physics of the elementary
particles may be much more profound. Perhaps the most ambitious scenario in
this respect is the one presented by the American physicist John Wheeler
almost twenty years ago. Using the fact that the Planck length is the
smallest length scale available today, he argues that the elementary parti-
cles — out of which the world is built — are themselves born out of the
quantum fluctuations of the space-time geometry. He envisages the space-time
geometry to be very turbulent and complicated at the Planck scale, perpetually
undergoing changes. He calls this small scale structure, "space-time foam".
How can the space-time be so foamy and tumultuous at the Planck length and yet
appear to be a nice, quiet continuum at our everyday scale and beyond? The
analogy is to an ocean. Seen from an airplane a few kilometers above the sea
level, the ocean seems calm and smooth, a plane, a continuum. Fly down to a
few hundred meters and you realize that the ocean is not as peaceful. You may perceive waves and the idealization by a smooth plane would no longer be valid. Come still closer and the structure would look even less peaceful. You may perceive all sorts of sub-structures, turbulences around rocks, foam around algae and what not. Wheeler's picture of space-time is similar. What do the waves and turbulences and the perpetual motion of water correspond to? They correspond to perpetual changes in the small scale topology of space-time. To understand what this means, let us consider a two-dimensional space. The picture which comes to mind immediately, is the x-y plane. But there are other possibilities. One can consider a sphere, the surface $x^2 + y^2 + z^2 = \text{const.}$, or a torus which corresponds to a sphere with a handle, or, a sphere with two handles, or, three, and so on. These are all two-dimensional spaces and yet they are distinct because one cannot continuously deform or bend one of them to another. One says that they have different topologies. The situation is similar in three dimensions, i.e., for space, as well as in four dimensions, i.e., for space-time. Wheeler envisages all sorts of topologies, co-existing at the Planck length and below. And these are perpetually changing. In the two-dimensional picture, we can envisage a plane with a lot of very tiny handles of Planck length which can appear and disappear perpetually. This gives space-time its foamy structure at the small scale. The appearance of a handle represents an excitation of the quantum gravitational field and the disappearance, a de-excitation. Finally, Wheeler imagines that such excitations appear to us as elementary particles. What distinguishes one of these particles from another is the topology associated with the excitation. One type of topological fluctuation may appear to us as a charged particle, while another type may appear as a spinning particle, and yet another may correspond to a particle ("quark") with "color", or "charm". Thus, in Wheeler's pic-
ture, to understand the phenomenology of elementary particles, one must understand the vacuum state — with all its fluctuations — of quantum gravity; the elementary particle phenomenology is "chemistry of geometry"! One must add however, that the picture has remained mostly speculative and concrete results have been very slow to emerge. But it may be that the picture came way ahead of its time. It is only recently that a mathematical model of any sort to represent space-time foam was proposed and only last year that even an approximate calculation of the topological transitions became feasible.

It is also of interest to know that these considerations were proposed some fifteen years before the importance of topological issues was appreciated in the elementary particle physics through gauge theories.

Somewhat more concrete expectations can be formulated in connection with symmetries. In addition to the obvious symmetries associated with motions in the physical space, e.g. rotations and translations, in quantum physics there are also the so-called "internal" symmetries which cannot be pictured so easily in terms of everyday notions. Motions corresponding to these symmetries take place in abstractly defined, "internal" spaces of physical attributes associated with particles. Perhaps the most familiar example is the "spin space", associated, e.g., with an electron. Now, if a physical system has a certain symmetry, it inherits a conservation law. For example, the conservation of momentum in the collision of billiard balls is a consequence of the fact that laws governing their motion are invariant under space-translations.

Among fundamental interactions it is an observed fact that weaker the interaction, smaller is the number of symmetries it respects (see table 2). One might, therefore, expect that quantum gravity would have less symmetries than the other three interactions; quantum gravity processes may violate the conservation laws that are respected by other interactions. One now has
concrete indications as to how this may come about. The strong, weak and electromagnetic interactions conserve the so-called baryon number. The situation is similar to the conservation of electric charge. Just as there are positively and negatively charged particles in Nature, there are baryons and anti-baryons, and, just as the algebraic sum of charges of all incoming particles in a reaction is the same as that of outgoing particles, the number of baryons minus anti-baryons is conserved in any reaction caused by one or more of these three interactions. If quantum gravity effects are included, however, the conservation would fail. For example, a star, consisting only of baryons may collapse to form a black hole. Once the black hole is formed, from outside one can only measure the total mass of the material that collapsed using the long-range gravitational field associated with it, as well as the total charge, using the long-range electric field. However, since there is no long-range field associated with the baryon number, one would not be able to tell if the collapsed matter consisted of baryons or anti-baryons. Now, the black hole can evaporate by quantum processes by emitting particles. (See the following section.) Since the gravitational interaction cannot distinguish between baryons and anti-baryons, they will be emitted in equal numbers. Thus, the net effect of the whole process would be creation of anti-baryons at the cost of baryons; this "reaction" would violate baryon conservation. This non-conservation is important conceptually but in practice it would be dwarfed by other processes which are predicted by the so called grand unified theories. The second possible symmetry violation associated with quantum gravity is much more important. It is called CPT violation. It is an observed fact that in any reaction mediated by strong, weak and electromagnetic forces, one has certain symmetries. Each elementary particle has an anti-particle which has the same mass, spin, etc. as the particle, but
electric charge of opposite sign. (Neutral particles such as the photon are their own anti-particles.) The operation of replacing all particles in a reaction by their anti-particles is called charge conjugation and denoted by C. Another operation is space-reflection (or Parity) P, which replaces spatial coordinates x, y, z of particles and fields by their reflected images through the origin, -x, -y, -z. Finally, one has the operation T, the time reversal, which replaces the time variable t in equations by -t. Newton's laws of motion, for example, are invariant under the operations P and T. Now, it is an observed fact that the composition CPT of these operations is a symmetry of strong, weak and electromagnetic forces: If, in any reaction mediated by these three forces, one replaces particles by their anti-particles, and carry out a space reflection and a time reversal, one obtains a reaction which has the same properties (e.g. the same rate of occurrence) as the first one. Furthermore, there is also a theorem which says that in any process described by the usual quantum field theory, CPT must be conserved. However, as we saw earlier, the usual quantum field theory is a product of special relativity and quantum mechanics. Thus, the theorem presupposes that the space-time is a flat inert background and uses crucially the symmetries of flat space. Hence it is simply not applicable when one replaces the flat space-time by a curved one, let alone by a probability distribution of curved space-times! Furthermore, in the quantum processes on a background black hole space-time, one knows that CPT is not conserved. In full quantum gravity, the situation is less clear although definite scenarios have been proposed to show how the CPT violation may come about. The issue is important both conceptually and phenomenologically because this violation concerns processes which are purely of quantum gravity origin; it would not be masked, for example, by the predictions of grand unified theories. All this discus-
sion serves only to illustrate the type of physical results expected from quantum gravity. As the history of major advances of this century - e.g. special relativity - suggests so eloquently, the most profound predications of quantum gravity may turn out to be precisely those which could not have been guessed before!

Black Hole Evaporation

*Truth is often stranger than fiction.*

-Dr. Johnson

Let us now turn to concrete achievements.

The most important physical effect unravelled in quantum gravity is undoubtedly the discovery by Stephen Hawking that black holes radiate quantum mechanically. This prediction relies on the so-called external potential approximation. Since much of the work in this area over the past decade also used this approximation, it is worthwhile to first discuss the physical issues involved in its use.

The main idea is to study the effects of a classical background, gravitational field on the propagation of quantum matter fields. If one ignores the gravitational effects, quantum field theory in Minkowski space provides the appropriate framework for describing fields and particles. As a first step to incorporate the gravitational effects, one may replace Minkowski space by a curved space-time and investigate the influence of the classical gravitational field, coded in the curvature of this space-time, on the resulting quantum theory of fields. Thus, in this scheme, the gravitational field is treated as
classical and is kept fixed throughout the calculation; one keeps track of the
effects of this background field on quantized matter, but ignores the back-
reaction of the matter on space-time geometry. In a sense, the spirit is
quite different from the original goal: instead of "quantizing" the gravita-
tional field of general relativity, one "general-relativizes" quantum field
theory. Therefore, one does not expect this scheme to give us a complete
picture of the physical effects associated with quantum gravity. Indeed,
there exist arguments\(^9\) which demonstrate that it would be inconsistent to
consider a semi-classical description (in which gravity is treated classically
and matter quantum mechanically) as the true or the correct theory, even if
one were to take into account the effect of matter back on space-time geo-
metry. However, experience from quantum electrodynamics suggests that the
external potential methods can prove to be a useful approximation to unravel
many physical effects predicted by the full theory; one can, for example,
compute the famous Lamb-shift in this approximation. In the context of
gravitation one can expect the semi-classical approximation to be a useful
tool when the space-time curvature is small compared to the Planck scale but
not so small as to be negligible compared to other parameters in the problem.
Examples of such situations are provided by the gravitational fields in the
early universe (say after Planck time but before the era dominated by grand
unified theories), and around the so-called mini black holes which may have
the mass of a mountain (about a hundred billion kilograms). Much work has
been done in both contexts.

Once one replaces Minkowski space by a curved space-time geometry, a
number of interesting phenomena occur. First of all, the quantum fields are
scattered in a non-trivial way by the background curvature. More importanty,
the vacuum state of quantum fields is no longer stable; physical particles can
be created in pairs *spontaneously*. In the usual Minkowskian quantum field theory, the stability of the vacuum state is ensured by the principle of conservation of energy: Since the vacuum is the only quantum state with zero energy, the field which is initially in the vacuum state must continue to be in that state. When the space-time is curved, however, energy of quantum fields is not necessarily conserved owing to interactions of these fields with curvature, which can affect their evolution without itself being affected. In particular, the background gravitational field can pump energy into the vacuum and create excited states which correspond to particles. Heuristically, this process may be pictured as follows. One can envisage the vacuum state of quantum fields as being filled with a sea of freely falling virtual particle-anti-particle pairs. Due to the tidal forces created by the background gravitational field, a given pair may be torn apart as it falls. If the curvature is strong enough, the pair may be torn apart to such an extent that the gravitational potential energy gained in the process exceeds twice the rest mass of particles associated with the quantum field. If this occurs, it is "energetically favorable for the two virtual particles to become real" and there is a spontaneous creation of pairs. This pictorial description is, of course, very rough. However, it makes the phenomenon plausible and is also useful in making order of magnitude estimates without having to do detailed quantum field theoretic calculations.

Let us now turn our attention to black holes\(^\text{10}\). A black hole is a region of space-time in which the curvature is so strong that not even light can escape from it. From astrophysical considerations, one knows that black holes can form by stellar collapse. However, compared to stars, black holes are exceedingly simple objects. Indeed, a black hole in equilibrium is completely characterized by just three numbers: its mass \(M\), its angular momentum \(J\) and
its charge $Q$. By contrast, to specify the gravitational field of a star in equilibrium, one needs to specify an infinite number of parameters, e.g., all the multipole moments. Furthermore, one has the explicit analytical expression of the most general space-time geometry representing such a black hole. Finally, black holes are subject to very simple laws, listed in Table 3. These laws have a striking similarity to the basic laws of thermodynamics: the surface gravity $g$ plays the role of temperature $T$ and the area $A$ of the black hole plays the role of the entropy $S$. However, in classical relativity, these similarities are only formal. For, the classical geometry of a black hole space-time is such that the black hole can only absorb radiation; it cannot emit anything. In other words, the implication is that a black hole must appear strictly black; it resembles a body with temperature zero, rather than $g$. This, coupled with the first law, implies that the entropy of a black hole should be infinite rather than $A$. Indeed, since entropy $S$ is dimensionless while the area $A$ of the black hole has dimensions of \((\text{length})^2\) and since one cannot construct a quantity with dimensions of length from the only available fundamental constants, $G$ and $C$, it is impossible to even conceive of a meaningful relation between the entropy and the area in the classical theory.

About ten years ago, the Israeli physicist Jacob Bekenstein suggested that the situation would be very different if one brought in quantum mechanics. Using Planck's constant $\hbar$ together with $G$ and $C$, one can form a fundamental length, the Planck length. Using the square of this length in the proportionality factor, one can require that the area $A$ be proportional to the entropy $S$. Bekenstein also pointed out that while it is possible classically to make a black hole with given values of parameters $M$, $J$, and $Q$ in an infinite number of ways — which accounts for the infinite value of entropy predicted classically — the possibilities are severely limited if one uses
matter which is subject to quantum mechanics. While these arguments provided plausibility to the idea that entropy should be proportional to the area of the black hole, the definitive derivation of the relation, \( S = \frac{c^3}{4GhA} \), came from Steven Hawking's detailed calculation in the framework of quantum field theory on a black hole background space-time. This calculation showed that black holes are not black after all. More precisely, one can study the quantum mechanical pair creation process in the gravitational field of a star which collapses to form a black hole. Since the field of the black hole is very special, at late times, the particle emission is especially simple:

particles are emitted to infinity with just the energy distribution that one would expect if the black hole were an ordinary black body with temperature

\[ T = \frac{g^2}{2\pi k} \] (Here \( k \) is Boltzmann's constant.) For astrophysical black holes, this radiation is quite negligible. For example, for a black hole with the mass of our sun, the temperature is about \( 3^\circ K \). However, since the surface gravity is proportional to the inverse of the mass of the black hole, the temperature rises as the black hole loses mass due to particle emission. When the mass goes down to that of a mountain on earth, the temperature would be about a thousand billion degrees and the species of particles that can be emitted at this stage is so great that the black hole could just explode away the rest of its mass. Although these considerations may eventually prove to be of interest to astrophysics and cosmology, the immediate interest stems from conceptual reasons. The black hole evaporation process is a direct consequence of the unification of principles of general relativity and quantum mechanics. What is astonishing is that the ramifications of the result extend to thermodynamics. From the quantum gravity point of view, a black hole is exactly like a black body. Hence, it is meaningful to speak of its temperature and entropy. Finally, the explicit expressions of these quanti-
ties are such that the laws of black hole mechanics fuse with the laws of thermodynamics to provide the laws of black hole thermodynamics. It is, of course, true that the detailed calculations which lead to this result use the external potential approximation. However, in the final result, the principles of general relativity, quantum mechanics and thermodynamics are so delicately interwoven that one is compelled to believe that it embodies an essential feature of full quantum gravity.

Frontiers of Research

And as imagination bodies forth
The form of things unknown, the Poet's pen
Turns them to shapes and gives to airy nothing
A local habitation and a name.

-William Shakespeare

Apart from black holes, perhaps the most important arena for quantum gravity is the early universe. Just after the big bang, the gravitational field was so strong that all physical processes must have been dominated by quantum gravity. A variety of possible effects have been studied during the past decade. The simplest framework is that of the external potential approximation. American physicist Leonard Parker and the Russian Academician Yakov Zeldovich pointed out, already in the late sixties, that the spontaneous creation of particles by the gravitational field must have been very significant in the early stages of the universe. Detailed calculations have been made since then and these show that this process played an important role in
making the universe uniform; the irregularities, if any, are quickly damped out. It was soon realized, however, that the external potential approximation is too crude to get a realistic picture of the situation, because it neglects the gravitational field produced by the created particles themselves. To obtain a more realistic picture, one must take into account the "back reaction" of the particles on the gravitational field. Over the past five years, a number of attempts have been made to develop techniques which can incorporate the back-reaction in a consistent way. The overall progress has been significant. For example, one now knows that the initial irregularities must have been damped out in about a tenth of the Planck time! These time scales are so ridiculously small compared to those we encounter in everyday life that one's first impulse may be to wonder why so much effort is devoted to finding out what happened during these tiny fractions of seconds. One must remember, however, that time scales must be judged relative to processes involved. For quantum gravity processes, an interval of Planck time is large and a lot of activity may take place during this miniscule period. And the key point is that the end result of this activity provides the initial conditions for the era that follows.

In these investigations, the gravitational field was treated classically; only the matter fields were quantized. During the last three years or so, however, the focal point of research is turning steadily away from the semi-classical approximation to quantum gravity proper. These developments touch many areas, not just cosmology. Furthermore, some of the work is quite technical and some of it represents only new directions for future research. We shall, therefore, conclude by giving only a flavor of the type of ideas that are being pursued.
Broadly, the emphasis is on qualitative features of quantum gravity rather than on numerical results. This seems appropriate since, as we saw in the beginning of the article, the numerical corrections due to quantum gravity effects are likely to be very small. The overwhelming concern is, therefore, with conceptual issues. What kind of non-perturbative effects is one likely to encounter? What would be the nature of the object that would replace space-time geometry in quantum gravity? Would one have to modify Einstein's equation even at the classical level before attempting to go to the quantum regime? Is it obvious that principles of quantum mechanics should remain intact in quantum gravity?

Much attention has been given to the nature of the ground state of quantum gravity. In most physical theories, the total energy is defined only up to an additive constant; one is free to choose the zero-level arbitrarily. In general relativity, however, one does not have this freedom because energy itself - rather than just differences in energy - is directly measurable in terms of the curvature of space-time. Recently, positivity of energy has been established in classical general relativity and one knows that the energy can vanish only if there is no curvature anywhere, i.e. if the space-time is flat. One may expect, therefore, that the quantum Hamiltonian operator should be non-negative and its unique eigenvector with zero energy would be a wave function peaked at flat space. That is, one may expect that a perturbative treatment around flat space would lead to a good approximation of the true ground state of quantum gravity. Detailed analysis has shown that this expectation is not likely to be correct: the perturbative ground state is unstable. The calculations involved make certain assumptions and use suitable approximations. So, it may be that the result is only an indication that the methods used are inappropriate. However, this seems to be unlikely because
the same result has been obtained by using quite different methods. Rather, the calculations suggest that non-perturbative effects are likely to be important in quantum gravity. In particular, one might expect the quantum vacuum to have not only curvature or field fluctuations but also the topological fluctuations mentioned earlier. Several independent ways of incorporating these have been suggested, although a definitive picture of the vacuum state is yet to emerge. In one approach, virtual black hole geometries can interpolate between the vacuum state in the past and the one in distant future\textsuperscript{13}. Another, more extensively followed approach is based on the idea of Feynman path integrals\textsuperscript{5}. Here, it is positive definite, Riemannian space-time geometries that play a dominant role. Both approaches lead to certain physical results. For example, they suggest that there cannot low mass, spin zero particles which are truly elementary, i.e. of the size of the Planck length or less; such particles acquire an effective, large mass via interactions with topological fluctuations. Perhaps the most complete description of the vacuum state to date is the one contained in the recent work of the American physicist Andrew Strominger\textsuperscript{6}. He works in the context of the so-called higher derivative theories of gravity in which Einstein's field equations are modified. The resulting picture of the quantum vacuum bears out many of John Wheeler's expectations: topological fluctuations appear and disappear perpetually. Using suitable approximations, Strominger has given quantitative estimates for processes involving topology changes. Furthermore, one can see that these quantum tunnelings can contribute to the values of the parameters associated to elementary particles.

Another development which also fits in Wheeler's general picture comes from quite a different analysis. The issue here is the origin of quantum numbers, such as the spin, associated with elementary particles. Three years
ago, American physicists John Friedman and Rafael Sorkin showed that, if one allows non-trivial spatial topologies, one can construct quantum states of gravity with half-integral angular momentum. This is a remarkable result because the gravitational field is normally thought of as being bosonic; the topological degree of freedom enables one to construct fermions from bosons! Phenomenological implications of this mechanism are, however, still unclear.

Finally, a number of ideas have been proposed to suggest that the space-time geometry itself should be regarded as a secondary object, which is both useful and meaningful at length scales much larger than the Planck length, and that the basic, microscopic variables, which dictate the physics at the Planck scale may be quite different. An outstanding example is the twistor theory. Here the basic variables - twistors - represent energy-momentum and angular momentum structure of elementary particles; space-time geometry arises as a derived, secondary concept. In a second, more recent approach, the starting point is the observation that although the Hamiltonian, which governs time-evolution, in Einstein's theory is quite complicated when expressed using the usual geometrical variables, it is simple when expressed in terms of certain spinorial variables which are non-local functions of the usual ones. One therefore regards the new variables as primary and attempts to describe physics at Planck length and below in terms of them. Another idea goes under the name "induced gravity" where one attempts to build the space-time geometry using fields and particles that feature in gauge theories. Here, Einstein's theory is obtained only in the long wave length limit, after applying certain quantum, radiative corrections. Still another idea is to try to obtain a unified theory of all interactions by enlarging the dimension of space-time. Two types of enlargements are being considered. The first involves attaching to space-time, dimensions which correspond to internal
degrees of freedom of elementary particles; it is modeled on ideas proposed by
German mathematician Theodor Kaluza and Swedish physicist Oskar Klein, over
fifty years ago, to unify gravity and electromagnetism. The extra dimensions
are supposed to have physical existence. However, "the thickness" in those
directions is assumed to be extremely small so that they would not be notice-
able in the everyday phenomena. The other enlargement goes under the name of
"supergravity". Here, the extra symmetries correspond to interchange of
bosons and fermions. The hope is that the imposition of this extra symmetry
would limit the possible interaction of gravity with matter severely, thereby
avoiding the infinities that can otherwise occur in a quantum field theory.

To a layman, many of these ideas may seem far-fetched. But then, many of
the new ideas that are at the base of general relativity as well as quantum
theory also seemed far-fetched when they first appeared. Perhaps the British
biologist J. B. S. Haldane hit the mark when he wrote over half a century
ago: "Now my own suspicion is that the universe is not only queerer than we
suppose, but queerer than we can suppose."
Planck Units

<table>
<thead>
<tr>
<th>Physical Quantity</th>
<th>Planck Unit</th>
<th>Habitual Scales</th>
</tr>
</thead>
<tbody>
<tr>
<td>length</td>
<td>(Gh/c^3) (1/2) (\sim 10^{-33}) cm</td>
<td>Atomic radius (\sim 10^{-8}) cm</td>
</tr>
<tr>
<td>time</td>
<td>(Gh/c^5) (1/2) (\sim 10^{-13}) s</td>
<td>Neutron mean life (\sim 100)s</td>
</tr>
<tr>
<td>density</td>
<td>(c^5/0^2\h) (\sim 10^{46}) g/cm^3</td>
<td>Nuclear density (\sim 10^{12}) g/cm^3</td>
</tr>
</tbody>
</table>

Table 1: Three fundamental constants feature in quantum gravity: Newton's constant of gravity G, velocity of light c, from special relativity, and Planck's constant h, from quantum mechanics. One can construct from these constants unique quantities of dimensions of length, time and density respectively. Their magnitudes represent scales at which quantum gravity effects are expected to be important.

Fundamental Interactions

- Strong
- Electromagnetic
- Weak
- Gravitational

Conserved Quantities

- \(I_3, I, P, C, T, B, Q, \bar{P}, \bar{E}\)
- \(I, P, C, T, B, Q, \bar{P}, \bar{E}\)
- \(C, P, T, B, Q, \bar{P}, \bar{E}\)
- \(Q, \bar{P}, \bar{E}\) only

Table 2: It is an observed fact that weaker the interaction, less is the number of symmetries it respects and of quantities it conserves. The symbols \(I_3, I, P, T, B, Q, \bar{P}, \bar{E}\) and CPT stand, respectively, for the third component of isospin, total isospin, parity, charge conjugation, time-reversal, baryon number, charge, momentum, energy and the composition of charge conjugation, parity and time-reversal.
Laws of Thermodynamics

0th In equilibrium, the temperature $T$ of the system is constant.

1st The change in the internal energy $U$ of a system equals the sum of the heat absorbed and the work done:
$$dU = T \, dS + \text{work terms}$$

2nd The area of a thermally isolated system never decreases:
$$dS \geq 0$$

3rd It is impossible to reduce temperature of a system to absolute zero through a finite number of steps.

Laws of black-hole mechanics

In equilibrium, the surface gravity $g_s$ at horizon of a black-hole is constant.

The change in the mass $M$ of a black hole equals the sum of $(g/8\pi)$ times the change in area and work terms:
$$dM = \left(\frac{g}{8\pi}\right) \, dA + \text{work terms}$$

The area of the black-hole horizon never decreases:
$$dA \geq 0$$

It is impossible to reduce the surface gravity of a black-hole to zero through a finite number of physical processes.

Table 3: Comparison between the laws of thermodynamics and classical black-hole physics.
Fig. 1 - The Cosmic Snake of Glashow. The long part of the tail represents the world of astronomy and cosmology, governed by gravity; the next part stands for the everyday world, dominated by electromagnetism; the last part of the body denotes the world of atomic nuclei and elementary particles, ruled by weak and strong interactions. The head, representing the Planck length and below, belongs to gravity once again; the snake eats its own tail.
References and Notes


15. A. Ashtekar, Phys. Rev. Lett. 46, 573 (1981). Also see Part II of these notes.


PART II

ASYMPTOTIC QUANTIZATION
II.A: INTRODUCTION

Traditional attempts at obtaining a quantum theory of gravity may be divided into two broad classes\(^1,2\): "covariant" approaches and "canonical" methods. In the first approach, one treats gravity in the spirit of other field theories and focuses on scattering processes involving gravitons, while in the second, one emphasizes the geometrical nature of the gravitational field and attempts quantization via Hamiltonian methods. Both avenues have led to a number of insights. However, they also have obvious limitations. In the covariant approach, for example, one begins by splitting the dual role played by the space-time metric \( g_{ab} \) in general relativity: one introduces a background metric \( \eta_{ab} \), usually chosen to be flat, which is to provide the "kinematic arena", and regards \( h_{ab} = g_{ab} - \eta_{ab} \) as the dynamical field. Thus, the geometrical role of \( g_{ab} \) is assigned to \( \eta_{ab} \) while the role of the gravitational potential is now played by \( h_{ab} \). Einstein's equations on \( g_{ab} \) provides a non-linear field equation on \( h_{ab} \). One first linearizes this equation and subjects the linear field to quantization following the usual rules of Minkowskian field theories. The resulting quanta are called gravitons. These are then subject to interactions dictated by the original non-linear equation. While this procedure may seem to be a natural one from a field-theoretic viewpoint, it appears to be rather artificial from the standpoint of general relativity. For, the dual role of the metric is a most essential feature of Einstein's theory and splitting of these roles violates the very spirit of the theory. Thus, it is unsettling to see the fictitious background \( \eta_{ab} \) play a significant role in the resulting theory: notions
such as the microcausality of field operators, asymptotic regions for in and out states, spin and mass of the particles involved, all refer to $\eta_{ab}$.

Indeed, the very name "covariant" refers to the Poincare' covariance with respect to this flat background. Secondly, the fact that an essential use of the linearization procedure is made in the introduction of the basic notion of a graviton is also unappealing\(^3\). Even if such aesthetic considerations are ignored, one is still faced with significant difficulties. These arise from the fact that, since the underlying manifold structure is required to be $\mathbb{R}^4$, one cannot hope to encompass processes involving non-trivial topologies such as the formation and evaporation of a black hole. And, presumably, it is precisely through such qualitatively new processes that quantum gravity will make its impact felt. Indeed, the detailed numerical predictions for scattering processes which have played a key role in the development of other field theories seem, at the moment, uninteresting in the gravitational case, given the weakness of the coupling constant. The issues of immediate interest are, rather, the conceptual ones. And, it appears that covariant approaches avoid these very issues by imitating Minkowskian quantum field theories. Canonical methods, on the other hand, are better equipped to handle these questions. For, the emphasis is now on understanding, already at the classical level, ways in which general relativity differs from other field theories due to the geometrical nature of the gravitational field. Here, one introduces neither a background metric nor a linearization of Einstein's equation.

The first step in the program is to cast exact general relativity in the Hamiltonian form. This was achieved a number of years ago and has shed much light on the geometrical significance of the constraints which must be satisfied by initial data on a space-like Cauchy surface. However, this program also has certain drawbacks. From an aesthetic point of view, the

"3+1 split" work done fixed, although an essentially equipped with difficulties, can become available to perform the quantitative extract properly. It wasn't consistent, and one has to not sufficient to fit them naturally.

The result is that, while as a symplectic structure $a$ background decomposes, the resulting structure is no longer the result of reducing the interest.
"4+1 splitting" of Einstein's theory is somewhat unappealing. Also, in the work done so far, the underlying manifold structure has again been kept fixed, although, in principle, this restriction could be removed, thanks to an essentially exhaustive analysis of topologies admitted by 3-manifolds equipped with asymptotically flat positive definite metrics, which has now become available. The more serious limitation of the program is the practical one. The constraints which arise in the Hamiltonian framework are difficult to manipulate and one is yet to find a large class of solutions to the quantum (operator) constraint equations. Hence it is difficult to extract physical information from the formalism. (In fact, until recently, it wasn't even clear that the operator constraints could be made mutually consistent through a suitable choice of factor ordering.) Thus, overall one has the uneasy feeling that the covariant approaches are pragmatic but not sufficiently deep for the basic problems while the canonical methods are broader in their goal but not sufficiently supple to maneuver. The question naturally arises: can one place oneself "in between" the two schemes?

The approach that we wish to present in Part II aims at this possibility. As in the covariant schemes, the goal is to obtain a superscattering-operator, while as in the canonical methods, the passage to quantum theory is via symplectic techniques. However at no stage in the analysis do we introduce a background metric, a linearization of Einstein's equation or a 3+1 decomposition of space-time. In particular, the underlying manifold structure is left arbitrary to a large extent. In spite of this generality the resulting framework is quite rich in structure: we are able to introduce the Hilbert spaces of asymptotic quantum states, identify physically interesting operators such as energy-momentum and angular momentum, discuss
the particle content of the theory and compute spin and mass of these particles.

To achieve this, one begins with the observation that for the Maxwell field one quantizes only the radiative degrees of freedom. One wishes to do the same in the gravitational case. Fortunately, there is available, since the early sixties, a rich mathematical framework describing gravitational radiation in exact general relativity. The idea is to use this framework as the point of departure for quantum gravity. Thus, it is only the asymptotic structure of the gravitational field at null infinity that enters the discussion directly. This fact enables one to avoid the fixation of the underlying manifold structure, the introduction of the flat background and the 3+1 splitting of Einstein's theory. Finally, the Bondi-Metzner-Sachs (BMS) group at null infinity provides the machinery essential for the introduction of familiar notions in terms of which one can pose questions of physical interest. Thus, for example, gravitons now arise as asymptotic notions in the exact theory rather than as exact notions in the linear theory and their properties such as spin and mass refer to the BMS group at null infinity rather than to the Poincare' group of flat background space-time. More generally, one exploits the fact that, asymptotically, Einstein's equation becomes "almost linear"; it is this simplification that enables a passage to quantum theory via symplectic techniques. Note, however, that these simplifications are not introduced by hand; the geometrical boundary conditions which must be imposed in order that one can meaningfully talk about gravitational radiation in exact general relativity themselves imply that most non-linearities of the exact theory are ironed out asymptotically. Finally, traces of non-linearities do persist even at infinity and these lead to new features in quantum theory.
The basic limitation of the approach arises from the very fact that it places itself "in between" the traditional schemes. Thus, for example, the approach is not likely to yield information on the quantum fluctuations at the Planck length except for the indirect effect of such fluctuations on the superscattering operator. The approach is also unsuitable in the cosmological contexts. Finally, and most importantly, at the present stage we only have a new kinematic framework; very little is known about the quantum scattering operator.

The plan of Part II is as follows. Chapter II.B begins with a brief introduction to the notion of null infinity and shows how one can isolate the true radiative modes of the Maxwell field in Minkowski space-time without recourse to space-time Fourier transforms. We then turn to the gravitational case. We show that the two gravitational degrees are coded in certain equivalence classes of connections at null infinity and summarize the known results about the structure available on the space of these equivalence classes. Chapter II.C addresses the problem of quantization of these radiative modes. Once again, we begin with the Maxwell field in Minkowski space-time and indicate how the quantum theory constructed at null infinity, without any reference to the interior, is, via field equations, isomorphic to the usual theory constructed from space-time fields. What is more, the structure at null infinity leads to a geometric interpretation of the requirement of finiteness of 1-particle norms, which, in turn, brings out the artificiality of the restriction to the Fock representation of photons when charged particles are present. Further, the structure at null infinity suggests a way to construct the Hilbert spaces of asymptotic states appropriate for full quantum electrodynamics which turn out to be precisely those proposed by Kulish and Faddeev to handle the infrared
problems, i.e. to obtain a well-defined S-matrix. We then turn to the gravitational field and carry out asymptotic quantization of the radiative modes. Fock representation of the (asymptotic) canonical commutation relations is easy to obtain. By studying the action of the BMS group on the 1-particle Hilbert space, we show that gravitons have zero rest mass and spin two in a well-defined sense. (Note again that these are the asymptotic particle states of the exact theory, rather than the quantization of the linearized gravitational field off a background flat metric.) We point out the relation between the helicity of 1-particle states, positive and negative frequency decomposition and the algebraic symmetries of the fields representing these states, providing a more complete picture of the situation than is available in the literature. In particular, we show that it is possible to incorporate both helicities using just "self-dual" (or just "anti self-dual") fields if one allows fields with both positive and negative frequencies. This fact plays a significant role in a new approach to canonical quantization based on certain spinorial variables.

We then discuss non-Fock representations which are analogs of the infrared sectors of QED and show that their existence is intertwined with the enlargement of the asymptotic symmetry group from the Poincaré to the BMS.

To conclude this introductory discussion, let me mention two applications of this framework. First, it provides the mathematical setting required in the discussion of whether or not the quantum gravity effects lead to CPT violation. To introduce the parity and time reversal operators, one often begins by introducing these mappings on the Hilbert space of non-interacting, free fields in Minkowski space, exploiting the simplicity of the space-time geometry. One then sets up a heuristic correspondence between the scattering states of the exact, physical theory and the states of the linearized theory, and, using it, transports the required operators to the scattering states of the exact theory. In the asymptotic quantization framework, these operators can be defined directly on the asymptotic
states at null infinity without reference to the space-time interior and the heuristic elements can be avoided. The second application involves topological fluctuations. By superposing all space-times—weighted by the exponential of i times the action—which induce vacuum configurations both at past and future null infinity, one can construct a semi-classical picture of the quantum vacuum. Now, since the framework allows arbitrary topologies in the interior, paths (i.e., classical 4-geometries) representing black holes can and do contribute significantly. One can investigate the effect of these contributions on the propagation of matter fields. One can conclude, for example, that there are no "elementary" low mass scalar particles; such particles acquire a large effective mass by interaction with virtual black-hole geometries. We will not discuss these applications in detail because they require a host of new technical ideas—the structure at the point 19 at spatial infinity for the discussion of P and T operators and details of the action of black-hole space-times and of certain symmetry breaking mechanisms for the discussion of topological fluctuations—which were beyond the scope of these lectures.
II.B: Radiative Modes of the Gravitational Field in Exact General Relativity

1. Maxwell fields in Minkowski space.

Let $\hat{\mathbb{M}} = \mathbb{R}^4$, $\hat{g}_{ab}$ denote Minkowski space-time. Then there exist coordinates $t, r, \theta, \phi$ in terms of which the metric takes the form:

$$d\hat{s}^2 = \hat{g}_{ab}dx^adx^b = -dt^2 + dr^2 + r^2( d\theta)^2 + \sin^2\theta (d\phi)^2$$  (B.1)

where $\theta$ and $\phi$ are the 2-sphere coordinates, $t$ takes values in $]-\infty, \infty[$ and $r$, in $[0, \infty[$. Set $u = t - r$. The surfaces $u = \text{const.}$ represent future null cones emanating from the central time line, $r = 0$. Zero rest mass fields, such as the Maxwell, propagate along such null cones. The structure of these fields is complicated in the interior - i.e. in the "source region" - and simplifies as $r$ goes to infinity along a $u = \text{const.}$ cone. Hence, to study (outgoing) radiation, it is convenient to attach a boundary, $\mathcal{I}^+$, to the physical space-time which corresponds to the limiting points "$r = \infty$; $u$, $\theta$, $\phi$, = const." Each future directed null geodesic would have an end point on $\mathcal{I}^+$; instead of taking limits $r \rightarrow \infty$ of various quantities and worrying about how the operations of taking limits and computing derivatives interact, one would just do local differential geometry near $\mathcal{I}^+$.

To carry out this program, let us introduce a new coordinate $\Omega = 1/r$ in the region $r > a$ for some positive constant $a$. Let $M$ be the manifold with boundary whose interior is $\hat{\mathbb{M}}$ and whose boundary, $\mathcal{I}^+$, is defined by $\Omega = 0$. Thus, $\mathcal{I}^+$ is topologically $S^2 \times \mathbb{R}$ with a global chart $u \in ]-\infty, \infty[$, $\theta, \phi$. In the $u, \Omega, \theta, \phi$ chart, the metric $\hat{g}_{ab}$ takes the form:

$$d\hat{s}^2 = \hat{g}_{ab}dx^adx^b = -(du)^2 + 2\Omega^{-2}du d\Omega + \Omega^{-2}((d\theta)^2 + \sin^2\theta (d\phi)^2)$$  (B.2)
Thus, while $\hat{g}_{ab}$ is well-defined in the interior $\hat{M}$ of $M$, it is ill-behaved on the boundary $\partial \hat{M}$ (where $\Omega = 0$). However, we can conformally rescale $\hat{g}_{ab}$,

$$
\text{d}s^2 \equiv g_{ab} \text{d}x^a \text{d}x^b : = \Omega^2 \hat{g}_{ab} \text{d}x^a \text{d}x^b = -\Omega^2 (\text{d}u)^2 + 2 \text{d}u \text{d}v + (\text{d}\theta)^2 + \sin^2 \theta (\text{d}\phi)^2
$$

(B.3)

to obtain a metric $g_{ab}$ which is well-behaved everywhere in the region covered by the chart $\Omega, u, \theta, \phi$ (i.e., in the region $r > a$). By smoothly extending $\Omega$ to a positive function in $r < a$, we obtain a smooth metric $g_{ab}$ on all of $M$, conformally related to $\hat{g}_{ab}$ on $\hat{M}$. Note that even though $\Omega$ vanishes at $I^+$, $g_{ab}$ is non-degenerate, (with signature $-+++$) throughout $M$. Thus, by conformally rescaling the physical metric $\hat{g}_{ab}$, we have brought its null infinity to a "finite region of $M$".

If one is interested in the incoming radiation, one can use the advanced null coordinate, $\nu = t + r$, in this construction in place of $u$ and obtain a past boundary $I^-$; every past-directed null geodesic of $\hat{M}$ would have an end point on $I^-$. (One can simultaneously attach $I^+$ and $I^-$ to $\hat{M}$. Then $I^+ \cup I^-$ appears as the null cone of the point $i^-$ at spatial infinity and the future (past) vertex of $I^+ (I^-)$ represents the future (past) time-like infinity. See Fig. 1. 2.15 From now on we shall use the symbol $I$ to denote $I^+$ or $I^-$ obtained via the above construction.

Let us examine the intrinsic structure of $I$. It is a 3-manifold with topology $S^2 \times \mathbb{R}$. If one moves within $I$, only $u, \theta, \phi$ change. Hence, by (B.3), the intrinsic metric $q_{ab}$ on $I$ is given by

$$
\text{d}s^2 \equiv q_{ab} \text{d}x^a \text{d}x^b = (\text{d}u)^2 + \sin^2 \theta (\text{d}\phi)^2.
$$

(B.4)

Thus, $q_{ab}$ is degenerate, with signature $(0,++)$, and is precisely the metric of a unit 2-sphere. It is called the pull-back of $g_{ab}$ to $I$ (whence the arrow in $\text{d}s^2$).
Let us now consider source-free Maxwell equations on \((\mathcal{M}, \hat{g}_{ab})\):
\[
\hat{F}_{[a} \hat{F}_{bc]} = 0 \quad \text{and} \quad \hat{\nabla}_a \hat{F}_{ab} = 0 \quad \text{(B.5)}
\]
It is easy to verify that \(F_{ab} := \hat{F}_{ab}\) satisfies Maxwell's equations
\[
\nabla[a \ n_{bc}] = 0 \quad \text{and} \quad \nabla_a \ n_{ab} = 0 \quad \text{(B.5')}
\]
(where indices are raised and lowered using \(g_{ab}\)) iff \(\hat{F}_{ab}\) satisfies (B.5).

Let us therefore study solutions to Maxwell's equations on \((\mathcal{M}, g_{ab})\). It is easy to verify that a smooth solution \(F_{ab}\) (i.e. one whose components in \(u, \Omega, \theta, \phi\) chart are smooth) on \(\mathcal{M}\) corresponds to a solution \(\hat{F}_{ab}\) on \(\mathcal{M}\) whose components in the \(u, r, \theta, \phi\) chart fall-off at the physically expected rates as \(r\) tends to infinity. (This is known as the Peeling property,\textsuperscript{14,16}) Let us therefore restrict ourselves to such solutions. Then \(I\) serves as the characteristic surface in \(\mathcal{M}\) on which one can specify initial data. More precisely, we have the following. Set:
\[
\begin{align*}
\hat{\tau}^a &= g^{ab} \nabla_b \hat{\Omega} \quad \text{on } \mathcal{M}, \text{ and} \\
\tau^a &= F_{ab} h^b \quad \text{at } I.
\end{align*}
\]

Then, the pull-back, \(\tau_a\), to \(I\) of \(\tau_a\) is the characteristic data which determines the solution \(F_{ab}\) uniquely. In the component language, \(\tau_a\) is completely characterized by the complex scalar
\[
\Phi^a := F_{ab} m^a n^b = \tau_a m^a \quad \text{(B.8)}
\]
on \(I\), where \(m^a\) is the complex vector
\[
m^a = \frac{2}{r} + i (\sin \theta)^{-1} \frac{\partial}{\partial \theta}
\]
(B.9)

Thus, each well-behaved solution of Maxwell's equation in Minkowski space \((\mathcal{M}, g_{ab})\) is completely characterized by the 2 independent components\textsuperscript{17} of \(\tau_a\) (or real and imaginary parts of \(\Phi^a\)) on \(I\). These are the two radiative modes of the Maxwell field. (In particular, the flux of energy across any "patch" \(\Lambda\) of \(I\) is given just by the integral
\[
\int_\Lambda |F_{ab}|^2 \sin \theta d\theta d\phi = \int_\Lambda |\Phi^a|^2 \sin \theta d\theta d\phi.
\]

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Let us now consider the regularity conditions on the characteristic data \( f_3 \) or \( \theta^2 \). In the completion discussed above analytically, (with \( \Omega = 1/r \), see Fig. 1), let \( \theta^2 \) be a smooth complex-valued function of \((u, \theta, \phi)\) on \( I \), which falls off as \( 1/u^{3+\epsilon} \) for some \( \epsilon > 0 \) as \( u \to \infty \). Then, one can use this \( \theta^2 \) as the characteristic data and obtain a globally regular, source-free solution to Maxwell's equations via Kirchoff integrals. Denote by \( \hat{F}_{ab} \) the restriction of this solution to the physical (i.e., Minkowski) space \( \mathbb{R} \). One can show that \( \hat{F}_{ab} \) has finite energy, momentum and angular momentum and "peels-off" properly. However, if we now consider the embedding of \( \mathbb{R} \) in the Einstein cylinder \( \mathbb{R} \) (Fig. 2), one finds a peculiar result: If \( \int_{\mathbb{R}} f_3^2(u, \theta, \phi) du = \mathcal{Q}(\theta, \phi) \) is not zero on \( I \), the (appropriately rescaled version of) the field \( F_{ab} \) diverges as one approaches the point \( i^+ \) at spatial infinity, although it admits smooth limits to \( I^+, I^-, i^+ \) and \( i^- \). In the classical theory, \( \mathcal{Q}(\theta, \phi) \) has no direct physical significance and there is no a priori reason for it to vanish. (In fact, if we permitted sources and considered retarded fields, \( \mathcal{Q}(\theta, \phi) \) would vanish on \( I^+ \) only in the physically unrealistic case in which the scattering of charges is trivial. This point will be discussed in Chapter II.C.) We shall see later that non-vanishing of \( \mathcal{Q}(\theta, \phi) \) is intimately related to the existence of the infrared sectors in the quantum theory. Indeed, the introduction of \( \mathcal{Q}(\theta, \phi) \) and the analysis of the global behavior of solutions on the Einstein cylinder with \( \mathcal{Q}(\theta, \phi) \) non-zero was carried out in the classical theory after one realized the significance of this quantity in the quantum theory.

To summarize, the notion of null infinity serves two purposes. First, it provides a natural arena for specifying incoming and outgoing states of zero rest mass fields in scattering theory. Second, the structure of these fields simplifies at null infinity so that one can simply read off the two radiative
modes from the values of the asymptotic fields without having to solve any
elliptic equations or fix gauge.

2. The Gravitational Field: Kinematic Structure at Infinity.

Let us now generalize the notion of null infinity to curved space-times.
The idea is to simply define a space-time to be asymptotically flat at null
infinity if it admits a conformal completion in which the boundary resembles
Minkowskian $I^\pm$. The key question is: how much of the structure available
in the case of Minkowski space can we carry over? And it is here that a
balance has to be struck: if we require too much, the definition may not be
satisfied by interesting examples, while if we require too little, we may
not have a sufficiently rich structure to introduce physically interesting
quantities or to establish that they have reasonable properties. This
problem is substantially more difficult in general relativity than in other
field theories because the dynamical field on which we wish to impose
suitable boundary conditions itself determines the space-time geometry which
one normally uses to specify the fall-off conditions. The problem has,
however, received substantial attention since the early sixties in the
context of the gravitational radiation theory\textsuperscript{20} and, although some questions
still remain unresolved, the overall picture is by now well established.\textsuperscript{21}

Let $\hat{M}$ be $C^\infty$ manifold equipped with a $C^\infty$ metric $\hat{g}_{ab}$ of signature (-+++). The pair $(\hat{M}, \hat{g}_{ab})$ will be referred to as a space-time. One can think of it
as representing the gravitational field of an isolated body which may emit
gravitational waves, or, of a pure gravitational radiation field itself.
Definition 1: \( \hat{\mathcal{M}}, \hat{\mathcal{g}}_{ab} \) is said to be asymptotically flat at null infinity if there exists a manifold \( \mathcal{M} \) with boundary \( \partial \mathcal{M} = \mathcal{I} \) equipped with a smooth metric \( \mathcal{g}_{ab} \) of signature \(-+++\) such that the interior, \( \mathcal{M} - \mathcal{I} \), of \( \mathcal{M} \) is diffeomorphic to \( \mathcal{N} \); and,

i) There exists a smooth function \( \Omega \) on \( \mathcal{M} \) with \( \Omega = 0 \) on \( \mathcal{I} \), \( \nabla_a \Omega \neq 0 \) on \( \mathcal{I} \), and \( \hat{g}_{ab} = \Omega^2 g_{ab} \) on \( \mathcal{N} \);

ii) \( \hat{g}_{ab} \) satisfies the vacuum Einstein's equation, \( \hat{R}_{ab} = 0 \), in the intersection with \( \mathcal{N} \) of a neighborhood of \( \mathcal{I} \) in \( \mathcal{M} \); and,

iii) \( \mathcal{I} \) is topologically \( S^2 \times \mathbb{R} \), the vector field \( n^a := g^{ab} n_b \Omega \) on \( \mathcal{I} \) is complete and the space of its orbits is diffeomorphic to \( S^2 \).

Remarks: i) Although we want physically reasonable space-times to be asymptotically flat both at future and past null infinity (as well as at spatial infinity \( \mathcal{I}^0 \)), to simplify notation, in this section we shall let \( \mathcal{I} \) stand for either future or past null infinity.

ii) The role of the various conditions in the definition is as follows. The first condition ensures that \( \Omega \) falls off as \( 1/r \) and that the boundary \( \mathcal{I} \) representing null infinity of the physical space-time is brought to a finite distance by a conformal rescaling of the physical metric. Note that, although \( g_{ab} \) and \( \hat{g}_{ab} \) belong to the same conformal equivalence class on \( \mathcal{N} \), they fail to do so on the boundary \( \mathcal{I} \); at the boundary, \( \Omega \) vanishes and \( \hat{g}_{ab} \) is not even well-defined. The second condition merely ensures that we do not have non-gravitational radiation fields at infinity. This condition can be easily relaxed. The essential results will continue to hold if the tensor field \( \Omega^{-1} \hat{R}_{ab} \) admits smooth limits to \( \mathcal{I} \) (i.e., if the components of this field in a chart which is well defined at \( \mathcal{I} \) admit smooth limits as \( \Omega \) tends to zero.) But in this case, it would be natural to study the radiative modes of both the gravitational and the matter (e.g. Maxwell, Yang--
Mills, ...) fields. The validity of the asymptotic field equations and the smoothness of $\varepsilon_{ab}$ at $I$ (actually, it suffices that $\varepsilon_{ab}$ be $C^1$) implies that $I$ is a null 3-surface, its normal $n^a := \varepsilon_{ab} V^b_\phi$ is tangential to it. The last condition in the definition specifies the topology of the boundary and ensures that we have "all of null infinity" and not just a piece of it.\textsuperscript{22}

iii) If we allow the presence of a cosmological constant, $\Lambda$, in Einstein's field equations, the situation changes quite drastically, both with respect to the kinematical structure of $I$ which is being discussed in this section and the radiative modes at $I$ to be discussed in the next section. Detailed analysis incorporating a non-vanishing $\Lambda$ has become available only recently.\textsuperscript{23} In these lecture notes, we shall assume that $\Lambda$ is zero.

Note that, given the physical metric $\tilde{g}_{ab}$, there is a certain conformal freedom in the choice of $\varepsilon_{ab}$, i.e., in the choice of $Q$: given a completion $(M, \tilde{g}_{ab})$ of $(\tilde{M}, \tilde{g}_{ab})$ satisfying the requirements of Definition 1, $(\tilde{M}, \tilde{g}_{ab})$ also satisfies these requirements if $\tilde{g}_{ab} = \omega^2 g_{ab}$ for some nowhere vanishing, smooth function $\omega$ on $M$. Using this freedom, one can always choose a "conformal frame" - i.e. a conformal factor - such that the rescaled metric satisfies Definition 1, and, such that $I$ is divergence-free, i.e., $V^a_{\phi}V^a_\phi = V^a_\phi n^a = 0$ on $I$. Using (asymptotic) field equations, it then follows that $n^a$ is in fact covariantly constant at $I$, i.e. $V^a_\phi n^a = 0$ on $I$. From now on we shall restrict ourselves to such conformal completions. What is the restricted conformal freedom? If $(M, \varepsilon_{ab})$ satisfies the definition and makes $I$ divergence-free, so does $(M, \tilde{g}_{ab} = \omega^2 g_{ab})$ iff $L_\omega = n^a V^a_\omega$ vanishes on $I$. ($L_\omega$ stands for "Lie derivative w.r.t. $n^a$. In the $u, \theta, \phi$ chart on $I$, the condition is satisfied iff $\omega$ is a function of $\theta$ and $\phi$ only.)

The integral curves of the null vector field $n^a$ on $I$ are called gener-
ators of $I$. The space of generators, $S$, is diffeomorphic to a 2-sphere. Let $g_{ab}$ denote the pull-back to $I$ of the metric $g_{ab}$. Since $I$ is a null 3-surface with normal $n^a$, $g_{ab}v^b = 0$ for some vector field $v^a$ tangential to $I$ iff $v^b$ is proportional to $n^b$. Further, since $v_ah^b = 0$ on $I$, $L^n g_{ab}$ and $L^n g_{ab}$ both vanish on $I$. That is, $g_{ab}$ can be projected down unambiguously to obtain a metric $q_{ab}$ of signature $(++)$ on $S$. In other words, $q_{ab}$, the degenerate metric on $I$ (signature (0++)) is the pull-back to $I$ of $g_{ab}$. Now, there is a general result that any (globally regular, positive definite) metric on a manifold which is a topological 2-sphere is conformally related to the standard, unit 2-sphere metric; the conformal structure of $S^2$ is unique.

Thus, we have the following structure. $I$ is topologically $S^2 \times \mathbb{R}$ and ruled by the integral curves of its null normal, $n^a$. The intrinsic metric $g_{ab}$ on $I$ is degenerate (signature 0++) and is the pull-back to $I$ of a metric $q_{ab}$ on the space $S$ of integral curves of $n^a$ which is conformally related to the standard unit 2-sphere metric. Finally, the permissible conformal rescalings are $Q \to \omega Q$ where $\omega$ is a nowhere vanishing, smooth function on $M$, satisfying $L_n \omega = 0$ on $I$. Under these rescalings, we have, on $I$, $g_{ab} \to \omega^2 g_{ab}$ and $n^a \to \omega^{-1} n^a$. The ruling of $I$ by its null generators and the collection of pairs $(q_{ab}, n^a)$, in which any two pairs are related by a $\omega$ as above, is called the universal structure of $I$. It is the structure common to the null infinity of all space-times satisfying the definition and captures the essence of the boundary conditions imposed by the definition.

The universal structure thus represents the "kinematic arena", which is immune to dynamical, radiation fields, and on which these fields, their inter-relation and equations can be specified. The asymptotic symmetry group at null infinity is therefore the group that preserves this universal structure. Let us work at the infinitesimal, or, Lie algebra level. A
vector field $v^a$ will be called (an infinitesimal, asymptotic) symmetry if the
diffeomorphism it generates leaves the integral curves of $n^a$ (i.e., the
"ruling of $J"$) invariant and maps a pair $(q_{ab}, n^a)$ in our collection to an
equivalent pair $(\omega^2 q_{ab}, \omega^{-1} n^a)$, where $L^n = 0$ on $J$. In terms of Lie
derivatives, these conditions become:

$$L_q n^a = -\alpha n^a,$$

and

$$L_q q_{ab} = 2\alpha q_{ab}$$

(B.10)

for some smooth function $\alpha$ (which may admit zeros) satisfying $L_q n = 0$. It is
trivial to check that if $v^a$ and $w^a$ are symmetries, $\mu v^a + \alpha w^a$ as well as
$[v, w]^a$ are also symmetries (i.e. satisfy (B.10)), where $\mu$ and $\alpha$ are any real
numbers and where $[,]$ stands for the Lie bracket. Thus, the symmetries form
a Lie algebra. Denote it by $B$ (for Bondi, who was the first to investigate
these issues).

Let us now analyze the structure of $B$. Consider vector fields $v^a$
of the form $v^a = \delta n^a$. Using the second equation in (B.10) and the fact
that $q_{ab} n^b = 0$ and $v_{a n^b} = 0$ on $J$ it follows that

$$L_v q_{ab} = 0.$$

Thus, $v^a$ can be a symmetry iff

$$L_v n^a = 0.$$

A direct computation yields

$$L_v n^a = -(L_{\delta n^a}) n^a$$

whence $v^a = \delta n^a$ can be a symmetry iff $L_{\delta n^a} = 0$; in the $u, \theta, \phi$
chart
on $J$, $B = \delta (\theta, \phi)$. Such symmetries are called supertranslations. Clearly,
they form a vector sub-space of $B$ which we denote by $ST$. Now, given
any symmetry $v^a$ and a supertranslation $\delta n^a$, we have

$$[v, \delta n^a] = (L_v \delta n^a) - \alpha \delta n^a.$$

Since we know that $B$ is closed under Lie bracket, it follows that $[v, \delta n^a]$ is
in $B$, and hence in $ST$. Thus, $ST$ is in fact a Lie-ideal of $B$. Let us:
consider the quotient, \( B/ST \). Given any \( v^a \) in \( B \), because of the first equation in (B.10), it projects down to the 2-sphere \( S \), the space of generators of \( I \). Denote the projection by \( v^a \). The second equation in (B.10) implies that \( v^a \) is a conformal killing field of \( q_{ab} \) on \( S \). Now, it is clear that each equivalence class, \( \{ v^a \} \), of symmetries — where two symmetries are considered as equivalent iff they differ by a supertranslation — is completely characterized by the projection \( v^a \) to \( S \) of any \( v^a \) in that equivalence class. (Supertranslations themselves project down to the zero vector field on \( S \).) Hence \( B/ST \) is the Lie algebra of conformal killing fields on \((S, q_{ab})\). Since a 2-sphere admits a unique conformal structure, this Lie algebra is unique, independent of the specific choice of \( q_{ab} \) on \( S \); it is the Lie algebra of the Lorentz group \((SO(3,1))\). Thus, the Lie algebra \( B \) admits (an infinite dimensional, Abelian) ideal \( ST \) and the quotient \( B/ST \) is the Lie algebra of the Lorentz group.

Let us ignore, for the moment, the problems associated with the infinite dimensionality of \( B \). Then, at the level of finite transformations, we have the following results. The group \( B \) of asymptotic symmetries at null infinity is a semi-direct product of the Lorentz group with an infinite dimensional, Abelian, normal group \( ST \) of supertranslations. \( ST \) is naturally isomorphic to the additive group of functions on a 2-sphere and the Lorentz group acts on \( ST \) via its action as the conformal group of the 2-sphere. Finally, it is known\(^20\) that \( B \) admits a unique 4-dimensional normal subgroup \( T \), called the subgroup of translations. \(( T \subset ST)\). \( B \) contains many Poincare’ subgroups (related to one another by elements of \( ST/T \)). However, with the boundary conditions under consideration, it is not possible to single out a preferred Poincare’ subgroup. Thus, in presence of gravitational radiation, although one expects the energy-momentum to be well-defined (since it
refers to $D$, one expects the notion of angular momentum (which normally refers to the Lorentz sub-groups of a given Poincare' group) to become ambiguous. The fact that the asymptotic symmetry group turned out to be the infinite dimensional group $E$ rather than the Poincare' group (the "symmetry group of the ground state") came as a surprise in the sixties. We shall see that the enlargement reflects the long-range, or, the infrared behavior of the quantum gravitational field. $E$ is called the Bondi-Metzner-Sachs Group.\textsuperscript{20}

To conclude this section, let us summarize the kinematical structure in terms of the physical space-time. Conditions of Definition 1 can be recast in terms of the components of the physical metric in a suitable asymptotic chart $u, r, \theta, \phi$: \( \hat{g}_{ab} \) is required to have the form

\[
d\hat{s}^2 = -2drdu + r^2((d\theta)^2 + \sin^2(\phi)(d\phi)^2)
+ \text{terms } O(r) \text{ in } (d\theta)^2, \, d\theta d\phi, \, (d\phi)^2
+ \text{terms } O(1) \text{ in } (du)^2, \, dud\theta, \, dud\phi
+ \text{terms which fall off as } r^{-n} \text{ (for some } n > 0). \quad (B.1)
\]

Exact symmetries of \( \hat{g}_{ab} \) are of course its isometries, whose number depends sensitively on its detailed structure. The asymptotic symmetries, on the other hand, are "asymptotic isometries", i.e., diffeomorphisms whose action preserves the asymptotic form of \( \hat{g}_{ab} \) (i.e. boundary conditions). Let $A$ be the Lie algebra of vector fields which generate such diffeomorphisms and $A'$ its Lie ideal consisting of vector fields whose action leaves the leading order term in the asymptotic metric (B.11) untouched. Then, $B = A/A'$. (For details see Ref. 24).
3. The Gravitational Field: Radiative Modes

Fix a space-time \((\hat{N}, \hat{g}_{\alpha\beta})\) satisfying Definition 1 and a conformal completion \((M, g_{\alpha\beta})\) thereof in which \(I\) is divergence-free (i.e. in which \(\nabla_{\alpha}n^{\alpha} = 0\) and hence, also \(\nabla_{\alpha}n_{\beta} = 0\) on \(I\)). The metric \(g_{\alpha\beta}\) defines various geometrical fields. We shall first focus on fields which are directly relevant to the gravitational radiation theory and investigate the structure that they induce on \(I\). In a second step, we shall see that this structure is delicately interwoven just in a way that makes the asymptotic quantization scheme feasible: one can first isolate the radiative modes of the non-linear gravitational field and using them introduce, intrinsically on \(I\), the entire structure necessary in the classical and quantum analysis of radiative aspects, without having to refer to the (completed) space-time at any intermediate stage.\(^{25}\) That such a simplification should occur is by no means clear a priori.

The "leading order" fields, introduced on \(I\), by \(\hat{g}_{\alpha\beta}\) are simply the intrinsic metric \(q_{\alpha\beta}\) and the null normal \(n^\alpha\):

\[
q_{\alpha\beta} = \hat{g}_{\alpha\beta} \quad \text{and} \quad n^\alpha = g^{\alpha\beta} v_\beta \Omega \quad (B.12)
\]

where, from now on, \(\equiv\) will stand for "equals, at points of \(I\) to". As we saw in Section 2, this leading order structure has no dynamical content; it is available at null infinity of any space-time which satisfies Definition 1. Characteristics of individual space-times begin to appear in the "second order" structure. This is coded in a connection \(\nabla\) defined intrinsically on \(I\). Let \(V_\alpha\) be any covector field in a neighborhood of \(I\) in \(M\). Set

\[
\nabla_\alpha V_\beta = \hat{V}_\alpha V_\beta \quad (B.13)
\]

Using the fact that \(\Omega \equiv 0\), \(\nabla_\alpha \equiv 0\) and \(\nabla_{\alpha}n_{\beta} \equiv 0\), it is easy to show that

\[
\nabla_{\alpha} V_\beta = \nabla_{\alpha} V_\beta \quad \text{if} \quad V_\alpha = \hat{V}_\alpha.
\]

Since every covector field on \(I\) is a
pull-back of some covector field on M. D is well-defined on all covector fields on I. Hence, using the properties of torsion-free connections (linearity, satisfaction of the Leibnitz rule, equality with the gradient operator on functions and vanishing of torsion) one can extend the action of D to all tensor fields on I. The resulting connection has the following interesting properties:

\[ D_a q_{bc} = 0, \quad \text{and} \quad D_a n^b = 0, \quad \text{and} \]  \hspace{1cm} \text{Equation (B.14)}

\[ D_a V_b = D_l (a V_b) + L_V q_{ab} \quad \text{if} \quad V_a n^a = 0. \]  \hspace{1cm} \text{Equation (B.15)}

In the last equation, the vector field \( V^a \) is defined by \( V^a = q^{ab} V_b \) where \( q^{ab} \) is any "inverse" of \( q_{ab} \), i.e. any second rank, symmetric tensor field on I satisfying

\[ q^{mn} q_{am} q_{bn} = q_{ab} \]  \hspace{1cm} \text{Equation (B.16)}

(Note that \( q^{ab} \) is not unique; one can add to it terms of the type \( n^a w_a \) for any vector field \( w^b \) on I. However, since \( V_a \) satisfies \( V_a n^a = 0 \), \( L_V q_{ab} \) is defined unambiguously.) \( (B.14) \) says that the induced connection on I is compatible with the "kinematical structure" thereon. \( (B.15) \) brings out the fact that the class of connections thus available on I is severely limited: since all these connections have the same action on co-vectors orthogonal to \( n^a \), the action of \( D_a \) on a covector field \( l_a \) on I, satisfying \( n^a l_a = 1 \), completely determines the connection.

The "third order" structure is obtained by pulling back to I the curvature tensor of \( g_{ab} \). Now, there is a general result \(14,20 \) that the Weyl tensor, \( C_{abcd} \), of of \( g_{ab} \) vanishes on I as a consequence of the conditions in Definition 1. Thus, only the Ricci part matters. It is convenient to consider the restriction to I of the combination

\[ S^a_{\ b} := R^a_{\ bc} - \frac{1}{6} R \delta^a_{\ b} . \]  \hspace{1cm} \text{Equation (B.17)}
The pull-back of $s_{ab}$ to $\mathcal{I}$,

$$s_{ab} := s_{\mathcal{I}}^{ab},$$

contains information about the flux of gravitational radiation across $\mathcal{I}$.

However to extract this information, one has to remove from $s_{ab}$ a certain piece which is "pure gauge" (in the sense specified below.) This is accomplished as follows. First, one can show\(^{16}\) that the kinematic structure of $\mathcal{I}$ enables the introduction of a second-rank tensor field $\rho_{ab}$: There is a unique $\rho_{ab}$ on $\mathcal{I}$ satisfying

$$\rho_{ab} = R_{(ab)} , \quad \rho_{ab} n^b = 0 , \quad D_{[a} ^b b]c = 0 ,$$

where $R$ is the (pull-back to $\mathcal{I}$ of the) scalar curvature of the metric $\sigma_{ab}$ on the 2-sphere $S$ of generators of $\mathcal{I}$. (If the conformal factor is so chosen that $\sigma_{ab}$ on $S$ is the unit, 2-sphere metric, $\rho_{ab}$ turns out to be equal to $\rho_{ab}$. However, under conformal rescalings, $\rho_{ab}$ has a complicated behavior.)

Set

$$N_{ab} = s_{ab} - \rho_{ab}.$$  

(\ref{Nab})

Then, one can show that

$$N_{ab} n^b = 0 \quad \text{and} \quad N_{ab} q^{ab} = 0.$$  

(\ref{Nab2})

The field $N_{ab}$ is called the Bondi news tensor. It is gauge invariant and its square, $q^{ac} q^{bd} N_{ab} N_{cd}$, defines the local flux density of energy carried away by gravitational waves.

The fourth - and, for our purpose, the last - order structure at $\mathcal{I}$ is provided by Weyl curvature of $s_{ab}$. Since the Weyl tensor, $\omega_{abcd}$, vanishes at $\mathcal{I}$,

$$K_{abcd} := \mathcal{R}^{-1} \omega_{abcd}$$

(\ref{Kabcd})
admits a smooth limit there. Set

$$K^c := K_{abcd} n^n_d$$

and

$$\kappa^c := \kappa_{abcd} n^n_d.$$ (B.23)

The fields $K^{ab}$ and $\kappa^{ab}$ are tangential to $I$ and are trace-free:

$$K^{ab}_{ab} = 0, \quad \kappa^{ab} n^a_{ab} = 0.$$ (B.24)

They can be thought of as the "electric" and the "magnetic" parts of the asymptotic Weyl curvature. If $I$ were space-like or time-like, one could have recovered the full $g^{abcd}$ from these two parts. However, since $I$ is null, this is not possible: Although $K^{ab}$ and $\kappa^{ab}$ each has five independent components, they are not all linearly independent. (In the Newman-Penrose notation, $K^{ab}$ has the same information as $\psi_1^2$, $\psi_2$ and $\text{Re}\psi_2$ while $\kappa^{ab}$ has the same information as $\psi_0$, $\psi_1$, and $\text{Im}\psi_2$.) The information in $K^{ab}$ is "purely radiative"; for instance $\kappa^{ab}$ vanishes identically in any stationary space-time. $K^{ab}$, on the other hand, has some information also about the "longitudinal part" of the gravitational field (through $\text{Re}\psi_2$) which features in the expression of the Bondi 4-momentum, the measure of the total 4-momentum of the isolated system at a retarded instant of time.

Field equations satisfied by $g^{ab}$ and the Bianchi identities imply that the various fields introduced on $I$ above have additional algebraic symmetries as well as inter-relations:

1. $$2D[aP_b]k_c = R_{abc}^d n^c_d \equiv (q_c [a s_b]^d + s_c [a s_b]^d) k_d$$ (B.26)
2. $$s_a b n^a \equiv c_n b,$$ (B.27)
3. $$s_{ab} n^a \equiv s(ab), \quad s_{ab} n^b \equiv 0, \quad s_{ab} n^b \equiv R$$ (B.28)
4. $$D[a s_b]^d \equiv 0, \quad s_{bc} \kappa^{bc}$$ (B.29)
5. $$D_{a} K^{ab} \equiv 0, \quad D_{a} \kappa^{mc} \equiv 0$$ (B.30)
Where \( c \) is a function on \( I \). Finally, recall that the kinematic structure 
\((q_{ab}, n^a)\) on \( I \) is not unique: we still have the restricted conformal freedom

\[
q_{ab} \rightarrow q'_{ab} = \omega^a q_{ab} \quad \text{and} \quad n^a \rightarrow n'^a = \omega^{-1} n^a
\]

with \( L_n \omega = 0 \) \hspace{1cm} (B.31)

Under these rescalings, the fields at \( I \) transform as follows:

\[
D_a k_b = D_a k_b - 2\omega^{-1} (k_a D_b) \omega + \omega^{-1} (\omega^m k_m) q_{ab}
\]

(B.32)

\[
s_a^b = \omega^{-1} s_a^b - 2\omega^{-2} D_a q_b + 4\omega^{-1} \omega^b D_b \omega - \omega^{-2} (\omega^m D_m) s_a^b
\]

(B.33)

\[
s'_{ab} = s_{ab} - 2\omega^{-1} D_a q_b + 4\omega^{-2} D_a q_b - \omega^{-2} (\omega^m D_m) q_{ab}
\]

(B.34)

\[
p_{ab} = p_{ab} - 2\omega^{-1} D_a q_b + 4\omega^{-2} D_a q_b - \omega^{-2} (\omega^m D_m) q_{ab}
\]

(B.35)

\[
N_{ab} = N_{ab}
\]

(B.36)

\[
K'_{ab} = \omega^{-2} K_{ab} \quad \text{and} \quad \star K'_{ab} = \omega^{-2} \star K_{ab}
\]

(B.37)

where \( \omega^a = g^{ma} n^m \). Note that, although \( s_{ab} \) has a complicated conformal behavior, \( N_{ab} \) is conformally invariant; subtraction of \( p_{ab} \) from \( s_{ab} \) just removes the irrelevent, conformal gauge dependent part of \( s_{ab} \).

To summarize, the relevant fields on \( I \) constructed from \( s_{ab} \) are: \( q_{ab}, n^a \), the connection \( D_a \), \( s_a^b \), \( \star s_{ab} \), \( N_{ab} \), \( K_{ab} \) and \( \star K_{ab} \). These enjoy several properties, some directly in virtue of their definition, and other due to (asymptotic) field equations and Bianchi identities (Equations 14, 15, 19, 21, 25-30). Under the change of the conformal factor, they have transformation properties (equations (31)-(37)) which follow directly from the conformal transformation properties of various fields on the completed space-time \((M, g_{ab})\). This entire suite of fields together with their transformation properties is needed in the gravitational radiation theory in exact general relativity. This concludes the first part of the present section.
In the second part, we wish to show that one can work intrinsically on \( I \) without having to constantly pull back fields from space-time.

For this, we first introduce the notion of "abstract \( I \)", without any reference to a specific space-time, which will serve as the kinematic arena; it is to be equipped with the structure which is common to all asymptotically flat space-times. Fix a 3-manifold \( I \), topologically \( S^1 \times \mathbb{R} \), equipped with a collection of pairs \((q_{ab}, n^a)\) of nowhere vanishing fields, such that:

1. \( q_{ab}v^b = 0 \) iff \( v^b \) is proportional to \( n^a \);
2. \( L_n q_{ab} = 0 \);
3. Pairs \((q_{ab}, n^b)\) and \((q'_{ab}, n'^b)\) are in the collection iff there exists a smooth function \( \omega \) on \( I \), s.t. \( q'^a_{ab} = \omega^2 q_{ab} \) and \( n'^a = \omega^{-1} n^a \); and
4. the vector \( n^a \) is complete and the manifold \( S \) of its orbits is diffeomorphic to \( S^2 \). (Note that i), ii), and iii) imply that the function of \( \omega \) of iii) automatically satisfies \( L_n \omega = 0 \).) This is the required "kinematic arena". In Minkowski field theories, the symmetry group of the kinematic arena is the Poincaré group which then plays an important role in the entire theory. What is the corresponding group in the present case? This issue has been analyzed in the literature in a different context.\(^{26,16}\) We have: The subgroup of the diffeomorphism group of \( I \) which preserves the given collection of pairs \((q_{ab}, n^a)\) is precisely the BMS group.

The fact that one obtains the BMS group rather than the Poincaré group will play an important role in what follows. For the present, we merely note that, given an asymptotically flat space-time \((\mathcal{N}, \hat{\omega}_{ab})\) and a completion thereof, there exists a diffeomorphism \( \psi \) between the null boundary \( I \) of this space-time and the 3-manifold \( I \) introduced here which preserves the preferred collection of pairs ("the universal structure") and that any two such diffeomorphisms are related by an element of the BMS group.
Next, we introduce certain fields on our kinematical arena. Denote by \( D \) the collection of torsion-free connections \( D \) on \( J \), satisfying:

\[
D_a q_{bc} = 0 \quad \text{and} \quad D_a n^b = 0 \quad (B.38)
\]

for some pair \((q_{ab}, n^a)\) in our collection. (Since \( J \) is now an abstract manifold, not imbedded in any space-time, it is unnecessary to distinguish between = and \( \equiv \); from now all equalities will hold just on \( J \).) Since \( q_{ab} \) is degenerate, Eq. (B.7) fails to determine \( D \) uniquely. To see the available freedom, first note a consequence of Eq. (B.7):

\[
D_a V_b = D[a V_b] + \frac{1}{2} \omega V_{ab} \quad \text{if} \quad V_c n^c = 0 \quad (B.39)
\]

where, as before, \( V^c \) is any vector field on \( J \) such that \( V_c q_{bc} = V_b \).

Thus, because of Eq. (B.7), the action of \( D \) or any covector field \( V_c \) satisfying \( V_c n^c = 0 \) is pre-determined. Hence, to specify the action of \( D \) on any covector field - and therefore on arbitrary tensor fields - on \( J \), we need to give only \( D_a q_b \) for a \( q_b \) satisfying \( q_a n^b = 1 \). This is the freedom available in the choice of \( D \) in any one "conformal frame" \((q_{ab}, n^a)\). Equation (B.38) is clearly motivated by Eq. (B.14) on the null infinity of an asymptotically flat space-time. However, now, since \( J \) is not imbedded in any space-time, Eq. (B.38) had just to be postulated; the derivation of Eq. (B.14) uses the properties of the space-time connection \( V \). It will turn out, somewhat surprisingly, that one does not need to introduce additional restrictions:

The entire structure necessary for the gravitational radiation theory can be introduced starting only from the connection \( D \). Let us first consider the curvature of \( D \):

\[
R_{abd}^c k_d := 2D[a D_b] k_c \quad (B.40)
\]
Using the algebraic symmetries of $\mathcal{R}_{abcd}$ and (B.38) it follows that there exists a unique tensor field $s_{ab}$ on $\mathcal{I}$ such that

$$s_{ab} = (s - \mathcal{R}) n^a; \quad s_{abc} = s_{ac} = s_{bc}$$

(B.41)

and

$$\mathcal{R}_{abcd} = (q_{c} s_{ab}) + s_{c} (s_{b})$$

Next, we can define the Bondi news tensor. Using the kinematical structure on abstract $\mathcal{I}$, one can show that there exists a unique tensor field $\rho_{ab}$ on $\mathcal{I}$ satisfying (B.19). Set

$$\mathcal{N}_{ab} = s_{ab} - \rho_{ab}$$

This is the Bondi news tensor on abstract $\mathcal{I}$. It is easy to check that it is symmetric and trace-free. Finally, using $s_{ab}$ we can simply define $\mathcal{N}_{ab}$ via

$$\mathcal{N}_{ab} = e^{amnp} s_{np}$$

(B.42)

and the properties of $\mathcal{D}$ and $s_{ab}$ immediately imply that $\mathcal{N}_{ab}$ is trace-free and divergence-free:

$$\mathcal{N}_{ab} + \mathcal{D}_{ab} = 0$$

Thus, we have recovered all the radiative information starting from $\mathcal{D}$.

Note, however, that we have restricted ourselves to a fixed pair $(q_{ab}, n^a)$ so far. We must now study the effect of the conformal rescalings (B.31) since we only have a conformal equivalence class of pairs on $\mathcal{I}$. The first step is the specification of the way in which the connection $\mathcal{D}$ is to transform. The specification is subject to two constraints. First, the statement of the transformation law can refer only to that structure which is available on $\mathcal{I}$; we do not have access to a preferred space-time geometry. Second, given an imbedding of $\mathcal{I}$ into the completion $(\hat{M}, \hat{\mathcal{R}}_{ab})$ of an asymptotically flat space-time $(\hat{M}, \hat{\mathcal{R}}_{ab})$ or, more precisely, given a universal structure preserving diffeomorphism between the null infinity of $(\hat{M}, \hat{\mathcal{R}}_{ab})$ and $\mathcal{I}$.
and \( I \) the transformation properties specified on \( I \) should reduce to Eq. (B.12) at null infinity. Let us therefore first examine Eq. (B.32).

Setting \( \omega = 1 \) on \( I \) in these equations, we have, for \( q_{ab} \), \( n^a \) and \( D \):

\[
q_{ab}^t = q_{ab} \quad ; \quad n'{}^a = n^a \quad ; \quad \text{and}
\]

\[
D_a^t b = D_a b + f \, q_{ab} \, n^m \, k_m
\]

where \( f \) is the function on \( I \) given by \( fn^m = \nabla^m \omega \). (Since \( \omega \) is constant on \( I \), \( \nabla^m \omega \) is necessarily proportional to \( n^m \) there.) Thus, because the derivative operator \( D \) on \( I \) contains information about space-time geometry to "second order" while \( q_{ab} \) and \( n^a \) contain information only to "first order", \( D \) can be affected by changes in the space-time conformal factor to which \( q_{ab} \) is insensitive. In terms of abstract \( I \), on the other hand, we cannot distinguish between two conformal factors which agree on \( I \); we do not have access to fields such as \( \nabla^a \omega \) which refer to space-time. We therefore introduce the equivalence relation:

\[
D = \tilde{D} \quad \text{iff} \quad (D_a - \tilde{D}_a) \, K_b = f \, q_{ab} \, n^c \, K_c \quad \text{(B.44)}
\]

where \( D \) and \( \tilde{D} \) are connections satisfying Eq. (B.38) in the conformal frame \( (q_{ab}, n^a) \). Denote by \( \{D\} \) the equivalence class to which \( D \) belongs. Then, \( \{D\} \) has a well-defined transformation property satisfying the two constraints: under \( (q_{ab}, n^a) \rightarrow (q_{ab}^t, n^a) = (\omega^t q_{ab}, \omega^{-1} n^a) \) we have

\[
(D_a^t)b = (D_a)b - 2\omega^{-1}K_b(D_a)\omega \quad \text{(B.45)}
\]

where \( (D_a)K_b \) denotes the equivalence class of tensor fields obtained by operating in \( K_b \) various elements of \( \{D\} \). It turns out that these equivalence classes, \( \{D\} \), are the basic dynamical variables representing the radiative modes of the (exact, non-linear) gravitational field. Trans-
formations $D$ and $\tilde{D}$ where $D$ and $\tilde{D}$ are related by Eq. (B.44) will therefore be regarded as a gauge transformation. Under such a change, we have

$$s^b_a \rightarrow s^b_a - 2(D_a f)b$$
$$s_{ab} \rightarrow s_{ab}, \quad N_{ab} \rightarrow N_{ab}$$
$$\Psi^{ab} \rightarrow \Psi^{ab} \quad (B.46)$$

Thus, while only equivalence classes $\{D\}$ and $\{s^b_a\}$ have direct, gauge independent significance, fields $s_{ab}$, $N_{ab}$ and $\Psi^{ab}$ are by themselves gauge invariant. Under conformal rescalings (II.31) (with $\omega$ not necessarily equal to one on $\mathcal{I}$), the physically meaningful quantities transform in a well defined way. Using the transformation law (B.45) for $\{D\}$, we have:

$$s^b_a \rightarrow \omega^{-2} \{s^b_a\} - 2\omega^{-1} D_a \omega^b + 4\omega^{-1} \omega^b D_a \omega - \omega^{-1} (\omega^m D_m \omega) s^b_a \quad (B.47)$$
$$s_{ab} \rightarrow s_{ab} - 2\omega^{-1} D_a \omega D_b \omega + 4\omega^{-1} D_a \omega D_b \omega - \omega^{-1} (\omega^m D_m \omega D_n \omega) q_{ab} \quad (B.48)$$
$$N^a_{\alpha} \rightarrow N^a_{\alpha} \quad (B.49)$$
$$\Psi^{ab} \rightarrow \omega^{-1} \Psi^{ab} \quad (B.50)$$

where $\omega^m$ is any vector field on $\mathcal{I}$ satisfying $\omega^m q_{mn} = \nabla_m \omega$. (Note that (B.47) is well-defined as an equality of equivalence classes in spite of the ambiguity in $\omega^m$.)

To summarize, then, the radiative information is coded in equivalence classes of connections, $\{D\}$, on the abstract $\mathcal{I}$ defined by (B.38) and (B.44). Tensor fields $s^b_a$, $s_{ab}$, $N_{ab}$ and $\Psi^{ab}$ which feature in the gravitational radiation theory can be recovered directly from $\{D\}$ on abstract $\mathcal{I}$ without having to refer to an imbedding of $\mathcal{I}$ in a (completed) space-time. Given an imbedding, of course, the abstractly defined fields coincide with the pull-backs to the null boundary of the appropriate fields on the completion. As we shall see in the next chapter, the fact that we can work just on abstract $\mathcal{I}$ is crucial to the asymptotic quantization program.
4. The Phase Space of Radiative Modes

As before, let \( I \) denote the abstractly defined 3-manifold equipped with the universal structure of null infinity of an asymptotically flat spacetime. \( I \) will serve as the kinematic arena in the asymptotic quantization scheme; its role will be analogous to the Minkowskian background space-time in the covariant quantization program, or, the fixed 3-manifold (the Cauchy surface) in the canonical approach. Thus, various physical fields are to be "painted" on \( I \) which itself is to remain inert, insensitive to the presence or absence of the dynamical fields. In particular, in the passage to quantum theory, \( I \) itself will be left untouched; only the radiation fields, painted on \( I \), will be subject to uncertainty relations and quantum fluctuations.

For technical convenience, let us first fix a conformal frame \( (g_{ab}, n^a) \) on \( I \), introduce the radiative phase space, and then, at the end, investigate the effect of the change of the conformal frame on various structures that arise. Denote by \( C \) the collection of connection \( D \) on \( I \) satisfying (B.38) and by \( \Gamma \) the space of equivalence classes, \( \{D\} \), subject to the equivalence relation (B.44).\(^{28} \) Then, \( C \) and \( \Gamma \) are both affine spaces.

Indeed, in virtue of (B.38), any two elements \( D \) and \( D' \) of \( C \) are related to each other via

\[
(D_a - D'_a) K_b = \Sigma_{ab} n^c K_c \tag{B.51}
\]

for all \( K_c \) on \( I \), for some symmetric tensor field \( \Sigma_{ab} \) satisfying \( \Sigma_{ab} n^b = 0 \).

Hence, in virtue of (B.44), it follows that the difference, \( \{D\} - \{D'\} \), of
the corresponding elements of $\Gamma$ is completely characterized by the trace-free part,

$$
\sigma_{ab} := \Gamma_{ab} - \frac{i}{2} (q^{mn} q_{mn}) q_{ab},
$$

(B.52)

of $\Gamma_{ab}$. We can use this fact to coordinatize $\Gamma$. Fix, once and for all, an element $\{D\}$ in $\Gamma$ and regard it as the origin. Then, a generic element $\{D\}$ can be labelled by the second rank, symmetric, trace-free tensor field $\sigma_{ab}$ which is transverse to $n^a$ (i.e. satisfies $\sigma_{ab} n^b = 0$). How many independent components does $\sigma_{ab}$ have? Since $I$ is 3-dimensional, algebraic symmetries of $\sigma_{ab}$ imply that the number is precisely 2. These are the two radiative modes. $\Gamma$ is the phase space of radiative modes in exact general relativity.\textsuperscript{29} Using results from the linearized theory as a guide, one can introduce on $\Gamma$ a symplectic structure - i.e. a (weakly non-degenerate\textsuperscript{30}) 2-form $\Omega$: Given any two tangent vectors, $\sigma_{ab}$ and $\sigma'_{ab}$ at a generic point $\{D\}$, of $\Gamma$, $\Omega(\{D\})$ associates with them the real number

$$
\Omega(\{D\}) \sigma (\sigma , \sigma ') := \frac{1}{8\pi} \int_I (\sigma_{ab} L^a_c \sigma'^d_b - \sigma \sigma') q^{ac} q^{bd} \epsilon_{\text{mpn}} d^m d^n d^p
$$

(B.52)

where $\epsilon_{\text{mpn}}$ is (up to sign) the unique 3-form on $I$ satisfying

$$
\epsilon_{\text{mpn}} \epsilon_{\text{mpn}} = 6, \quad \text{and} \quad \epsilon_{\text{mpn}} a_{\text{abc}} q_m a q_n b = n^p n^c
$$

(B.53)

one can show\textsuperscript{31} that there is a precise sense in which $\Omega$ on $\Gamma$ defined by (B.52) is the same as the canonical symplectic structure involving Cauchy data on a space-like hypersurface which one normally uses in general relativity.\textsuperscript{1,2}

Let us now examine the effect of conformal rescalings (B.31) on the phase-space $\Gamma$ and the symplectic structure $\Omega$. Under these rescalings, elements $\{D\}$ of $\Gamma$ transform via (B.45), whence the tangent vectors $\sigma_{ab}$ can be shown to satisfy:

$$
\sigma_{ab} + \sigma'_{ab} = m \sigma_{ab}
$$

(B.54)

Since we also have $q^{ab} \sigma_{ab} = 0$, it follows that $q^{ab} \sigma'_{ab} = 0$, one expects in particular

Finally, $H_\gamma(\{D\})$ is a real-valued functional which preserves the endoscopic relation (B.46), an additional symmetry property.

It is easy to see that

$$
H_\gamma(\{D\}) = \frac{1}{8\pi} \int_I L_\gamma q_{ab} = 2 \pi \gamma q_{ab} \quad \text{on } \Gamma
$$

where $L_\gamma$ is an operator on $\gamma$.

One can also check that the conformal frame $\gamma$ satisfies (2D)\textsuperscript{28}, which states that $\gamma_{ab} n^a n^b$ is the flux density of electric current.

This quantity is regular at $\gamma=0$ on
Since we also have
\[ q_{\text{ab}}' = \omega^{-2} q_{\text{ab}} \quad , \quad c_{\text{abc}}' = \omega^3 c_{\text{abc}} \quad , \]
(B.55)

it follows that the symplectic structure (B.52) is conformally invariant, as one expects it to be on physical grounds.

Finally, let us consider the action of the BMS group on the phase space. Recall that a vector field \( V^a \) is said to be a (BMS) symmetry if the diffeomorphism it generates preserves the universal (or, kinematic) structure of \( I \). One can proceed step by step and show that these diffeomorphisms preserve the collection of connections on \( I \), respect the equivalence relation (B.44) (and is continuous on \( \Gamma \) w.r.t. the topology introduced in footnote 28), and leaves the symplectic structure invariant. Thus, each BMS symmetry provides us with an infinitesimal canonical transformation on \( \Gamma \).

It is easy to compute the corresponding generating functions, i.e. Hamiltonians \( H_V(\{D\}) \), one has
\[ H_V(\{D\}) = -\frac{1}{16} \int_I \left[ N_{ab} (L_V D_c - D_c L_V) k_d + 2 N_{ab} \bar{\omega} D_d \omega \right] q^{ac} q^{bd} \epsilon_{\text{mnp}} \epsilon_{\text{amnp}} \quad , \]
(B.56)

where \( L_V \) is any covector field on \( I \) satisfying \( L_V q^{\text{ab}} = 1 \) and \( q^{\text{ab}} \) is defined by \( L_V q^{\text{ab}} = 2 \alpha q^{\text{ab}} \). As one expects on physical grounds, the function \( H_V(\{D\}) \) on \( \Gamma \) is independent of the particular choice of the covector field \( L_V \), the conformal frame \( (q_{\text{ab}}, \omega^a) \) and the inverse \( q^{\text{ab}} \) of \( q_{\text{ab}} \) made in its evaluation.

One can also argue that the integrand represents the local flux-density of the "conserved quantity" associated with the BMS symmetry \( V^a \). In the particular case when \( V^a \) is a BMS translation, i.e., \( V^a = B n^a \), where \( B \) satisfies (2D_a b^b + D_b a^a) \( q_{\text{ab}} \) on \( \mathbb{R}^4 \), the expression simplifies: The flux density associated with \( B n^a \) is simply
\[ \frac{1}{32\pi} B n_{ab} n_{cd} q^{ac} q^{bd} \quad . \]
(B.57)

This quantity represents the flux density of the component of the Bondi 4-momentum along the translation represented by \( B n^a \). We shall return to
these Hamiltonians in the discussion of quantum theory. (For details concerning the action of the BMS group on \( \mathcal{J} \); see Ref. 32.)

5. Discussion

1) Existence of classical solutions admitting null infinity. One does know that there exist many stationary solutions to Einstein's vacuum equation which satisfy all the conditions of Definition 1. Of particular interest are the black hole solutions which, as we shall see in Appendix B, can be regarded as the (Lorentzian) paths that contribute significantly to the vacuum to vacuum amplitude in the quantum theory. However, the main interest in the null infinity framework stems from the gravitational radiation theory. Therefore, it is natural to ask if there exist "sufficiently many" asymptotically flat, radiative solutions to the vacuum equations. The current situation can be summarized as follows. Certain explicit radiative solutions admitting \( \mathcal{J} \) which is topologically \( S^2 \times \mathbb{R} \) are known.\(^{33}\) However, they all have a 2-parameter group of isometries -boost-rotation symmetry - which leads to certain pathologies: some generators of \( \mathcal{J} \) are incomplete and, what is worse, the total energy-momentum of the space-time vanishes identically.\(^{34}\) Thus, although these space-times provide certain insights, they do not have direct physical interest. Attempts have been made to obtain general existence results, without looking for explicitly analytic solutions, along two different lines. First, there exist "perturbative treatments" to test whether the notion of asymptotic flatness is "stable", i.e., if, given a space-time satisfying...
Definition 1, there exist "nearby" space-times all of which also satisfy the
Definition. Although these investigations have clarified several issues in
the linearized theory, their implications to full general relativity are
inconclusive because they suggest quite different scenarios. The first
result is that, given a space-time satisfying Definition 1, every $C^0$
perturbation (i.e. a solution to the linearized vacuum equation) whose
initial data on a Cauchy surface is of compact support preserves the
conditions of Definition 1 to first order. This suggests that there
should exist lot of solutions satisfying the Definition. A second result,
however, casts a doubt on this expectation. If one drops the compact
support restriction on initial data, one can find solutions to linearized
equations, already off Schwarzschild background, which are ill-behaved at $I$
(irrespective of the choice of gauge.) These perturbations have the
property that their initial data affects even the leading order terms in the
metric - e.g. the "mass aspect" - at $i^0$; it is the conformal singularity at
$i^0$ that spoils the regularity of linearized solutions at $I$. Since these
analyses ignore non-linear effects which can accumulate over the infinite
time interval between $I^-$ and $I^+$, it is difficult to evaluate the
relevance of their conclusions as far as full general relativity is con-
cerned. The second set of attempts to gain insight into the question of
existence of radiating solutions involves the initial value problem in exact
general relativity. Considerable work has been done on setting up a
characteristic data on $I$ and a null hypersurface in the space-time which
intersects $I$ in a 2-sphere cross-section. This work establishes the
existence of a large class of radiating, vacuum solutions which admit a
topologically $S^2 \times R$ null infinity. However, it is yet to face the

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problem of existence of complete $I^*$, $I^-$ joined together by $i^*$. The most promising line of attack available is based on some recent re-examination of the standard Cauchy problem with small initial data. There are strong indications that such data will evolve to globally regular space-times which admit not only $I^*$ and $i^*$ but also $i^\pm$. Furthermore, the indications are that although it may be difficult to establish that the rescaled metric is $C^\infty$ at $I$ for a large class of space-times, it is likely to be sufficiently differentiable to enable the construction of the phase space $\Gamma$ of radiative modes.\textsuperscript{39}

11) Classical vacua and the Poincare' reduction of the BMS group.

Following the terminology often used in gauge theories, we shall call an element $\{D^a\}$ of $\Gamma$ a \textit{classical vacuum} if its curvature is "trivial", i.e., if the corresponding fields $N_{ab}$ and $\psi_{ab}$ (Equations (B.40)-(B.42)) vanish identically. This terminology is also suggested by the fact that the Hamiltonians $H_{\nu}((D))$ of Eq. (B.56) vanish at these points $\{D^a\}$ of $\Gamma$ for all BMS symmetries $V$. In particular, in the state of the gravitational field represented by a classical vacuum, there is no flux of energy across $I$. We shall denote by $D^a$ any connection in $\mathcal{C}$ which belongs to the equivalence class $\{D^a\}$. Given $D^a$ and $D'^a$, one can show that they are related by

\begin{equation}
(D^a - D'^a) K_b = \Sigma_{ab} C K_c
\end{equation}

where

\begin{equation}
\Sigma_{ab} = D_a D_b f + f \rho_{ab}
\end{equation}

for some function $f$ on $I$, satisfying $\Delta f = 0$. Recall\textsuperscript{16} that, since $\Delta f = 0$, $f\rho_{ab}$ is a BMS supertranslation and that it is a BMS translation if $f$ satisfies in addition, $(D_a D_b f + f \rho_{ab}) = \rho_{ab}$. Thus, $D^a$ and $D'^a$ in (B.58) define the same element of $\Gamma$ (i.e. the same equivalence class in (B.44)) iff $f$ is a BMS translation. In fact there is a more direct relation between the BMS group and the space of classical vacua. First, the action of the BMS group $B$ on $I$ maps class $D^a$ into class $D'^a$ under transformation $B$. Further, the space of all such classes $D^a$ is the Poincare' group $\mathcal{P}_0$. Details of the proof in this case can be found in the appropriate section of the text.
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ii) Momentum maps. Let us first recall some terminology from symplectic geometry. Let \((\Gamma, \Omega_{GB})\) be a symplectic manifold. (For simplicity, in this general introductory discussion we shall assume that all manifolds are finite-dimensional and all fields are smooth.) Let a Lie group \(G\) act on \(\Gamma\) and let the action preserve \(\Omega_{GB}\). Then, (if the first homotopy of \(\Gamma\) is trivial), the Hamiltonians generating the corresponding canonical transformations provide us with a linear mapping from \(\Gamma\) to the dual \(\mathcal{L}^*\) of the Lie algebra \(\mathcal{L}\) of \(G\). This is called the momentum map associated with the given action of \(G\) on \(\Gamma\). Let \(\theta\) be a symplectic potential, i.e., satisfy \(\Omega_{GB} = 2\pi [\eta^B]\) and let us suppose that \(\theta\) is also left invariant by the action of the group. Then, if an element \(\mathcal{V}\) of \(\mathcal{L}\) is represented by a vector field \(\mathcal{V}^B\) on \(\Gamma\), a Hamiltonian which generates the corresponding canonical transformation is simply the function \(H_\mathcal{V} := \mathcal{V}^B \theta\) on \(\Gamma\). Denote
by $\gamma$ an arbitrary point in $\Gamma$ and by $\Theta^*$, the element of $L^*$ which maps $V$ in $L$ to the real number $\langle \Theta^* \Theta \rangle(\gamma)$. Then $\gamma \mapsto \Theta^*$ is the required momentum map. Let us now return to the phase space $\Gamma$ of radiative modes $(D)$ on $I$.

The BMS group $B$ on $I$ has a natural representation on $\Gamma$ which leaves the symplectic structure $\Omega$ on $\Gamma$ invariant. The momentum map is given by Eq. (B.56) which defines the Hamiltonians $H_V(D)$. This map is, in fact, of the type $\gamma \mapsto \Theta^*$. To see this, we must introduce the symplectic potential, the analog of $\Theta^*$. Consider the 1-form $\Theta$ on $\Gamma$ defined by

$$\Theta(X) := \frac{1}{16\pi} \int_J N_{ab} X_{cd} q^{ac} q^{bd} d^{3}l$$

(B.59)

for all tangent vectors $X$ to $\Gamma$ (i.e. second rank, symmetric tensor fields $X_{ab}$ satisfying $X_{ab} = 0$ and $X_{ab}q^{ab} = 0$ on $J$). It is straightforward to verify that $\Theta$ is a symplectic potential for $\Omega$; $\Omega = d\Theta$. Furthermore, from (B.56) it follows that $H_V(D) = \Theta(V)$, where $V$ is the vector field on $\Gamma$ defined by the BMS symmetry $V^a$ on $I$. The symplectic potential $\Theta$ is a natural one in that it vanishes at all classical vacua. However, to my knowledge it is not "canonical", e.g., in the sense in which the natural symplectic potential on a cotangent bundle is.

iv) "Charge" integrals. The Hamiltonian densities -- the integrands in (B.56) -- are to be interpreted as fluxes of "conserved quantities" associated with the BMS group. The question naturally arises: what are these "conserved quantities"? Since the integrands in (B.56) are 3-forms, we are led to look for 2-forms $V_{ab}$ on $I$ whose curl yields these 3-forms. Then, given any 2-sphere cross-section $C$ of $I$, one could set

$$Q^C := \int_C V_{ab} d^{3}a$$

(B.60)

$Q^C$ can be interpreted as the "conserved charge" associated with the BMS symmetry $V$ and the cross-section $C$. Given any two cross-sections $C$ and $C'$, the difference $Q^C - Q^{C'}$ would be given by the integral of the Hamiltonian density $\Theta(V)$.

Thus, $Q^C$ is another "charge" integral, which is conserved in the framework of the BMS symmetry. It is a "negative" integral, as expected, since for the AdS superspinor, which is negative, the "charge integral" is interpreted as the "integrated momentum", available at the superspinor boundary (see the discussion of "integral representations" in the superspinor approach for additional details).

In fact, this is a simple charge integral, but it is stated as a "negative" charge integral of magnitude $Q^C - Q^{C'}$, although it is not allowed to isolate it.
density in (B.56) over the 3-dimensional region of \( I \) bounded by \( C \) and \( C' \). Thus, one is led to ask: can one "integrate" the flux 3-forms to obtain the charge 2-forms which depend only \textit{locally} on various physical fields? Within the framework of abstract \( I \) and radiative modes, the answer is in the negative. Indeed, already in the case when \( V^a \) is a BMS translation, one expects \( Q^n^c \) to yield the (corresponding component of the) Bondi 4-momentum which involves a longitudinal part \((\Re \psi^n)\) of the asymptotic Weyl curvature, part that is not contained in the radiative modes \((D)\). If one considers \( I \) as the boundary of an actual 4-dimensional space-time, however, the desired "integration" is possible because one has a greater number of fields available. For BMS translations, one recovers the Bondi 4-momentum. For supertranslations, one obtains new charge integrals, called supermomenta (see the reference in footnote 28). For general BMS symmetries, the problem of "integration" was resolved only recently;\textsuperscript{40} the corresponding charges represent generalized (or, BMS as opposed to Poincare') angular momentum.

v) \textit{The NUT 4-Momentum.}\textsuperscript{41} In gauge theories, there is considerable interest in states with a non-zero magnetic charge. One may therefore look for asymptotic states of the gravitational field with a "magnetic 4-momentum". In fact, explicit solutions with a magnetic 4-momentum are known. The simplest of these is the Newman Unti-Tamburino - or NUT - solution\textsuperscript{42}, which is static and has two parameters, \( m \) and \( n \), representing (the usual or "electric") mass and the magnetic mass respectively. However, the presence of magnetic mass introduces certain wire singularities in the metric although the curvature is everywhere regular. It turns out that such solutions can be incorporated in the asymptotic quantization framework by allowing the connections \((D)\) on \( I \) to develop wire singularities along isolated generators of \( I \), keeping the (curvature) fields \( N_{ab} \) and \( \Psi_{ab} \).
everywhere smooth. One can then write down an unambiguous expression for the NUT 4-momentum. However, the expressions has several unexpected features. First, while the "electric" (or, Bondi) 4-momentum involves the longitudinal field \( \Phi_\ell \), the "magnetic" (or, NUT) 4-momentum involves \( \Phi_\varphi \), which can be recovered entirely from \( (D) \). This is similar to the situation in non-Abelian gauge theories in which, again, the electric type charge involves longitudinal fields while the magnetic charge can be recovered purely from the radiative information in the connection. However, there is also a major difference: while both types of charges can be radiated away across \( \mathcal{I} \) in non-Abelian gauge theories, only the Bondi 4-momentum gets radiated away in the gravitational case; NUT 4-momentum is absolutely conserved even in presence of radiation! One would expect the NUT charge to lead to new superselection rules in the quantum theory.\(^{41}\)

vi) **Relation to the Newman-Penrose notation.** Let us begin by recalling the Newman-Penrose notation.\(^{43}\) Fix a space-time \((\mathcal{R}, \mathcal{S}_{ab})\) which is asymptotically empty and flat at null infinity. Consider a conformal completion \((\mathcal{M}, \mathcal{g}_{ab})\) such that the induced \( \mathcal{g}_{ab} \) on \( \mathcal{I} \) is the lift to \( \mathcal{I} \) of a unit 2-sphere metric on the space \( \mathcal{S} \) of generators. Then \( \mathcal{m}_a \) generates a BMS time translation. Fix a cross-section of \( \mathcal{I} \) and obtain a foliation of \( \mathcal{I} \) by translating this cross-section along \( \mathcal{m}_a \). Introduce a coordinate \( u \) on \( \mathcal{I} \) such that the 1-parameter family of cross-sections is labelled by \( u = \text{constant}. \) Thus, on \( \mathcal{I} \), \( u \) satisfies \( L_{\mathcal{m}} u = 1. \) Denote by \( \mathcal{J}^a \) the unique null vector field on \( \mathcal{I} \), orthogonal to these cross-sections, which satisfies \( \mathcal{g}_{ab} \mathcal{J}^a \mathcal{J}^b = 1 \). Finally, introduce a complex vector field \( \mathcal{m}^a \), tangential to the cross-sections, satisfying \( L_{\mathcal{m}} \mathcal{m}^a = 0, \mathcal{m} \cdot \mathcal{m} = 0 \) and \( \mathcal{m} \cdot \mathcal{m} = 1 \). Then, \( (\mathcal{m}^a, \mathcal{J}^a, \mathcal{m}^a, \mathcal{m}^a, \mathcal{m}^a) \) gives us a null tetrad at any point of \( \mathcal{I} \). Set \( \sigma^\alpha = \mathcal{m}^a \mathcal{m}^b \nabla_a A_b. \) \( \sigma^\alpha \) is independent of the particular extension of \( A_b \) off \( \mathcal{I} \), chosen in its

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evaluation.) $\sigma^0$ is the asymptotic shear. $N = -2 \mathcal{L}_\mathbf{a} \sigma^0$ is called the Bondi news function. Finally, the ten components of the asymptotic Weyl curvature, $\hat{\eta}_{abcd}$, are captured in five complex functions $\psi^0$, $\psi^1$, $\psi^2$, and $\psi^3$.

Five of these ten quantities are completely determined by the shear $\sigma^0$:

\[
\begin{align*}
\psi^0 &= \sigma^0 \\
\psi^1 &= \sigma^0 \\
\psi^2 &= \sigma^0 \\
2 \mathcal{L}_a \mathcal{L}_b \sigma^0 &= \sigma^0 - \sigma^0 + \sigma^0 - \sigma^0
\end{align*}
\] (B.61)

where 'dot' stands for $n^a \mathcal{L}_a$ and where $\equiv$ stands, as before, for "equals at points of $I$ to". The remaining five components contain "longitudinal" information and are not captured in $\sigma^0$ alone.

Consider the pull-back to $I$ of the covector field $\mathcal{L}_a$. Set

\[
\sigma^0_{ab} \equiv D_a \mathcal{L}_b - \frac{i}{2} q_{ab} q_{mn} D_m \mathcal{L}_n.
\]

(Note that $D_a \mathcal{L}_b \equiv D_a (\mathcal{L}_b)$ and that $q_{mn} D_m \mathcal{L}_n$ is well-defined because $n^m D_m \mathcal{L}_n \equiv 0$.)

Clearly, $\sigma^0_{ab}$ contains the same information as $\sigma^0$ although it is independent of the choice of $m^a$. We shall now show that, together with $\mathcal{L}_a$, $\sigma^0_{ab}$ contains the same information as the element (D) of $\Gamma$.

Define, on $I$, a symmetric connection $\hat{\mathcal{B}}$ satisfying Eq. (B.38) and the condition $D_a \mathcal{L}_b = 0$. From the discussion following Eqs. (B.38) and (B.39), it is clear that $\hat{\mathcal{B}}$ is completely specified by these conditions. A direct calculation shows that $\hat{\mathcal{B}}$ has trivial curvature whence $(\hat{\mathcal{B}})$ is a classical vacuum. (The $4$-parameter family of cross-sections obtained from $u = 0$ by the action of BMS translations is shear-free w.r.t. the connections in $(\hat{\mathcal{B}})$. As one might expect, the Poincare' sub-group of the BMS group which leaves this family invariant is the same as that selected by $(\hat{\mathcal{B}})$ in the remark ii) above.) Let us use this $(\hat{\mathcal{B}})$ as origin. Then, any element $\{D\}$ of $\Gamma$ can be labelled by a tensor field $\gamma_{ab}$ satisfying $\gamma_{ab} \equiv \gamma(ab)$, $\gamma_{ab} \equiv 0$ and $\gamma_{ab} q_{ab} \equiv 0$. Consider the connection $\mathcal{D}$ induced on $I$ by $\gamma$. We have $(\mathcal{D}_a - \hat{\mathcal{L}}_a) \mathcal{L}_b \equiv D_a \mathcal{L}_b - \varepsilon_{ab} n^c \mathcal{L}_c \equiv \varepsilon_{ab}$, whence $\{D\} - \{\hat{\mathcal{B}}\}$ is character-
ized by the trace-free part of $E_{ab}$. Thus, $\{D\}$ is labelled by $\gamma_{ab}$ defined by

$$\gamma_{ab} = D_{a} l_{b} + q_{ab} q^{mn} D_{m} l_{n}.$$ 

Hence $\gamma_{ab} = \sigma^{a}_{ab}$; $\{D\}$ and $\sigma^{a}_{ab}$ - or equivalently, $\sigma^{a}$ - contain identical information. Next, since $N = L_{m} \sigma^{m}$, it follows that $N = m^{a} m^{b} N_{ab}$. Finally, Eqs. (B.61) are equivalent to Eqs. (B.40) - (B.42).

Thus, by choosing a cross-section of $I$, Newman and Penrose first introduce an origin in $\Gamma$ and then, using this origin, represent the connections, news tensors and asymptotic curvatures in terms of spin and boost weighted scalars. The choice of origin, however, is not a natural one. For example, under a supertranslations, the origin is shifted and the scalars undergo transformations reflecting this change. In particular, the vector space structure that the space of shears appears to have is not a natural one: the notion of zero shear fails to be invariant under supertranslations. Hence, the choice of a cross-section is similar to the choice of a gauge in Yang-Mills theory: it simplifies many computations but complicates the analysis of conceptual issues since it introduces auxiliary structure which is not naturally available. (Thus, for example, the use of shears $\sigma$ as the basic dynamical variable had led to an incorrect expression for the symplectic structure and the Hamiltonians generating the BMS symmetries; it is rather easy to overlook the fact that the vector space structure on the space of shears is illusory.)

In the framework presented in Sec. 3, gauge makes it appearance via transformations $D \rightarrow \tilde{D}$ where $\tilde{D}_{a} K_{b} = D_{a} K_{b} - f q_{ab} n^{c} K_{c}$. It is this freedom that led us to equivalence classes $\{D\}$. What is the status of this freedom in the Newman-Penrose formalism? Since this freedom arises from that of conformal rescalings $g_{ab} \rightarrow \omega^{2} g_{ab}$ (with $\omega = 1$ on $I$ but $\nabla g_{ab} \neq 0$ on $I$), it persists in the Newman-Penrose formalism as we have summarized it here. However, in the conformal metric basis, the $D_{[ab]} = 0$ on the boundary of the gauge manifold is a natural condition obtained by the one and only one condition to satisfy this additional constraint that arises in this additional condition for null conformal vector fields. This self-consistent condition, which is a second order constraint for the vector field, can be interpreted as requiring the null vector field to be non-singular at the boundary of the gauge manifold.

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Here. However, it can be and often is eliminated by requiring that the
conformal factor $\Omega$ be so chosen that not only should $g_{ab}$ be a unit 2-sphere
metric but also $\Lambda_a$ be divergence free on $\mathcal{I}$. One can use a similar technique
on the abstract $\mathcal{I}$. We could have fixed a covector field $\lambda_a$ on $\mathcal{I}$ satisfying
$D_a A_b = 0$, $\mathcal{L}_\lambda \lambda_a = 0$ and $\lambda_a n^a = 1$ and permitted only those connections $D$
in $\mathcal{C}$ which satisfy $(D_a A_b) g^{ab} = 0$. It is easy to verify that there exists
one and only one connection $D$ in each equivalence class $\{D\}$ on $\mathcal{I}$ satisfying
this additional requirement. However, for aesthetic reasons, we shall not
employ such a gauge fixing procedure.
II.C: Quantization of the Radiative Modes

1. Maxwell Fields in Minkowski Space: Infrared Sectors

We shall first carry out the asymptotic quantization of the radiative modes $F_\alpha$ on $\mathcal{I}$, then point out that the restriction to the Fock representation is a severe requirement and finally present the appropriate asymptotic Hilbert spaces on $\mathcal{I}^-$ and $\mathcal{I}^+$ between which the $S$-matrix is well defined (in the perturbative, order by order sense), i.e., is free of infrared problems. (For details, see Refs. 7 and 8.)

Recall that the symplectic structure on the space of well-behaved solutions $\mathbf{F}_{ab}$ in Minkowski space $(\hat{\Omega}, \hat{\mathcal{F}}_{ab})$ is given by:

$$\hat{\Omega}(\hat{\mathbf{F}}, \hat{\mathbf{F}}') = \frac{1}{4\pi} \int_{\Sigma} \left( \hat{\mathbf{F}}_{ab} \hat{A}^b - \hat{\mathbf{F}}_{ab} \hat{A}^b \right) dS^a$$

where $\Sigma$ is any spacelike hyperplane in Minkowski space; $\hat{A}_{\alpha}$, any vector potential of $\hat{\mathbf{F}}_{ab}$ which tends to zero at spatial infinity (say as $1/r^{1+\epsilon}$ for some $\epsilon > 0$); and $\hat{\mathbf{E}}$, the electric field induced on $\Sigma$ by $\hat{\mathbf{F}}_{ab}$. The integrand in (C.1) is divergence-free, whence the integral is independent of the choice of the surface $\Sigma$ made in its evaluation. Therefore, if $\hat{\mathbf{F}}_{ab}$ and $\hat{\mathbf{F}}_{ab}'$ are two solutions obtained via Kirchoff integrals from well-behaved characteristic data $\mathbf{F}_\alpha$ and $\mathbf{F}_\alpha'$ on $\mathcal{I}$ (Sec. 1, ch. II.B), we can rewrite (C.1) as an integral on $\mathcal{I}$. (As before, $\mathcal{I}$ stands for $\mathcal{I}^+ \cup \mathcal{I}^-$.) For this it is convenient to go to a gauge such that

$$A_\alpha n^\alpha = 0$$

where $\Delta$ and $\hat{\mathcal{R}}$ denote a quantization of a universal connection in the space. The pull-back $\mathbf{F}_{ab}$ on $\mathcal{I}$.) Instead subject

$$[\mathbf{F}_{ab}]$$

We have the Schwartz property (Thus, if $\mathbf{F}_{ab}$ are $C^\infty$ functions on $\mathbf{F}_{ab}$ and $\mathbf{F}_{ab}'$ for any $\mathbf{V}_{a}(u, \theta, \phi)$ positive

We shall indicate by the subscript
where as before, \(\approx\) stands for "equals, at points of \(I\), to". (It is easy to check that (C.3) implies \(\mathcal{E}_a = L_\alpha \Lambda_a\).) Now, one has:

\[
\Omega(\mathcal{F}, \mathcal{F}') = \frac{1}{4\pi} \int_I (\mathcal{E}_a A^a - \mathcal{E}'_a A^a) \, d^3I = \frac{1}{4\pi} \int_I \int_J (\mathcal{E}_a(u, \theta, \phi) A^a(u', \theta', \phi') - \mathcal{E}'_a(u', \theta', \phi')) \times \\
\delta(\theta, \phi; \theta', \phi') \Delta(u-u') \, d^3I \, d^3I' = \Omega(\mathcal{E}, \mathcal{E}') \tag{C.4'}
\]

where \(\Delta(u-u')\) is the step function. This is the starting point for quantization of the radiative modes.

Consider \(I\) as an abstractly defined 3-manifold equipped with the universal structure of null infinity, without any reference to Minkowski space. (We shall therefore drop the tilde on equalities and the explicit pull-back arrows; all equalities and fields will be defined intrinsically on \(I\).) Introduce on \(I\) an operator valued distribution \(\mathcal{E}_a(u, \theta, \phi)\), subject to the (null surface) canonical commutation relations (CCR's):

\[
[\mathcal{E}_a(u, \theta, \phi), \mathcal{E}_b(u', \theta', \phi')] = \frac{i}{\hbar} q_{ab} \delta(\theta, \phi; \theta', \phi') \Delta(u-u') \tag{C.5}
\]

We have to find representations of these CCR's. For this, consider the Schwartz space \(\mathcal{S}\) of real test fields \(V_a\) on \(I\), satisfying \(V_a(n) = 0\).

(Thus, in the \(u, \theta, \phi\) chart, \(V = V_\theta d\theta + V_\phi d\phi\), the components \(V_\theta\) and \(V_\phi\) are \(C^\infty\) and they and all their derivatives fall-off faster than \(1/|u|^n\), for any \(n\), as \(u \to \infty\).) Every \(V_a(u, \theta, \phi)\) in \(\mathcal{S}\) admits a Fourier transform \(V_a(u, \theta, \phi)\) which also belongs to the Schwartz space. Denote by \(V_a^\pm\) the positive frequency part of \(V_a\):

\[
V_a^\pm(u, \theta, \phi) = \frac{1}{(2\pi)^{3/2}} e^{-i\omega u} (2\pi)^{-3/2} d\omega \tag{C.6}
\]

We shall say that \(V^+\) belongs to the pre-Hilbert space \(h\) if its norm defined by the hermitean inner product:

\[
\langle V^+, W^+ \rangle := \frac{-i}{\hbar} \Omega((V^+)^*, W^+) = \int \int d\theta d\phi \sin\theta d\omega (V_a)^* (\overline{W}_a)\tag{C.7'}
\]

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is finite. Denote by $H$ the Hilbert space obtained by Cauchy completion of $h$. $H$ is to be the space of $1$-photon asymptotic states. Let $F$ be the symmetric Fock space based on $H$. Then, the representation mapping $A$ from the algebra generated by the distribution $\varepsilon_a$ to the algebra of linear operators on $F$ is given by:

$$
A \varepsilon(V) := \hbar [C(V^+) + A(V^+)]
$$

where $\varepsilon(V) = \int \varepsilon_a V^a d^3 \nu$ and where $C(V^+)$ and $A(V^+)$ are the linear operators on $F$ which, respectively, create and annihilate a photon in the state $V^+$. It is easy to verify that, in virtue of our choice of the inner-product (C.7), $A$ preserves the CCR.

How is this structure related to the usual Fock representation of the CCR in Minkowski space-time? Let us now regard $I$ as null infinity of Minkowski space. Then using field equations and (C.4) - (C.5), one can verify that the algebra generated by $\varepsilon_a(u, \theta, \phi)$ is naturally isomorphic to the usual algebra generated by the operator valued distribution $F_{ab}(x)$ in the physical space-time. Next, using Kirchoff integrals, one can set up a 1-1 correspondence between the Schwartz space $S$ and the space of well-behaved solutions to Maxwell's equation. This correspondence preserves the notion of positive frequency fields and, because of (C.7), also the hermitean inner-product. Consequently, although we worked exclusively on $I$, we have the same algebra of operators and the space of quantum states as in the (physical subspace of the) textbook Fock representation.

We are now ready to see how the infrared problems arise. The key step in the transition to quantum theory is the introduction of the hermitean inner-product (C.7). Thus, for example, whereas any element $V_a$ in $S$ is an admissible characterisitic data for a well-behaved solution of Maxwell's equation, $V_a^+$ can represent a $1$-photon wave function only if $\langle V^+, V^+ \rangle \neq 0$. Is this
requirement stringent? One's experience with massive fields may suggest that it is not. That is, one might expect that almost all "respectable" classical fields would lead to 1-photon wave functions. This expectation turns out to be wrong: finiteness of norm is a severe requirement. This becomes transparent in the $I$-framework because the requirement can be easily translated into a geometrical constraint on the fields $V_a$ on $I$ themselves, rather than on their Fourier transforms. To see this, let us re-examine the expression of the norm:

$$\langle V^+, V^+ \rangle = \frac{1}{4\pi^2 \hbar} \int d\theta \, d\phi \, \sin \theta \int_0^{\infty} \frac{du}{u} \langle \bar{V} \rangle^* \cdot \langle \bar{V} \rangle$$  (C.9)

Since $V(u, \theta, \phi)$ belongs to the Schwartz space, there is no convergence problem in the high frequency limit; there are not ultra-violet divergences. However, it is clear that the integral would diverge at the lower end, ($u=0$), unless $\bar{V}(u=0, \theta, \phi) = 0$. Since $\bar{V}_a$ is in the Schwartz space, we can just compute its value at $u=0$ via Fourier transforms. We have:

$$\bar{V}(0, \theta, \phi) = \frac{1}{(2\pi)^3} \int_0^{\infty} V_a(u, \theta, \phi) \, du$$  (C.10)

Thus, a necessary and (because $V$ is $C^\omega$) sufficient condition for finiteness of the norm $\langle V^+, V^+ \rangle$ is simply that

$$Q_V(\theta, \phi) := \int_0^{\infty} V_a(u, \theta, \phi) \, du = 0$$  (C.11)

along each generator of $I$, i.e. for all $\theta$ and $\phi$. From the point of view of the classical theory, this is clearly an artificial requirement on the characteristic data; it has no physical basis. Indeed, if one examines the retarded fields which result at $I^+$ due to scattering of charged particles, one finds that (C.11) can be satisfied only in very exceptional cases\(^{45}\), e.g., when the incoming velocity (at $i^-$) of each charged particle is the same as its outgoing velocity (at $i^+$). It is therefore not very surprising that in full QED, where non-trivial scattering of charges does occur, the $S$-matrix is ill-defined if one uses just the tensor product, $F_M \otimes F_D$, of the
Maxwell and the Dirac Fock spaces to represent the incoming and the outgoing quantum states.

The geometrical interpretation of the requirement of finiteness of norm itself suggests how one may improve on this situation. Introduce the following equivalence relation on $S$:

$$ V \cong V' \iff Q_V(\theta, \phi) = Q_{V'}(\theta, \phi) $$

i.e.,

$$ \int_0^\infty e^{-u^2} \left| V_a(u, \theta, \phi) - V'_a(u, \theta, \phi) \right| du = 0 $$

Then, each equivalence class is labelled by a function $Q(\theta, \phi)$ on a 2-sphere; the space $S$ of classical states (i.e. of well-behaved characteristic data) is thus divided into sectors, each labelled by a $Q(\theta, \phi)$. It is only the trivial sector, with $Q(\theta, \phi) = 0$, that can give rise to 1-photon states in the Fock representation. What we need is quantum states corresponding to other sectors. These can be obtained as follows. On the algebra of operators generated by $\varepsilon_a(u, \theta, \phi)$ on $I$, consider the following automorphisms:

$$ \Pi_{Q} : \varepsilon_a(u, \theta, \phi) \rightarrow \varepsilon_a(u, \theta, \phi) + V_a(u, \theta, \phi)I $$

where $V_a(u, \theta, \phi)$ is an element of $S$ in the $Q(\theta, \phi)$-sector, and $I$ is the identity operator. This automorphism is unitarily implementable in the Fock representation iff $V^*$ has finite 1-particle norm, i.e., $Q_V(\theta, \phi) = 0$.

If $Q(\theta, \phi) \neq 0$, one can use the Gelfand-Naimark-Segal construction to obtain representations $A_Q = A_0 \Pi_{Q}$ of the CCR which are unitarily inequivalent to the Fock representation. (Note that it is only the value of $Q(\theta, \phi)$ that matters; if $V_a \cong V'_a$, the representations of the CCR they yield via (C.13) are unitarily equivalent. Because of this, there is no natural vacuum (or, no particle) state in $O^P_M$ except when $Q(\theta, \phi) = 0$.) Thus, we now have as many (distinct) spaces $O^P_M$ of quantum states of the Maxwell field as there are functions $Q(\theta, \phi)$ on a 2-sphere.
One can imagine taking "direct sum" \( Q\mathcal{F}_M \) of all these representation spaces and using its tensor-product, \( \otimes Q\mathcal{F}_M \otimes \mathcal{F}_D \), with the Dirac Fock space \( \mathcal{F}_D \) as the Hilbert space of asymptotic states. Such a procedure was followed in the early work on infrared problems in QED.\(^{47}\) However, the resulting space, being non-separable, is technically inconvenient to use. Fortunately, there is a more subtle avenue available.\(^{9}\) To be specific, let us construct the space of incoming states. Let \( |\psi\rangle = |\vec{p}_1, \vec{p}_2, \ldots, \vec{p}_n\rangle \) be the incoming state of \( n \) Dirac particles with momenta \( \vec{p}_1, \vec{p}_2, \ldots, \vec{p}_n \). Then, using the classical Green's functions and the values \( \vec{p}_1, \ldots, \vec{p}_n \) of momenta, one can unambiguously select a \( Q(0,\psi) \)-sector on \( \mathcal{F}^\perp \). That is, associated with the incoming state \( |\psi\rangle \) of Dirac particles, there exists a non-Fock sector \( Q\mathcal{F}_M \) of the incoming quantum states of the Maxwell field, where \( Q \) is completely determined by \( |\psi\rangle \). The Hilbert space of incoming states of the combined Maxwell-Dirac system is \( \mathcal{H}^- := \otimes Q\mathcal{F}_M \otimes |\psi\rangle \), where the direct sum extends over the (momentum eigenstate) basis \( \{|\psi\rangle \} = \{|\vec{p}_1, \ldots, \vec{p}_n\rangle\} \) in the Dirac Fock space \( \mathcal{F}_D \). Thus, the Hilbert space of incoming states has a fibre-bundle type structure. The "base space" is the Dirac Fock space \( \mathcal{F}_D \) and the "fibres" are the various \( Q\mathcal{F}_M \) spaces, where the value of \( Q(0,\psi) \) depends on the point of the "base space" over which the "fibre" is constructed. The "fibres" are isomorphic, but not naturally; there are no preferred "horizontal subspaces". The fact that each Dirac state \( |\vec{p}_1, \ldots, \vec{p}_n\rangle \) carries its own Hilbert space of photons and that, for \( Q \neq 0 \), \( Q\mathcal{F}_M \) has no natural vacuum state, is often clothed in the phrase "electrons are always surrounded by a cloud of soft photons". Finally, note that \( \mathcal{H}^- \) is "of the same size as \( \mathcal{F}_D \otimes \mathcal{F}_N \)". The outgoing Hilbert space \( \mathcal{H}^+ \) is constructed in an analogous way. (For details,
see Ref. 8.) After renormalization (which takes care of the ultra-violet problems), the quantum $S$-matrix gives a mapping from $H^{-}$ to $H^{+}$ to any order in perturbation expansion.\footnote{8}

2. \textbf{Gravitational Field: The Fock Representation}

The first step in quantization will be the construction of the algebra of quantum operators using the classical symplectic description given in ch. II.B. Recall that, due to factor ordering problems, not all classical observables can admit unambiguous quantum analogs. Therefore, one must first single out a preferred vector space of classical observables (closed under Poisson brackets) which are to be directly promoted to quantum operators. How is this selection to be made? Quantization schemes applicable to simpler systems provide a hint. Consider for example non-relativistic systems from mechanics. Not only is their phase space a symplectic manifold but it also admits a natural cotangent bundle structure. It is precisely those classical observables which generate canonical transformations preserving this additional structure - i.e. observables which are either independent of or linear in momenta - that admit natural quantum analogs.\footnote{8} The situation is similar also for relativistic free fields. In this case, the phase space has a natural vector space structure and it is again precisely the linear observables that lead unambiguously to field operators.\footnote{8} In the case of the phase space $\Gamma$ of radiative modes of the gravitational field, the additional structure is that of an affine space. We are therefore led to look for generators of canonical transformations which preserve the affine character of $\Gamma$. In the infinitesimal form, these transformations correspond to constant vector fields on $\Gamma$, i.e., to symmetric, second rank tensor fields $f'_{ab}$ and $f_{ab}$. Defining a large number of new observables together with their conjugate momenta $P_{ab}$, $P_{a}$, $P_{b}$ as usual,

where $N(f)$, $N(f')$, $N(ab)$, $N(a)$, $N(b)$ are functions of the symplectic observables $f$, $f'$, $ab$, $a$, $b$.

Thus, beginning with a suitable set of new observables $\{N(f), N(f'), N(ab), N(a), N(b)\}$ over to the classical observables $\{f, f', ab, a, b\}$, observables $\{N(f), N(f'), N(ab), N(a), N(b)\}$ are obtained by evaluating $N(f)$, $N(f')$, $N(ab)$, $N(a)$, $N(b)$ at $f$, $f'$, $ab$, $a$, $b$ respectively.

To summarize, the construction of new observables is done as follows:

\begin{align*}
N(f) &= \text{expression involving $f$} \\
N(f') &= \text{expression involving $f'$} \\
N(ab) &= \text{expression involving $ab$} \\
N(a) &= \text{expression involving $a$} \\
N(b) &= \text{expression involving $b$}
\end{align*}

where, $f$, $f'$, $ab$, $a$, $b$ are the original classical observables.
f_{ab} and \( I \) satisfying \( f_{ab}n^a = 0 \), and \( f_{ab}g^{ab} = 0 \).

Denote by \( \mathcal{S} \), the space of \( C^\infty \), real-valued tensor fields \( f_{ab} \) on \( I \) with the above algebraic symmetries all of whose components in a \( u, \theta, \phi \) chart, together with all their derivatives, fall off faster than \( 1/|u|^n \), for any \( n \), as \( u \to \infty \). For any \( f_{ab} \) in \( \mathcal{S} \), define an observable \( N(f) \) on \( \Gamma \) as follows:

\[
(N(f))(\{D\}) = -\frac{1}{8\pi} \int_I N_{ab} f_{cd} \, q^{ac} q^{bd} \, d^3I
\]

(C.14)

where \( N_{ab} \) is the news tensor field of \( \{D\} \). Using the expression (B.52) of the symplectic structure \( \Omega \), it is straightforward to compute the Hamiltonian vector field \( X_N(f) \):

\[
X_N(f)|_{\{D\}} = f_{ab}(u, \theta, \phi)
\]

(C.15)

Thus, because of the affine structure of \( \Gamma \), it is precisely the smeared-out news observables that constitute the preferred class, to be directly carried over to the quantum theory. Let us compute Poisson brackets between these observables. We have:

\[
\{N(f), N(g)\}_{PB} = \frac{1}{8\pi} \int_I (f_{ab} L_n \, \delta_{cd} - g_{ab} L_n f_{cd}) \, q^{ac} q^{bd} \, d^3I
\]

\[
= \Omega(f, g)
\]

(C.16)

where, in the last step, we need not specify the point in \( \Gamma \) where \( \Omega(f, g) \) is evaluated since \( \Omega, f \) and \( g \) are all constant w.r.t. the affine structure of \( \Gamma \), i.e., since \( \Omega(f, g) \) is a constant function on \( \Gamma \).

To construct the algebra \( \mathcal{A} \) of quantum operators, therefore, we proceed as follows. Introduce on \( I \) an operator valued distribution \( \mathcal{N}_{ab}(u, \theta, \phi) \) - to be called the news operator - subject to the canonical commutation relations:

\[
\{N(f), N(g)\} = \frac{\hbar}{i} \Omega(f, g) \mathcal{X}
\]

(C.17)

where, \( \mathcal{N}(f) := \int_I \mathcal{N}(x) \, f_{ab}(x) \, d^3I \), is the smeared-out news.
operator and \( I \) is the identity operator. The CCR (C.17) are equivalent to:

\[
[N_{ab}(u,\theta,\phi), N_{cd}(u',\theta',\phi') - \frac{i}{\hbar} q_a(c_q d_b) \delta(\theta,\theta';\phi,\phi') \Delta(u,u') I \quad (C.17)
\]

where, as in (C.5), \( \Delta(u,u') \) is the step function. Thus \( N_{ab} \) plays the same role in Einstein's theory as \( \epsilon_a \) plays in Maxwell's, and \( \{D\} \) or, if an origin \( \{D'\} \) is chosen, the "coordinate" \( q_{ab} \) of \( \{D\} \) (see discussion following Eq. (B.52)) - plays the same role as the vector potential \( A_a \). The algebra \( A \) of quantum operators is the \( \star \)-algebra generated by the smeared out operators \( W(f) \) as usual.

We can now construct the Fock representation of \( A \). The procedure is completely analogous to that followed in Sec. 1. Given any element \( \epsilon_{ab}(u,\theta,\phi) \) in \( S \), its Fourier transform, \( \tilde{\epsilon}_{ab}(u,\theta,\phi) \), w.r.t. \( u \) is again in the Schwartz space. Set

\[
\epsilon^+_{ab} = \left( \frac{i}{\hbar} \right)^{1/2} \int \tilde{\epsilon}_{ab}(u,\theta,\phi) \sin u \, du \quad (C.18)
\]

and introduce, on the complex vector space of these positive frequency fields, the following hermitean inner product:

\[
\langle \epsilon^+_{ab}, \epsilon^+_{cd} \rangle := \frac{1}{\hbar} \Omega(\epsilon^+_{ab}, \epsilon^+_{cd}) \quad (C.19)
\]

\[
= \frac{1}{\hbar^2} \int d\phi \, d\theta \sin \theta \int du \, \text{exp}(u) (\tilde{\epsilon}_{ab})^\star (\tilde{\epsilon}_{cd}) \quad (C.19')
\]

where \( \epsilon^+_{ab} = (\epsilon^+_{ab})^\star \) is the negative frequency part of \( \epsilon_{ab} \). Denote by \( H \) the Cauchy completion of the complex pre-Hilbert space so obtained. Elements of \( H \) represent the 1-graviton states. Let \( F \) denote the symmetric Fock space based on \( H \). Then, associated with each 1-graviton state, i.e. element of \( H, \epsilon^+_{ab} \), there is a creation operator \( C(f^+) \) and an annihilation operator \( A(f^+) \) defined on \( F \). We have:

\[
[A(f^+), C(f^')] = \langle f^+, f' \rangle \quad \text{or} \quad [A(f^+), \tilde{C}(f^+)] = \langle f^+, f' \rangle
\]

We can now define the \( \{D'\} \) representation of the \( A \) algebra:

\[
[A(f^+), \tilde{C}(f') = \langle f^+, f' \rangle \quad (C.17)
\]

Let us introduce:

\[
[A+f^+], \tilde{C}(f') = \langle f^+, f' \rangle
\]

where \( \langle f^+, f' \rangle \) is the positive frequency projection:

\[
\langle f^+, f' \rangle = \frac{1}{\hbar^2} \int d\phi \, d\theta \sin \theta \int du \, \text{exp}(u) (\tilde{\epsilon}_{ab})^\star (\tilde{\epsilon}_{cd}) \quad (C.19')
\]

The mass and the negative frequency projection is the negative part of the mass and the negative frequency projection:

\[
[A+f^+], \tilde{C}(f') = \langle f^+, f' \rangle
\]

The Schwartz space is based on \( H \) and the \( \{D'\} \) representation:

\[
[A+f^+], \tilde{C}(f') = \langle f^+, f' \rangle
\]

We can also define the negative frequency projection:

\[
[A+f^+], \tilde{C}(f') = \langle f^+, f' \rangle
\]

This completes the algebra of quantum operators and the corresponding Fock representations.
on $F$. These are (densely defined) linear operators on $F$ satisfying:

\[ A(f^+ + \lambda g^+) := A(f^+) + \lambda^* A(g^+) \quad ; \quad C(f^+ + \lambda g^+) := C(f^+) + \lambda C(g^+) \]  
(\text{C.20})

\[ [A(f^+), A(g^+)] = 0 \quad ; \quad [C(f^+), C(g^+)] = 0 \]  
(\text{C.21})

\[ [A(f^+), C(g^+)] = \langle f^+, g^+ \rangle I = -\frac{i}{\hbar} \Omega(f^+, g^+)I \]  
(\text{C.22})

We can now specify the representation mapping $A$ from the *-algebra $A$ to the *-algebra of linear operators on $F$:

\[ A \circ N(f) := h [C(f^+) + A(f^+)] \]  
(\text{C.23})

Let us check that the mapping $A$ preserves the CCR (\text{C.17}). We have:

\[ [A \circ N(f), A \circ N(g)] = h^2 [C(f^+), A(g^+)] + h^2 [A(f^+), C(g^+)] \]

\[ = h^2 \left( \frac{i}{\hbar} \Omega(g^-, f^+) - \frac{i}{\hbar} \Omega(f^-, g^+) \right) \]

\[ = i \hbar \left( \Omega(g^-, f) + \Omega(g^+, f) \right) \]

\[ = \frac{i}{\hbar} \Omega(f, g) \]

\[ = A \circ [N(f), N(g)] \]  
(\text{C.24})

where, in the first step we have used (\text{C.21}), in the second, (\text{C.22}), in the third, the fact that the "symplectic product" $\Omega(g^-, f^+)$ between any two negative (or positive) frequency fields vanishes identically, and, in the fourth, anti-symmetry of $\Omega$ in its two arguments.

Having constructed the Fock representation of the CCR, let us compute the mass and spin of 1-particle states. For this, consider the action of the ENS group $H$ on the 1-particle Hilbert space $\mathcal{H}$. Since the ENS action leaves the Schwartz space $S$, the space of positive frequency fields $f^+_{ab}$ with $f_{ab}$ in $S$, and the symplectic structure $\Omega$ invariant, it preserves the inner-product (\text{C.19}). That is, $H$ provides us with an unitary representation of $\mathcal{H}$, and hence, also of any Poincaré' subgroup of $\mathcal{H}$. The idea is to use this unitary action to compute the values of the Casimir operators, mass and spin, of an arbitrarily chosen but fixed Poincaré' subgroup $P$ of $\mathcal{H}$. It will turn out that the values are independent of the specific choice, $P$. 

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To compute mass, we have to examine only the action of the translation subgroup \( T \) of \( B \). It is convenient to carry out the computation in a Bondi frame - i.e. to use a conformal factor such that the metric \( g_{ab} \) on the 2-sphere \( S \) of generators of \( T \) is a unit 2-sphere metric - although the final results are of course conformally invariant. With this choice, the \( X_0, X_1, X_2, \) and \( X_3 \) directional translations are represented, respectively, by the vector fields \( \alpha^a \cdot n^a \) on \( I \), where,
\[
\alpha_0 = 1, \quad \alpha_1 = \sin \theta \cos \phi, \quad \alpha_2 = \sin \theta \sin \phi, \quad \text{and} \quad \alpha_3 = \cos \theta.
\]
Hence, the mass-squared operator, \( M^2 \), is given by
\[
M^2 = \alpha^{ab} \alpha_{ab} = \left( -P_0^2 + P_1^2 + P_2^2 + P_3^2 \right) = \alpha^{ab} \alpha_{ab} = \left[ \frac{1}{r^2} (1 - \sin^2 \theta \cos^2 \phi + \sin^2 \sin^2 \phi + \cos^2 \theta) \right] \sim f_{ab}^+ f_{ab}^+ = 0.
\]

Thus, \( H \) represents quantum helicity states of zero rest mass particles. We therefore have to compute their helicity rather than spin. Consider the subspaces \( H^+ \) of \( H \) containing fields \( f^+_{ab} \) satisfying
\[
\epsilon^{mnlp} p_{mn} f^+_{lm} = i f^+_{ab}
\]  
(C.26)
where \( p_i \) is any covector field on \( I \) such that \( p^P_i = 1 \). (The condition (C.26) is independent of the specific choice of \( p_i \). Also, note that the left side is automatically symmetric in \( a \) and \( b \) because \( f^+_{ab} \) is trace-free.) By inspection, \( H^+ \) and \( \bar{H} \) are separately invariant under the action of \( B \), and hence, also of \( F \). Thus, the unitary representation of \( F \), provided by \( H \) is reducible.

It turns out that each of \( H^+ \) provides an irreducible representation, corresponding to helicities \( \pm 2 \).

The spin operator is defined by
\[
S^a := \frac{i}{2} \epsilon^{abcd} P_b M_{cd}
\]  
(C.27)
where \( P_b \) is, as before, the 4-momentum operator, and, \( M_{cd} \), the angular momentum operators to \( P^a \) and \( F^a \), respectively.

To compute these, from \( S^a = f^+_{ab} \cdot \bar{S}^b \) where \( \bar{S}^b \) is the conjugate of \( S^a \). We have
\[
S^a = f^+_{ab} \cdot \bar{S}^b = \frac{1}{2} \left( \frac{1}{r^2} (1 - \sin^2 \theta \cos^2 \phi + \sin^2 \sin^2 \phi + \cos^2 \theta) \right) \sim f^+_{ab} \cdot \bar{S}^b = 0.
\]

Thus, the helicity operators are light-like vectors in \( \mathbb{R}^3 \).
momentum operator. For zero rest mass fields, $S^a$ is in fact proportional to $p^a$ and the helicity $s$ is defined by

$$S^a = s \ h \ p^a$$  \hspace{1cm} (C.28)

To compute $s$, therefore, it suffices to compute just one component, say $S^a$, of $S^a$. We have:

$$S^a = f_{\alpha \beta}^a = (\varepsilon^{0123} P_1 M_{23} + \varepsilon^{0231} P_2 M_{31} + \varepsilon^{0312} P_3 M_{12}) \cdot f_{\alpha \beta}^a$$

$$= (\frac{1}{c} a_1 L_n) (\frac{1}{c} R_1) + (\frac{1}{c} a_2 L_n) (\frac{1}{c} R_2) + (\frac{1}{c} a_3 L_n) (\frac{1}{c} R_3)$$

$$= \frac{1}{c} L_n \left[ a_1 L_1 + a_2 L_2 + a_3 L_3 \right] \cdot f_{\alpha \beta}^a$$  \hspace{1cm} (C.29)

where the vector fields $R_1^\alpha, R_2^\alpha, R_3^\alpha$ on $I$ are BMS symmetries representing rotations around $X_1, X_2, \text{and } X_3$ axes. One can evaluate the quantity in the square bracket in the last step of (C.29) by using the explicit expressions of $a_i$ and $R_i$ ($i=1,2,3$), to obtain

$$f_{\alpha \beta}^a = \varepsilon^{\alpha \mu \nu} \lambda^\mu_{\alpha} q_{n\beta} f_{\lambda \mu}^a + \varepsilon^{\alpha \mu \nu} \lambda^\mu_{\alpha} q_{n\alpha} f_{\lambda \mu}^a$$

$$= 2 \varepsilon^{\alpha \mu \nu} \lambda^\mu_{\alpha} q_{n\beta} f_{\lambda \mu}^a$$

$$= 2i f_{\alpha \beta}^a \quad \text{if} \quad f_{\alpha \beta}^a \in \mathbb{H}$$  \hspace{1cm} (C.30)

Substituting (C.30) into (C.29) we obtain

$$S^a = f_{\alpha \beta}^a = (\pm 2i) \frac{1}{c} L_n f_{\alpha \beta}^a$$

$$= \pm 2i \ h \ p^a \cdot f_{\alpha \beta}^a \quad \text{if} \quad f_{\alpha \beta}^a \in \mathbb{H}$$  \hspace{1cm} (C.31)

Thus the helicity of states in $\mathbb{H}^+$ is $+2$ and that of states in $\mathbb{H}^-$ is $-2$.

**Remarks:** 1) Fix any classical vacuum $\{D^a\}$ as the origin in the phase-space $\Gamma$. Then, any other point $\{D^a\}$ of $\Gamma$ can be labelled by a symmetric, second rank, trace-free tensor field, transverse to $n^a$. Hence, each positive frequency field $f_{\alpha \beta}^a$ in $\mathcal{H}$ can be regarded as defining an equivalence class of complex connections. In the Newman–Penrose notation summarized in Sec. 5 of ch. II.B, $f_{\alpha \beta}^a$ can be interpreted as the complex shear tensor. The corresponding (complex) Bondi news tensor is given by $N_{ab} = (-2) \frac{1}{c} L_n f_{ab}$. The helicity condition is easy to translate: $f_{\alpha \beta}^a$ is of positive helicity iff
and of negative helicity iff \( f_{ab}^{a \alpha b} = 0 \). One of the most remarkable results obtained in recent years in mathematical relativity is that one can use precisely such helicity states - i.e. complex tensor fields \( f_{ab} \) with algebraic symmetries of the shear tensor, satisfying \( f_{ab}^a b = 0 \) (respectively, \( f_{ab}^b = 0 \)) - as "initial data" on \( \mathcal{I} \) to obtain a complex, self-dual (respectively, anti self-dual) solution to the full, non-linear Einstein's equation.\(^{51}\) These complex space-times are called \( \mathcal{H} \)-spaces. If the field \( f_{ab} \) happens to be of positive frequency at \( \mathcal{I} \), the solution is referred to as an asymptotically flat, non-linear graviton.\(^{52}\) More recently, an astonishing property of asymptotically flat \( \mathcal{H} \)-spaces was discovered: It was shown that, in spite of non-linearities of the field equations, the classical scattering matrix is trivial; the data sets, \( f_{ab} \), on \( \mathcal{I}^- \) and \( \mathcal{I}^+ \) of an asymptotically flat \( \mathcal{H} \)-space are related to one another exactly as in the linearized version of Einstein's theory, although in the "interior" of course, the \( \mathcal{H} \)-space bears no resemblance to the linearized solution.\(^{53}\) This "asymptotic solitonic behavior" of \( \mathcal{H} \)-spaces suggests that one should perhaps regard them as fundamental entities, say "dressed" gravitons of a given helicity, and approach the problem of quantum dynamics as the problem of interaction between \( \mathcal{H} \)-spaces of one helicity and those of another. Such program would complement very nicely the kinematic framework provided by the asymptotic quantization scheme outlined above. However, up to now, very little concrete progress has been made along these lines.

\( \text{ii) In ch. II.B, we saw that the action of the BMS group } \mathcal{B} \text{ on } \Gamma \text{ preserves the symplectic structure } \Omega, \text{ and gave expressions of the Hamiltonians generating the corresponding canonical transformations. (Eqs. (B.56) - (B.57).) These classical observables can be readily promoted as quantum operators on the Fock space } \mathcal{F}; \text{ one has only to use the normal ordering}\)
prescription to regularize the products of operator-valued distributions that appear in the expressions. Thus, the BMS group can be realized as a symmetry group also in the quantum theory. Let us examine the generator (B.57) of translations. It can be interpreted as the asymptotic energy momentum operator; its expectation values give us a measure of the energy momentum carried away by (asymptotic) gravitons. Note, however, that since \( I \) represents null infinity rather than a space-like Cauchy surface, the time evolution generated by these Hamiltonians has a somewhat unconventional interpretation. For simplicity, let us consider the classical theory. Let \((\hat{\mathcal{H}}, \hat{\mathcal{E}})\) be asymptotically flat at null infinity and induce the equivalence class \( [D] \) of connections of \( I \). Let \( t^a \) be a smooth, time-like vector field on \( M \) whose extension to \( I \) is a BMS time-translation \( \tau^a \). Since \( \tau^a \) is tangential to \( I \), it generates a 1-parameter group of motions within \( I \).

Under the action of this group, \( [D] \) is mapped to a 1-parameter family of equivalence classes of connections; on \( \Gamma \), we obtain a parametrized curve passing through \( [D] \). The points of this curve represent the radiative modes of the 1-parameter family of metrics on \( \hat{\mathcal{H}} \), obtained from \( \hat{\mathcal{E}} \) by the action of the 1-parameter group of diffeomorphisms generated by \( \tau^a \) on \( M \). The parametrized curve within \( \Gamma \) is an integral curve of the Hamiltonian vector field generated by (B.57). Thus, although the Hamiltonian does generate a time-translation on \( \Gamma \) in a well-defined sense, it does not directly let one evolve off \( I \). The situation is completely analogous in the quantum theory.

It is for this reason that we only have a kinematic framework and need new ideas - such as the possibility of using \( N \)-spaces discussed above - to obtain the quantum S-matrix.

iii) It is well known that the notions of self-duality (of curvature) and helicity are intimately intertwined: self-duality corresponds to positive
helicity and anti self-duality to negative. (For a clear derivation, without the use of Fourier transforms, see Ref. 10.) One would therefore expect that, in full quantum gravity, one would have to use both self-dual and anti self-dual fields. We wish to point out that this is not necessarily the case: one can in fact incorporate gravitons of both helicities just from self-dual or anti self-dual fields. (This point will play a significant role in Part III of these notes where self-duality will be imposed as the "polarization condition" à la geometric quantization in the passage from the classical to the quantum theory.) To see how this comes about, let us examine the notions of self-duality and helicity separately. The duality operation \( F_{ab} \rightarrow \ast F_{ab} = \frac{i}{2} \epsilon_{abmn} F_{mn} \) for the Maxwell tensor and \( C_{abcd} \rightarrow \ast C_{abcd} = \frac{i}{2} \epsilon_{abcd} \ast C_{abcd} \) for the Weyl tensor - can be carried out at the classical level; it makes no reference to the structure introduced in the passage to quantum theory. The calculation of helicity, on the other hand, depends on the unitary representation of the Poincare' group on the Hilbert space of 1-particle states. In the discussion above, we chose to represent 1-particle states by positive frequency fields \( f_{ab}^+ \). What would have happened if we had chosen to work with negative frequency fields instead? In order to maintain positivity of energy (i.e. of the spectrum of \( P^2 \)), we would have been forced to change the signs of operators generating Poincare' transformations on the 1-graviton Hilbert space \( H \); the self-adjoint generator corresponding to a Poincare' vector field \( V^a \) on \( I \), acting on negative frequency fields \( f_{ab}^- \), would be \( i\hbar \underline{L}^V \ast f_{ab}^- \) (rather than \( \frac{i}{2} \underline{L}^V \ast f_{ab}^- \)). Consequently, the operators \( P_a \) as well as \( \mathcal{H}_{ab} \) would change sign. This change does not affect the mass operator \( M^2 = P_a p_a \). However, it does change the sign of helicity \( s \) defined by

\[
\mathcal{S}_a = \frac{i}{4} \epsilon_{abcd} p_b \mathcal{H}_{cd} \equiv \pm p_a
\]

(C.32)
That is, in the unitary representation on negative frequency fields, "self-dual" fields $f_{ab}^\gamma$ satisfying:
\[ \varepsilon^{\alpha\beta\gamma\delta} g_{\alpha\gamma} f_{\beta\delta}^\gamma = i f_{ab} \]  \hspace{1cm} (C.33)

have negative helicity rather than positive. The situation is summarized in the following table.

<table>
<thead>
<tr>
<th></th>
<th>Self-Dual</th>
<th>Anti Self-Dual</th>
</tr>
</thead>
<tbody>
<tr>
<td>+ve frequency</td>
<td>+ helicity</td>
<td>- helicity</td>
</tr>
<tr>
<td>-ve frequency</td>
<td>- helicity</td>
<td>+ helicity</td>
</tr>
</tbody>
</table>

Thus, one can work just with self-dual fields, satisfying (C.33) on $J$ and still have gravitons of both helicities if one admits fields of both positive and negative frequencies.

3. Gravitational Field: Infrared Sectors

The treatment of infrared problems in the gravitational case will be similar to that in the electromagnetic case; the apparent disparity will arise only from the technical differences in the strategies adopted in the construction of Fock representations in the two cases.

Let us therefore begin with a discussion of these differences. Since the Maxwell theory is Abelian, in the free-field case one can work entirely with field strengths without any reference to potentials (i.e., U(1) connections).
In the asymptotic framework, the phase-space of radiative modes can be constructed from the radiative data, $\gamma_a$, for fields and one can write out asymptotic field equations without any reference to potentials. (If needed for computational convenience, the potentials $A_a$ can be introduced as derived quantities, e.g., via $\mathcal{L}_nA_a = \gamma_a$ in the $A_a \delta^n = 0$ gauge.) From physical considerations such as finiteness of energy, one is led to require that the elements $\gamma_a$ of the phase space should go to zero at an appropriate rate as $u \to \infty$. In the quantum theory, we used the radiative data to construct the $1$-photon states: the fields $V_a(u,\theta,\phi)$ in the Schwartz space $S$ with which we began represent characteristic data for field strengths, rather than for their potentials. In the gravitational case, the situation is different due to the "non-Abelian character" of the field. In the asymptotic framework, the phase-space $\Gamma$ of radiative modes was constructed from equivalence classes of connections, $\{D\}$, rather than from field strengths, the Bondi news tensors, $N_{ab}$; one cannot write down even the asymptotic field equations ((B.26) - (B.30)) if one does not have access to a connection. The boundary conditions on connections are of course different from those on field strengths. For example, the finiteness of energy requires only that the equivalence class $\{D\}$ approach some classical vacua, $\{D^+\}$ and $\{D^-\}$ as $u \to \infty$. Thus, if we fix a classical vacuum $\{D^0\}$ as the origin in the affine space $\mathcal{N}$, the shear $\sigma$ in the Newman-Penrose notation$^{43}$ (Sec. 5, ch. II.B) will only remain bounded, and, in general, not tend to zero as $u \to \infty$. (The news tensor $N_{ab}$ of any $\{D\}$ in $\mathcal{N}$ will go to zero at least as $1/u^3$ as $u \to \infty$. See footnote 28.) In the quantum theory, the Hilbert space $H$ of 1-graviton states was constructed from fields $f_{ab}$ in the Schwartz space $S$. However, $f_{ab}$ now represent constant vector fields on $\Gamma$, i.e., (differences of) connections, rather than field strengths. This is why the fields $f_{ab}$ (C.7') and the tensor $\bar{\xi}_{\alpha \beta}^{+\mu}$ (C.7) are $2 \times 2$ matrices.

Classically, $\bar{\xi}_{\alpha \beta}^{+\mu}$ is defined to be zero off at a sufficient distance, e.g., $\xi_{\alpha \beta}^{+\mu} = 0$ in a certain configuration of physical fields at the origin $\{D^0\}$ in $\Gamma$.

Let us consider the reverse process. Let us take a well-defined set of constant frequency particle waves $\{\Phi_j\}$ that do so if $\{\Phi_j\}$ is a complete orthonormal set. This is the news tensor $\bar{\xi}_{\alpha \beta}^{+\mu}$ where, as before,

$$\xi_{\alpha \beta}^{+\mu} = \sum_j \Phi_j^{\dagger} \bar{\xi}_{\alpha \beta}^{+\mu} \Phi_j.$$

Since, by assumption, the transformation $\bar{\xi}_{\alpha \beta}^{+\mu} \to \bar{\xi}_{\alpha \beta}^{+\mu}$ is a bounded end. However, $\bar{\xi}_{\alpha \beta}^{+\mu}$ is a bounded end unless $\bar{\xi}_{\alpha \beta}^{+\mu}$ is a bounded end for all $\{\Phi_j\}$. The integral is well-defined in the $\{\Phi_j\}$ expansion near the boundary. By the elements of the $\{\Phi_j\}$, we have: $\bar{\xi}_{\alpha \beta}^{+\mu}$ is...
is why the factor \( w \) appears in different places in the photon inner product (C.7') and the graviton inner-product (C.19').

We are now ready to discuss the gravitational infrared problems.

Classically, any equivalence class \((D)\) whose curvature \((N_{ab} \text{ and } K_{ab})\) falls off at a sufficiently fast rate as \( u \to \infty \) represents a "respectable" classical configuration. Do all such configurations define a 1-graviton state? Fix an origin \((D^0)\) in \( \Gamma \) and represent equivalence classes \((D)\) in \( \Gamma \) by the tensor fields \( f_{ab} = (D) - (D^0) \). Denote by \( F_{ab} \) the news tensor of \( f_{ab} \); \( F_{ab} := -\frac{2\pi}{n} f_{ab} \).

Let us consider a fixed tensor field \( f_{ab} \) such that \( F_{ab} \) belongs to the Schwartz space \( S \). The corresponding \((D)\) is in \( \Gamma \) and does represent a "respectable" classical configuration; in particular, all the BMS Hamiltonians (B.51) are well-defined at such a configuration. Let us now ask: does the positive frequency part, \( f_{ab}^+ \), of \( f_{ab} \) belong to the 1-graviton Hilbert space \( H_1 \)? It does so iff its norm (C.19) is finite. Rewriting this expression in terms of the news tensor \( F_{ab} \) we have:

\[
\int_0^\infty |F_{ab}(u, \theta, \phi)|^2 \, du < \infty \quad (C.34)
\]

where, as before, \( \tilde{F}_{ab}(u, \theta, \phi) \) is the Fourier transform of \( F_{ab}(u, \theta, \phi) \) w.r.t. \( u \).

Since, by assumption, \( F_{ab}(u, \theta, \phi) \) is in the Schwartz space, so is its Fourier transform \( \tilde{F}_{ab}(u, \theta, \phi) \). Hence the integral is convergent at the high frequency end. However, it is clearly logarithmically divergent at the low frequency end unless \( \tilde{F}_{ab}(u, \theta, \phi) \) vanishes at \( u = 0 \). If \( \tilde{F}_{ab}(0, \theta, \phi) \) does vanish, the integral is well-defined because \( \tilde{F}_{ab}(u, \theta, \phi) \), being \( C^0 \), admits a Taylor expansion near \( u = 0 \). Finally, using the fact that the Fourier transform maps elements of the Schwartz space in \( u \) to those in the Schwartz space in \( u \), we have:

\[
\int_0^\infty F_{ab}(u, \theta, \phi) \, du \equiv (2\pi)^\frac{n}{2} \tilde{F}_{ab}(0, \theta, \phi) = 0 \quad (C.35)
\]
i.e., iff,

\[ [f_{ab}] := [f_{ab}]_{u \to -\infty} = 0 \]  \hspace{1cm} (C.36)

From a classical viewpoint, this is a severe restriction on the configuration \( \{D\} \) in \( \Gamma \) defined by \( f_{ab} \). (C.36) requires that \( \{D\} \) should tend to the same classical vacuum as \( u \to +\infty \) as it does in the limit \( u \to -\infty \), i.e., that \( \{D^0\}^+ = \{D^0\}^- \). In the linearized approximation, for example, the radiation fields from sources will not satisfy this requirement if there is a non-trivial scattering. In the full theory the situation can be only worse. It would appear that back-scattering (or, lack of Huygen's principle) will lead to a violation of this requirement even in absence of sources. Thus, new tensors satisfying (C.35) on both \( I^+ \) and \( I^- \) are not likely to occur in classical scattering. Consequently, one expects that, due to infrared problems, the quantum \( \hat{S} \)-matrix will not be well defined between the incoming Fock space on \( I^- \) and the outgoing one on \( I^+ \).

Let us therefore construct non-Fock representations of the CCR (C.17) which are likely to occur in the \( \hat{S} \)-matrix description. Fix a classical vacuum \( \{D^0\} \) and consider only the space \( \Gamma_0 \) of connections \( \{D\} \) in \( \Gamma \) which tend to \( \{D^0\} \) in the distant past, i.e., in the limit \( u \to -\infty \), and whose new tensor belongs to the Schwartz space \( S \). The condition \( \{D\} \to \{D^0\} \) as \( u \to -\infty \) is analogous to the gauge condition \( A \_a \to 0 \) as \( u \to -\infty \) in the Maxwell theory and can be imposed without loss of generality. To see this, consider any asymptotically flat space-time \( (\mathbb{R}, \hat{g}_{ab}) \) and isomorphisms between the (say future) null infinity of \( (\mathbb{R}, \hat{g}_{ab}) \) and abstract null infinity \( I \). Recall that there are as many (universal structure preserving) isomorphisms as there are elements of the BMS group. We can use this freedom to ensure that the image of the vacuum \( \{D^0\}^- \) to which the connection \( \{D\} \) on the future null infinity of \( (\mathbb{R}, \hat{g}_{ab}) \) tends in the distant past is the given \( \{D^0\} \) on the abstract \( I \). Thus, we do
not exclude any physically interesting space-times by requiring \( \{D\} \rightarrow \{D^0\} \) in the distant past. Note, however, that unless the news tensor of the physical space-time satisfies the stringent condition (C.35), the vacuum \( \{D^0\}^+ \) to which \( \{D\} \) will tend in distant future will not be \( \{D^0\} \). The difference, \( \{D^0\}^+ - \{D^0\} \), is characterized by the tensor

\[
Q_{ab}(\theta,\phi) = [f_{ab}]_{u=\infty}^{u=-\infty}
\]

where \( f_{ab} \) is the coordinate label of \( \{D\} \) in \( \Gamma \) w.r.t. an arbitrarily chosen but fixed origin. (Note that \( Q_{ab} \) is symmetric, trace-free and transverse to \( n^a \).

Hence it is completely characterized by the complex function \( Q(\theta,\phi) := m^{ab}Q_{ab}(\theta,\phi) \). In the Newman-Penrose notation, \( Q \) is the difference between the asymptotic shears of \( \{D\} \). Note that, unlike shear itself the difference is independent of the Bondi slicing used in its construction.) Introduce on \( \Gamma_o \) the following equivalence relation:

\[
\{D\} \approx \{D'\} \quad \text{iff} \quad Q_{ab} - Q'_{ab} = 0
\]

Each equivalence class defines an affine subspace of \( \Gamma_o \), labelled by the value of \( Q_{ab}(\theta,\phi) \). It is only the subspace with \( Q_{ab}(\theta,\phi) = 0 \) that gives rise to 1-graviton states in the Fock representation. To obtain the non-Fock representations of the CCR, consider the automorphism \( \Pi_f \) on the \( \ast \)-algebra \( A \) generated by the news operator-value distribution \( N_{ab} \):

\[
\Pi_f N_{ab}(u,\theta,\phi) = N_{ab}(u,\theta,\phi) + f_{ab}(u,\theta,\phi) I
\]

where \( f_{ab} \) is any \( C^1 \) symmetric, trace-free field, transverse to \( n^a \). Can this automorphism be unitarily implemented in the Fock representation? Suppose it is. That is suppose there exists an unitary operator \( U \) on \( \mathcal{F} \) such that

\[
U B U^{-1} = A \circ \Pi_f \circ B
\]

for all \( B \) in \( A \). Choose for \( B \) the annihilation operator \( A(h^+) = \langle h^+, \bar{\omega} \rangle \) for some 1-graviton state \( h^+ \) and apply (C.40) to the image, \( U \mid 0 \rangle \), of the Fock
vacuum under $U$. We obtain:
\[ 0 = \langle A(h^+) + h^+ f^+ \rangle |0\rangle \]  \hspace{1cm} (C.41)
for all $h^+ \in H$, whence $U|0\rangle$ must be given by
\[ U|0\rangle = N[\exp C(f^+)]|0\rangle \]  \hspace{1cm} (C.42)
where $N$ is the normalization factor which is to make the right side of (C.42) a state with unit norm. A direct computation yields:
\[ \|\exp C(f^+)|0\rangle\| = \exp\langle f^+, f^+ \rangle, \]  \hspace{1cm} (C.43)
whence $U|0\rangle$ is in the Fock space iff $\langle f^+, f^+ \rangle < \infty$, i.e., iff $f_{ab}(\theta, \phi)$ belongs to the $Q_{ab}(0, 0) = [f_{ab}] = 0$ sector. (In fact, one can show that $\langle f^+, f^+ \rangle < \infty$ is the necessary and sufficient conditions for $U$ to exist.) Thus, if we choose $f_{ab}$ in the trivial sector, $U|0\rangle$ is the coherent state defined by the 1-particle state $f_{ab}^+$. If $f_{ab}$ is in a non-trivial sector, $U|0\rangle$ still "wants to be" the coherent state corresponding to $f_{ab}^+$, and is "forced out" of the Fock space simply because $f_{ab}^+$ is not a 1-graviton state in $H$. We can use this state which lies outside the Fock space to construct a new representation of the CCR (C.17). Define an expectation-value function $\langle 0 | f | 0 \rangle_f$ on the *-algebra $A$ via:
\[ \langle 0 | f | 0 \rangle_f := \langle 0 | A \circ f = B | 0 \rangle_f. \]  \hspace{1cm} (C.44)
It is easy to check that (C.44) defines a positive linear function on $A$, (i.e., a linear mapping from $A$ to the complexes which sends any element of $A$ of the type $B^* B$ to a non-negative, real number.) Hence, using the Gelfand-Naimark-Segal construction, one can obtain a representation $A_f$ of the *-algebra $A$ which is unitarily inequivalent to the Fock representation.

Let us choose two fields $f_{ab}$ and $h_{ab}$ in the same sector, i.e., such that $|f_{ab}| = |h_{ab}| = Q_{ab}(0, \phi)$. How are the two representations, $A_f$ and $A_h$ related? Since $|f_{ab} - h_{ab}| = 0$, it follows that $f_{ab} - h_{ab}$ belongs to the 1-graviton Hilbert space $H$. This in turn implies that the representations $A_f$ and $A_h$ is

unitarily equivalent to the same representations labeled $f_{ab}$ or $h_{ab}$, respectively. Thus, $A_f$ is an automorphism of the CCR algebra constructed from $f_{ab}$.

One can show that $Q_{ab}(0, \phi) = 0$ by using the discussion above. Therefore, the CCR algebra is well-defined in the Cauchy domain $D$.

The conclusion is that in the Cauchy domain, all divergent integrations do yield finite results without divergences. However, in the well-defined domain of the gravitont field, the charge, which is well-defined, belongs to $D$. Therefore, it is the domain of the Cauchy integral which is the source of the paradoxes.
unitarily equivalent to the Fock representation $A$ and $A_F$ is unitarily equivalent to $A_H$. Thus, up to unitary equivalence, the representations are in fact labelled by the value of $Q_{ab}(\theta, \phi)$ rather than by individual configurations, $f_{ab}$ or $h_{ab}$, in the $Q_{ab}(\theta, \phi)$-sector. Consequently, in a non-Fock representation, $A_Q$ there is no natural vacuum state: If $A_Q$ is constructed using the automorphism $\Pi_F$, $|0\rangle_F$ plays the role of the quantum vacuum while if it is constructed using $\Pi_H$, a different state, $|0\rangle_H$ plays the role of the vacuum. One cannot use the Hamiltonian operator to single out the vacuum because, if $Q_{ab}(\theta, \phi) \neq 0$, the $A_Q$ sector does not admit an eigenstate of the Hamiltonian with the discrete eigenvalue zero; zero is only the lower bound of the continuous spectrum. Using a holomorphic-function (or Bargmann) representation $^4$ of the CCR, one can visualize various representations as follows: quantum states in the $A_Q$ representations can be regarded as holomorphic functions $^5$ on (the Cauchy completion of) the $Q_{ab}(\theta, \phi)$ sector of $F$. 

The status of the $S$-matrix theory in gravity is of course less clear than that in electrodynamics because no one knows how to handle the ultraviolet divergences of the theory perturbatively. In addition, there are complications due to the non-Abelian nature of the theory. In the Maxwell case without sources, one can restrict oneself to Fock representations and get a well-defined - in fact, trivial - $S$-matrix between $I^-$ and $I^+$. In the gravitational case, since the gravitons themselves carry gravitational charge, i.e. energy momentum, one does not expect the outgoing states to belong to the Fock space even if the incoming do. In quantum electrodynamics, it is the asymptotic incoming and outgoing momenta of the charged particles - which are distinct from photons - that determine the infrared sectors to which the incoming and outgoing Maxwell states belong. In pure gravity without sources, gravitons are analogous to both charged particles and photons because
they themselves are carriers of the gravitational charge; the two roles get mixed up. Hence it is difficult to imagine the analog of the separable Hilbert space \( \mathcal{H}_Q \otimes |F_1, \ldots, F_n\rangle \) used in quantum electrodynamics. It would seem that one would have to use the (non-separable) Hilbert space consisting of a direct sum of all \( Q_{ab}(\theta, \phi) \)-sectors for incoming as well as outgoing states. Finally, even if one were to use these large spaces, because of ultraviolet problems, it is far from being clear that we will obtain a \( S \)-matrix which is well-defined in a suitable sense. What we have done is to proceed by analogy with electrodynamics. There, the need for infrared sectors for photons became obvious in the \( J \) framework because we could translate the condition for finiteness of norm of 1-photon states directly in terms of fields on \( J \), rather than their Fourier transforms. It then turned out that these sectors were just the correct ones for the \( S \)-matrix to be well-defined. In the gravitational case, the need for infrared sectors was suggested by the \( J \)-framework in a completely analogous way. We are therefore led to conjecture that these sectors would be essential in the \( S \)-matrix description.

We conclude this section by pointing out that the emergence of the infrared sectors in the quantum theory is intimately intertwined with the enlargement of the asymptotic symmetry group from the Poincare' group to the BMS. For the purpose of this discussion it is convenient to reverse the "quantization" strategy followed so far and take the classical limit of the quantum theory. Imagine, for a moment, that the Fock representation sufficed in the quantum theory. Since coherent states in the Fock space \( F \) are peaked at those classical configurations whose positive frequency parts have a finite l-particle norm, in the classical limit we would obtain only those configurations \( \{D\} \) on \( J \) for which \( \{D^{+}\} \) in the asymptotic future is same as the classical vacuum \( \{D^0\} \) that we fixed in the distant past. Therefore, on, say,
\( f \), we would have available a \textit{preferred} vacuum, whence, from the discussion in Sec. 5 of ch. II.B, the BMS group would be reduced to the Poincare'. Unfortunately, due to infrared problems, we cannot restrict ourselves to the Fock space \( F \); we have to allow \( Q_{\mathbb{R}}(\theta, \phi) \)-sectors. Now, the quantum Hilbert space does admit "coherent states" peaked at configurations \( \{ D \} \) for which \( \{ D \}^+ \neq \{ D \}^\circ \).

Therefore, we expect that such configurations will arise also in the classical theory. For these configurations, there are \textit{two distinct} ways of reducing the BMS group to the Poincare', one using \( \{ D \}^\circ \) (i.e. the structure at \( i^- \)) and the other using \( \{ D \}^+ \) (i.e. the structure at \( i^+ \)). Since \( \{ D \} \neq \{ D \}^\circ \), the two Poincare' subgroups of the BMS group are distinct. Thus, the enlargement of the symmetry group from the 10-dimensional Poincare' group to the infinite dimensional BMS is an imprint, left on the classical theory, by the infrared behavior of the quantum gravitational field. Physically, we can picture this enlargement as follows. Gravitons, being sources of gravitational charge, are themselves surrounded by "soft gravitons". This phenomenon is not peculiar to gravitation; it is shared by all non-Abelian fields. However, in the gravitational case, the field also determines the space-time geometry. The clouds of soft gravitons correspond to ripples in the geometry which do not go away even asymptotically, thereby making the asymptotic symmetry group different from the isometry group of the flat space.
Fig. 1 - Conformal completion of Minkowski space. We have suppressed the angular variables for simplicity. The conformal boundary, $\Omega = 0$, has three parts: $I^+$, future null infinity, consists of future end-points of null geodesics; $I^-$, past null infinity, consists of past end-points of null geodesics; and, the point $i^0$ denotes spatial infinity.

Fig. 2 - Conformal imbedding of Minkowski space in Einstein cylinder. The shaded region is the interior of Minkowski space.
References and Notes


3. As Roger Penrose puts it, "if we remove life from Einstein's beautiful theory by steam-rollering it first to flatness and linearity, then we shall learn nothing from attempting to wave the magic wand of quantum gravity over the resulting corpse"; Gen. Rel. and Grav. 7, 31 (1976).


6. This idea first appeared in the literature in the early sixties (see, in particular, R. K. Sachs, Phys. Rev. 128, 2851 (1962), and A. Komar, Phys. Rev. 134, B1430 (1964)). More recent contributions in this direction have come through the H-space theory developed by E. T. Newman and collaborators. A complete, systematic analysis was carried out by A. Ashtekar, Phys. Rev. Lett. 46, 573 (1981); J. Math Phys. 22, 2885 (1981); and, in: Quantum Gravity 2, volume cited in Ref. 5.


8. A. Ashtekar and K. S. Narain, to be published.


15. To incorporate the point \( i^0 \) at spatial infinity and the points \( i^\pm \) at timelike infinity, one needs a conformal factor \( \Omega \) with asymptotic behavior \( \Omega \sim 1/\nu \), rather than \( \Omega \sim 1/r \). With the conformal factor behaving as \( 1/r \), \( \mathcal{I}^0 \) are cylinders with intrinsic metrics given by (8.4). To incorporate \( i^\pm \) and \( i^0 \), one has to "pinch" the end-points of these cylinders, which is achieved by a change of \( \Omega \). The resulting completed manifold with boundary \( i^\pm \), \( \mathcal{I}^\pm \), \( i^0 \), \( \mathcal{I}^- \), \( i^- \) is easily exhibited by imbedding it in the Einstein cylinder.


17. The pull-back, \( \xi_a \), to \( \mathcal{I} \) of \( E_a \) is a 1-form intrinsically defined on the 3-manifold \( \mathcal{I} \). Since \( \xi_a n^a = F_{ab}n^a n^b = 0 \), \( \xi_a \) has two independent components.


19. S. Chakravarty and E. T. Newman, (to be published). More precisely, the claim is that the components of \( F_{ab} \) in a chart which is well-behaved near the boundary \( \mathcal{I}^0 \), \( i^\pm \), \( i^0 \), has the property that they diverge at \( i^0 \) (but admit \( C^\infty \) limits at \( \mathcal{I}^0 \) and \( i^0 \)) if \( Q(\theta, \phi) \) fails to vanish. If \( q(\theta, \phi) \) does vanish, and \( \Phi \) is analytic on \( \mathcal{I} \), the solution is analytic and extends beyond the Minkowski space region of the Einstein cylinder analytically.


25. A. Ashtekar, last three papers in Ref. 6.

27. The derivation of (B.26) in Ref. 16 crucially used the fact that \( I \) is imbedded in \((M, g_{ab})\) and that the Weyl tensor \( C_{abcd} \) of \( g_{ab} \) vanishes on \( I \). It is therefore somewhat surprising that (B.41) could be derived on the abstract \( I \), without reference to any space-time.

28. We must also specify regularity conditions on connections \( \Gamma \). Off-hand, it appears that there is a considerable freedom here; the optimal choice will become clear only after the characteristic initial value problem on \( I \) is solved completely. For the present purpose we shall use the regularity conditions of Ashtekar and Stuebel, Proc. R. Soc. London A376, 585 (1981). Fix a global chart \((u, \theta, \phi)\) on \( I \) (where \( u \) satisfies \( \epsilon_{\theta} u = 1 \)). Consider the Frechet space \( F \) of \( C^0 \) second rank, symmetric, traceless tensor fields \( \sigma_{ab} \) transverse to \( n^a \) for which

\[
\|\sigma\| := \sup_{\sigma = 0} \frac{1}{2}\left[ \frac{1}{1 + (1 + u^2)\sqrt{\epsilon}} \frac{(\partial u_{\alpha})}{\partial u_d} (\partial_{\alpha} \sigma) \right] \rightarrow \infty
\]

where \( \alpha \) is a non-negative integer, \( \sigma_{\alpha\beta} \) denotes any combination of \( u, \theta, \phi \) derivatives of various components of \( \sigma \) and \( \beta \) is the number of \( u \)-derivatives in this combination. Let \( \Gamma \) consist of equivalence classes \( \{D\} \) such that \( \{D\} - \{D'\} \) is characterized by an element \( \sigma_{ab} \) of \( F \), where \( D' \) is any "trivial" connection, i.e., one for which \( \nabla_{\alpha} \sigma_{ab} \) and \( \sigma_{\alpha\beta} \) vanish.

29. Normally one works with Cauchy data on space-like surfaces and finds that zero rest mass fields have 2 degrees of freedom in the configuration space, and hence \( \sigma \) in the phase space. However, if one works on null surfaces using the characteristic initial value problem, one has only 2 degrees of freedom in the phase-space. This holds for any zero rest mass field; not just for gravitation.

30. A 2-form \( \Omega \) is said to be weakly non-degenerate if \( \Omega(u,v) = 0 \) for all \( v \) implies \( u = 0 \). Note that this implies that \( \Omega \) admits an inverse (i.e., is strongly non-degenerate) only if the underlying vector space is naturally isomorphic to its double dual. In our case, the space \( F \) (of footnote 28) does not satisfy this condition whence \( \Omega \) is only weakly non-degenerate.

31. A. Ashtekar and A. Magnon, Commun. Math. Phys. 86, 55 (1982). This assertion assumes the existence of (radiative) solutions to Einstein's equation which satisfy Definition 1 and admit a regular \( I^4 \).


34. More precisely, the Barrow-Newman-Misner 4-momentum of the space-time vanishes, whence there are cross-sections of \( I \) on which the Bondi energy is negative. This does not contradict positive energy theorems because these space-times have 2-dimensional sheets of nodal singularities outside event horizons.

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38. In particular, under the assumption that i^+ is regular, Friedrich establishes the existence of vacuum, radiating solutions in a neighborhood of (i^+)U(t^+). However, this region does not admit an asymptotically flat Cauchy surface since it excludes i^- (Private communication).

39. D. Christou and S. Kleinerman and S. T. Yau (Private communication by D.C.)


44. We also assume, as before, that A^a \to 0 as one approaches i^- Note, incidently, that since (B.63) implies: \text{E}_t = \mathcal{E}_A, the symplectic structure (B.64) is quite analogous to the gravitational symplectic structure (B.52), where \sigma_{at} plays the role of A^a.

45. Ludwigsen, M. Gen. Rel. and Grav. 73, 7 (1981).


47. See, e.g., T. Kibble, Phys. Rev. 173, 1527 (1968); 174, 1882 (1968); 175, 1624 (1968).


49. See, e.g., A. Ashtekar and A. Magnon, Gen. Rel. and Grav. 12, 205 (1980).

50. To compute the expression in the square bracket, we can just restrict ourselves to a cross-section C in S to which R^a, R^c, \bar{R}^a are tangential and regard f^a as a symmetric, trace-free tensor field on C. Let us now imbed C as the unit 2-sphere in the 3-dimensional Euclidean space. Then \bar{a} on C are the restrictions to C of the Cartesian coordinates X^i and the rotational vector fields, R^a, are given by \epsilon^{ab}\partial_b X^i, where \epsilon^{ab} and \partial are, respectively, the natural alternating tensor and the derivative operator induced on C by the...
Euclidean metric. Now, the expression in the square bracket reduces to

\[ X_i X_i = f_{ab}^+ \]
\[ = x_i^a \left[ \left( \varepsilon^{mn} a_m X_i \right) a_n f_{ab}^+ + f_{am}^+ \left( a_b \varepsilon^{mn} a_n X_i \right) + f_{mn}^+ \left( a_m \varepsilon^{mn} a_n X_i \right) \right] \]
\[ = \varepsilon^{mn} a_n (x_i^a X_i) a_m f_{ab}^+ + f_{am}^+ \varepsilon^{mn} q_{bn} + f_{mn}^+ \varepsilon^{mn} q_{an}, \]

where \( q_{ab} \) is the natural metric on \( C \). Now, since \( C \) is the unit 2-sphere, \( X_i X_i = 1 \) on \( C \), whence the first term vanishes. If we now regard \( C \) as being imbedded in \( I \), we have \( \varepsilon^{mn} = \varepsilon^{mp} q_{np} \), where \( \varepsilon^p \) is the normal covector to \( I \), whence the last two terms yield, on \( I \):

\[ \varepsilon^{mp} \varepsilon^p q_{bn} f_{am}^+ + \varepsilon^{mp} \varepsilon^p q_{an} f_{mb}^+. \]


52. Actually, non-linear gravitons were first obtained via twistor methods by R. Penrose, Gen. Rel. and Grav. 7, 31 (1976). Its connection to H-spaces was analyzed later; see, e.g., Sec. 6 of Ref. 51.


54. To speak of holomorphic functions, we have to endow \( \Gamma_0 \) with a complex structure. Fix any point \( \{ D \} \) of \( \Gamma_0 \) and consider its tangent space. Given any vector \( F_{ab} \) in the tangent space, we set \( J * f = i f^+ + (-1) f^- \), where \( f^+ \) and \( f^- \) are, respectively, the positive and negative frequency parts of \( f \). Thus, \( J \) is a linear operator in the tangent space satisfying \( J^2 = -1 \). This almost complex structure is integrable because it is constant w.r.t. the affine structure of \( \Gamma_0 \). We can now define holomorphicity: \( \psi : \Gamma_0 \to C \) is holomorphic iff \( J \psi \circ df = i df \), where \( d \) is the exterior differentiation operator on \( \Gamma_0 \).
Selected topics on quantum gravity are treated emphasizing more geometrical methods and conceptual issues than functional analysis, perturbative expansions and computation of numbers. The first part deals with Asymptotic Quantization shedding light on the origin of the Bondi-Metzner-Sachs group in the gravitational radiation theory. The second part is devoted to canonical quantization, which provides the detailed quantum dynamics and complements the purely kinematical asymptotic description presented in the first part.

Abhay Ashtekar received his Ph. D. from the University of Chicago. Currently, he is Professor of Physics at Syracuse University and Professeur de Gravitation at Université de Paris VI. During the preparation of this work, he held an Alfred P. Sloan Research Fellowship.